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Math 2210 Final Examination

By signing your name below, you agree to follow the Student Code of Conduct regarding "Academic Honesty" as you take this exam. You agree to take this exam without any outside help of any form. (Example: graphing calculator, book, notes, search engines, websites, friends or family etc help is NOT allowed).

Signature: Tyler Trotter

Show all appropriate work and simplify all answers.

1. (3 pts each) Let $\mathbf{u} = \langle -3, -2, 4 \rangle$ and $\mathbf{v} = \langle 5, -1, -2 \rangle$. Find each of the following:

(a) $2\mathbf{u} - 3\mathbf{v}$

$2\langle -3,-2,4\rangle - 3\langle 5,-1,-2\rangle$

$\langle -6-15, -4+3, 8+6\rangle$

$= \boxed{\langle -21, -1, 14\rangle}$

(b) $|\mathbf{u} + \mathbf{v}|$

$u+v = \langle -3+5, -2-1, 4-2\rangle$

$= \langle 2, -3, 2\rangle$

$|u+v| = \sqrt{4+9+4}$

$= \boxed{\sqrt{17}}$

(c) Vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$

$u \times v = \langle -3,-2,4\rangle$

$\times \langle 5,-1,-2\rangle$

$\langle 4+4, -(6-20), 3+10\rangle$

$= \boxed{\langle 8, 14, 13\rangle}$

(d) $\text{proj}_{\mathbf{u}}\mathbf{v}$ *Hint:* $\text{proj}_{\mathbf{u}}\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u}$

$u \cdot v = -15+2-8$

$= -21$

$|u| = \sqrt{9+4+16}$

$\sqrt{29}$

$-\frac{21}{29}\langle -3,-2,4\rangle$

$= \langle \frac{63}{29}, \frac{42}{29}, -\frac{84}{29}\rangle$

2. (12 pts) Suppose the acceleration of a particle is given by $\mathbf{a}(t) = e^{-t}\mathbf{i} - 6\sin(3t)\mathbf{j}$. Find the position function $\mathbf{r}(t)$ given that $\mathbf{v}(0) = \mathbf{i} - 2\mathbf{k}$ and $\mathbf{r}(0) = 3\mathbf{j}$.

$a(t) = \langle e^{-t}, -6\sin(3t), 0 \rangle$

$\int a(t)\,dt = \langle -e^{-t} + c, 2\cos(3t) + c, c \rangle = v(t)$; $v(0) = \langle 1, 0, -2 \rangle$

$\int v(t) = \langle -e^{-t} + 2, 2\cos(3t) - 2, -2 \rangle = r(t)$

$r(t) = \langle e^{-t} + 2t + c, \frac{2}{3}\sin(3t) - 2t + c, -2t + c \rangle$, $r(0) = \langle 0, 3, 0 \rangle$

$$\boxed{r(t) = \langle e^{-t} + 2t - 1, \frac{2\sin(3t)}{3} - 2t + 3, -2t \rangle}$$

3. (12 pts) Find the directional derivative of $f(x, y) = \dfrac{x^2}{x^2 + y^2}$ at the point $(1, -2)$ in the direction toward the point $(5, -3)$. *Hint:* $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$ = scalar

$u = \langle 5 - 1, -3 + 2 \rangle = \langle 4, -1 \rangle$

$\nabla f = \left\langle \frac{2x(x^2 + y^2) - 2x^3}{(x^2 + y^2)^2}, \frac{0 - 2x^2y}{(x^2 + y^2)^2} \right\rangle$ $\vec{u} = \left\langle \frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle$

$\nabla f(1, -2) = \left\langle \frac{2xy^2}{(x^2 + y^2)^2}, \frac{-2x^2y}{(x^2 + y^2)^2} \right\rangle = \left\langle \frac{8}{25}, \frac{4}{25} \right\rangle \cdot \left\langle \frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}} \right\rangle$

$\frac{32}{25\sqrt{17}} - \frac{4}{25\sqrt{17}} = \boxed{\frac{28}{25\sqrt{17}}}$

