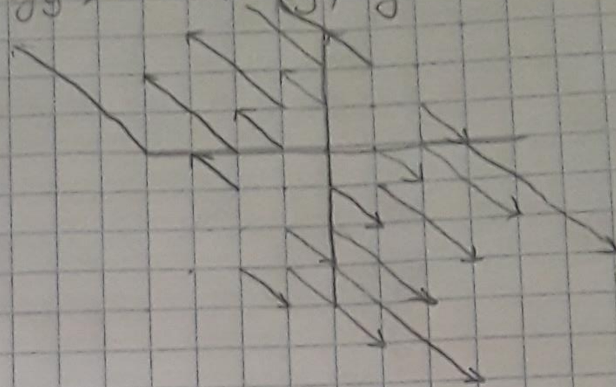
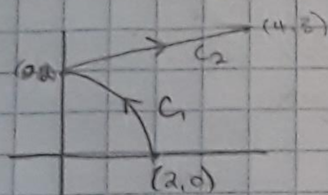
16.1 #25. $f(x,y) = \frac{1}{2}(x-y^2) = \frac{x^2}{2} - xy + \frac{y^2}{2}$

$\vec{F} = \nabla f = \langle f_x, f_y \rangle = \langle x-y,\ y-x \rangle$



16.2 #8 $\int_C x^2\,dx + y^2\,dy$




$C_1$: $x = r\cos(t)$ $y = r\sin(t)$ $r_{C_1}(t) = \langle 2\cos(t$

$r'(t) = \langle -2\sin(t),\ 2\cos(t) \rangle$

$C_2$: $x = 0 + t = t$ ; $y = \frac{x}{4} + 2 = \frac{t}{4} + 2$

$r_{C_2}(t) = \langle t,\ \frac{t}{4} + 2 \rangle$

$$\int_{0}^{\pi/2} \left[4\cos^2(t) \cdot -2\sin(t) + 4\sin^2(t) \cdot 2\cos(t)\right] dt \quad (C_1)$$

$$= 8\int_0^{\pi/2} \sin^2(t)\cos(t) - \cos^2(t)\sin(t) = 8\int \sin^2 t\cos t\,dt - 8\int c$$

$u = \sin$

$du = \cos t$

$\frac{8\sin^3(t)}{3} + \frac{8}{3}c$

$\left(\frac{8}{3} - 0\right) - \left(0\right.$

$= \boxed{0}$

$$\int_0^4 t^2 + \left(\frac{t^2}{16} + t + 4\right)\frac{1}{4}\,dt \quad (C_2)$$

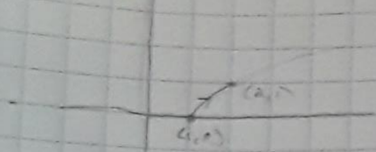
$t^2 + \frac{t^2}{64} + \frac{t}{4} + 1$

$$\frac{1}{3} \cdot \frac{65}{64} t^3 + \frac{t^2}{8} + t \Big|_0^4 = \frac{65}{3} + 2 + 4$$

$$= \frac{65}{3} + \frac{18}{3} = \boxed{\frac{83}{3}}$$

$C_1 + C_2 = 0 + \frac{83}{3} = \boxed{\frac{83}{3}}$

16.2 #40 $F(x,y) = x^2 i + y e^x j$




$x = y^2 + 1 \quad y = y_0 + t$

$y = t \; , \; x = t^2 + 1$

$dy = dt \quad dx = 2t\,dt$

$$\int_0^1 (t^4 + 2t^2 + 1)(2t\,dt) + t e^{t^2+1}\,dt$$

$$\int t^5 + 2t^3 + t\,dt = 2\left(\frac{t^6}{6} + \frac{t^4}{2} + \frac{t^2}{2}\right)\Big|_0^1$$

$$2\left(\frac{1}{6} + \frac{1}{2} + \frac{1}{2}\right)$$

$$\frac{1}{3} + 2 = \boxed{\frac{7}{3}}$$

$$+ \int_0^1 t e^{t^2+1}\,dt$$

Let $u = t^2 + 1$

$du = 2t\,dt$

$\frac{1}{2}du = t\,dt$

$$\frac{11}{6} + \frac{e^2}{2} - \frac{1}{2}$$

$$\frac{e^{t^2+1}}{2}\Big|_0^1 \quad \frac{e^2}{2} - \frac{e}{2}$$

$$= \boxed{\frac{e^2 - e}{2}}$$

Thus,

$$= \boxed{\frac{7}{3} + \frac{e^2 - e}{2}}$$

16.3 #12 $F(x,y) = \langle 3 + 2xy^2, 2x^2y \rangle$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ; $4xy = 4xy$ ✓ Conservative

$f = \int P\,dx = 3x + x^2y^2 + g(y)$

$f_y = 2x^2y + g'(y) = 2x^2y \quad g'(y) = 0$

a) $\boxed{f = 3x + x^2y^2 + C \text{ where } C = 0}$

b) $f(b) - f(a) = 12 + 1 - 4 = \boxed{9}$

16.3 #29 $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} = \langle P, Q, R \rangle$

$F = \nabla f$, thus $F = \langle f_x, f_y, f_z \rangle$

$P = f_x \quad Q = f_y \quad R = f_z$

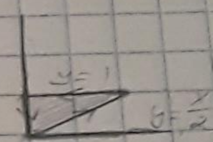
$\boxed{P = \frac{\partial f}{\partial x} \quad Q = \frac{\partial f}{\partial y} \quad R = \frac{\partial f}{\partial z}}$

$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \quad \frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$, thus $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ ✓