4. (14 pts) Use Lagrange multipliers to find the maximum and minimum values, if they exist, of $f(x, y) = 4x^2 + y$ subject to the constraint $xy = 1$, where x and y are both nonnegative.
Hint: $\nabla f = \lambda \nabla g$

g:

$\nabla f = \langle 8x, 1 \rangle \quad \nabla g = \langle y, x \rangle$

$8x = y\lambda \; ; \; \lambda = \frac{8x}{y} \, , \; y \neq 0$

$1 = x\lambda \; ; \; \lambda = \frac{1}{x} \, , \; x \neq 0$

$\lambda = \frac{8x}{y} = \frac{1}{x} \; ; \; 8x^2 = y$

$x(8x^2) = 1 \; ; \; 8x^3 = 1$

$x = \frac{1}{2} \, , \; y = 2$

crit.

$(\frac{1}{2}, 2)$, plug into f

$= 4(\frac{1}{2})^2 + 2 = 3$

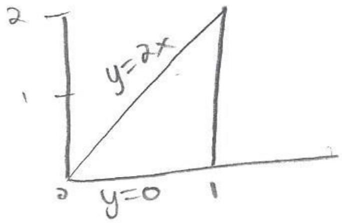
Check $(1,1) = 5$; Thus, $\boxed{3 \text{ is the minimum}}$

Addressing conditions $y \neq 0, x \neq 0$

If x or y are zero, we disobey the constraint

5. (12 pts) Evaluate the integral by switching the order of integration.

$$\int_0^2 \int_{y/2 = x}^{1 = x} \sqrt{x^2+4}\ dx\, dy.$$



$$\int_0^1 \int_0^{2x} \sqrt{x^2+4}\ dy\, dx$$

$$= \int_0^1 2x\sqrt{x^2+4}\ dx$$

let $u = x^2+4$

$du = 2x\,dx$

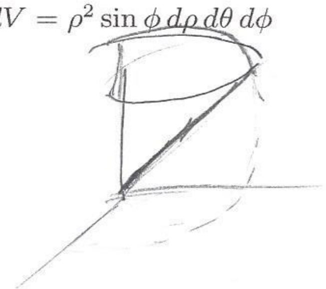
$$\int_0^1 u^{\frac{1}{2}}\, du = \frac{2u^{\frac{3}{2}}}{3}\Bigg| = \frac{2(x^2+4)^{\frac{3}{2}}}{3}\Bigg|_0^1$$

$$= \boxed{\left[\frac{2(5)^{\frac{3}{2}}}{3} - \frac{2(4)^{\frac{3}{2}}}{3}\right]}$$

$$= \boxed{\frac{2(5)^{\frac{3}{2}} - 16}{3}}$$

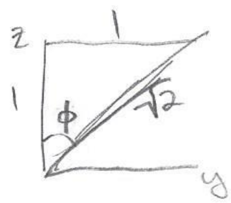
V

6. (12 pts) Use spherical coordinates to evaluate $\iiint_H xyz\,dv$ where H is the region in the first octant inside the sphere $x^2+y^2+z^2=2$ and above the cone $z=\sqrt{x^2+y^2}$.
Hint: $x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$, $x^2+y^2+z^2=\rho^2$,
$dV=\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$



$\rho^2=2\;;\;\rho=\pm\sqrt{2}$

$$=\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{\sqrt{2}}(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\;\partial\rho\,d\theta\,\partial\phi$$



$\tan^{-1}(1)=\pi/4$

$$=\iiint_0^{\sqrt{2}}\rho^5\sin^3\phi\cos\phi\sin\theta\cos\theta\,d\rho\,d\theta\,\partial\phi$$

$$=\frac{\rho^6}{6}\Big|_0^{\sqrt{2}}=\frac{8}{6}=\frac{4}{3}$$

$$\frac{4}{3}\int\sin^3\phi\cos\phi\,d\phi\int_0^{\pi/2}\underset{u}{\sin\theta}\,\underset{du}{\cos\theta\,d\theta}$$

$$=\frac{\sin^2\theta}{2}\Big|_0^{\pi/2}=\frac{1}{2}$$

$$=\frac{2}{3}\int_0^{\pi/4}\underset{u^3}{\sin^3\phi}\,\underset{du}{\cos\phi\,d\phi}$$

$$=\frac{2}{3}\left(\frac{\sin^4\phi}{4}\Big|_0^{\pi/4}=\frac{2}{3}\cdot\frac{1}{16}=\boxed{\frac{1}{24}}\right.$$

7. (14 pts) Evaluate $\iint\limits_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = \frac{1}{2}x^2\mathbf{i} - xy\mathbf{j} + xz\mathbf{k}$ and S is the surface of the paraboloid $z = x^2 + y^2$ with $z \le 4$ in the first octant with upward orientation.

Hint: $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\, dA$ or $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$

$$\iint -\left(\tfrac{1}{2}x^2\right)(2x) - (-xy)(2y) + (xz)\, dA$$

$$xz \rightarrow x(x^2+y^2)$$

$$= \iint -\cancel{x^3} + 2xy^2 + \cancel{x^3} + xy^2\, dA$$

*convert to polar: $x^2+r^2 \le 4$

$r^2 \le 4 : -2 \le r \le 2$

$$= \int_0^{\pi/2}\int_0^2 3xy^2 \cdot r\,dr\,d\theta$$

$$\rightarrow 3(r\cos\theta)(r\sin\theta)^2 \cdot r\,dr\,d\theta$$

$$= 3\int_0^{\pi/2}\int_0^2 r^4 \sin^2\theta \cos\theta\, dr\,d\theta$$

$$\frac{r^5}{5}\Big|_0^2 \int_0^{\pi/2} \sin^2\theta\cos\theta\, d\theta$$

$$= \frac{32}{5}\int_0^{\pi/2} \sin^2\theta \cos\theta\, d\theta$$

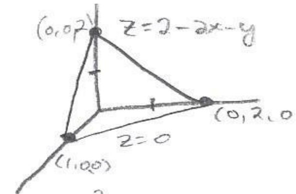
$$= \frac{32}{5}\,\frac{\sin^3\theta}{3}\Big|_0^{\pi/2} = \left(\frac{32}{15}\right)(3) = \boxed{\frac{32}{5}}$$

8. (12 pts) Use the Divergence Theorem to compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = x^2\,\mathbf{i} - xy\,\mathbf{j} + (x^3 - 2y)\,\mathbf{k}$ and S is the positively oriented surface of the tetrahedron bounded by the plane $2x + y + z = 2$ and the coordinate planes in the first octant.
Hint: $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div}\mathbf{F}\,dV$, $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$

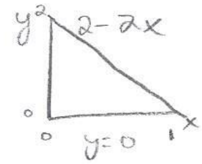
$2x+y+z=2$

$z=2-2x-y$

$\operatorname{div} F = \nabla F = 2x - x + 0 = x$



$$\int_0^1\int_0^{2-2x}\int_0^{2-2x-y} x\,dz\,dy\,dx$$



$$= \int_0^1\int_0^{2-2x} xz\Big|_0^{2-2x-y} dy\,dx \quad = x(2-2x-y)$$
$$= 2x-2x^2-xy$$

$$= \int_0^1\int_0^{2-2x} 2x-2x^2-xy\;dy\,dx = 2xy-2x^2y-\frac{xy^2}{2}\Big|_0^{2-2x}$$

$4-8x+4x^2$

$$= 2x(2-2x)-2x^2(2-2x)-\frac{x}{2}(2-2x)^2$$
$$= \cancel{4x}-\cancel{4x^2}-4x^2+\cancel{4x^3}-\cancel{2x}+\cancel{4x^2}-2x^3$$
$$= 2x-4x^2+2x^3$$

$$\int_0^1 2x-4x^2+2x^3\,dx$$

$$= x^2-\frac{4}{3}x^3+\frac{x^4}{2}\Big|_0^1 = 1-\frac{4}{3}+\frac{1}{2} = \frac{3}{2}-\frac{4}{3} = \boxed{\frac{1}{6}}$$

Extra Credit: (2 pts) What was your favorite part of the course? (Come on...you can think of *something*!)

Whiskey jokes, critical points and flux.

Thanks for the great semester, I learned a lot!