$\frac{\partial P}{\partial z} = \frac{\partial^2 f}{\partial z \partial x} \quad \frac{\partial R}{\partial x} = \frac{\partial^2 f}{\partial x \partial z}$, thus $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$ ✓

$\frac{\partial Q}{\partial z} = \frac{\partial^2 f}{\partial z \partial y} \quad \frac{\partial R}{\partial y} = \frac{\partial^2 f}{\partial y \partial z}$, thus $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$ ✓

16.4 #6 $\int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$




$= \iint_R \int_0^2 \int_{x/2}^{1} x - y \; dy\, dx$

$\int xy - \frac{y^2}{2} \Big|_{x/2}^{1} = 2\left(x - \frac{1}{2}\right) - \left(\frac{x^2}{2} - \frac{x^2}{8}\right)$

$2x - 1 - x^2 + \frac{x^2}{4}$

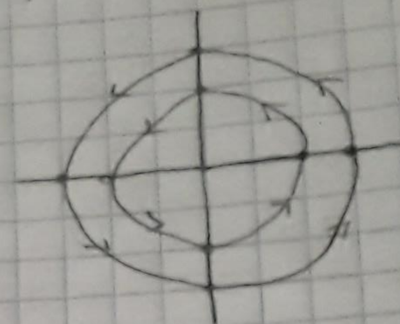
$\int -\frac{3x^2}{4} + 2x - 1$

$= -\frac{x^3}{4} + x^2 - x \Big|_0^2$

$-2 + 4 - 2 = -4 + 4 = \boxed{0}$

16.4 # 10 $\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy$ C is boundary between $x^2+y^2=4$ & $x^2+y^2=9$

$r^2=4$ $r^2=9$



$$\iint 3x^2+3y^2\,dA$$

$$= \int_0^{2\pi}\int_0^{3} 3(r^2)\cdot r\,dr\,d\theta$$

$$= \iint 3r^3\,dr\,d\theta$$

$$2\pi\left(\frac{3r^4}{4}\Big|_2^3\right) \quad 2\pi\left(\frac{3(81)}{4}-12\right)$$

$$\frac{243\pi}{2}-24\pi = \frac{243\pi-48\pi}{2}$$

$$= \boxed{\frac{195\pi}{2}}$$

5 #1 $F(x,y,z) = \langle xy^2z^2, x^2yz^2, x^2y^2z\rangle$

$$\nabla\times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = \langle \overbrace{2x^2yz-2x^2yz}^{0}, -(\overbrace{2xy^2z-2xy^2z}^{0}), \overbrace{2xyz^2-2xyz^2}^{0}\rangle$$

Curl $\vec{F} = 0$

7 $F(x,y,z) = \langle e^x\sin y, e^y\sin z, e^z\sin x\rangle$

a) $$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\sin y & e^y\sin z & e^z\sin x \end{vmatrix} = \langle 0-e^y\cos(z), -(e^z\cos(x)-0), 0-e^x\cos y\rangle$$

$$= \boxed{\langle -e^y\cos z, -e^z\cos x, -e^x\cos y\rangle}$$

b) $\text{div}\,F = \nabla\cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle\cdot\langle e^x\sin y, e^y\sin z, e^z\sin x\rangle$

$$= \boxed{e^x\sin y + e^y\sin z + e^z\sin x}$$

16.5 # 13 $F(x,y,z) = y^2 z^3 i + 2xyz^3 j + 3xy^2 z^2 k$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{vmatrix} = \langle \overset{0}{6xyz^2 - 6xyz^2}, -\overset{0}{(3y^2z^2 - 3y^2z^2)}, \overset{0}{2yz^3 - 2y} \ldots$$

$\text{Curl } \vec{F} = 0$, cons. ✓

$f = \int P\,dx = xy^2z^3 + g(y,z)$

$f_y = Q = 2xyz^3 = f' = 2xyz^3 + g_y$

$\int Q\,dy = xy^2z^3 + h(z)$ $\quad 3xy^2z^2 + h_z = R = 3xy^2z^2$

$$\boxed{f = xyz^3 + C \text{ where } C = 0}$$

16.6 # 20 Passes through $(0,-1,5)$

contains $\langle 2,1,4\rangle$ and $\langle -3,2,5\rangle$

$\langle 2,1,4\rangle$

$\times \langle -3,2,5\rangle$ $\quad \langle -3,-22,7\rangle$; $-3x - 22y - 22 + 7z - 35 = 0$

let $x = u$ $\quad z = \frac{57}{7}u + \frac{22}{7}v$ $\quad -3x - 22y + 7z = 57$

$y = v$ $\quad z = \frac{57x}{7} + \frac{22y}{7}$

$$\boxed{r(u,v) = \langle u, v, \tfrac{57u}{7} + \tfrac{22v}{7} \rangle}$$

16.6 # 34 $x = u^2+1$ $\quad y = v^3+1$ $\quad z = u+v$ at $(5,2,3)$

$r(u,v) = \langle u^2+1, v^3+1, u+v\rangle$ $\quad u^2+1 = 5$, $v^3+1 = 2$, $u+v = 3$ $\quad u = 2$, $v = 1$

$r_u = \langle 2u, 0, 1\rangle \times r_v = \langle 0, 3v^2, 1\rangle =$

$r_{u_0} = \langle 4,0,1\rangle \times r_{v_0} = \langle 0,3,1\rangle$

$\langle 0,3,1\rangle$ $\quad \langle -3,-4,12\rangle$

$$\boxed{-3(x-5) - 4(y-2) + 12(z-3) = 0}$$

$= -3x + 15 - 4y + 8 + 12z - 36 = 0$

$$\boxed{-3x - 4y + 12z = 13}$$

16.6 #40 $\quad r(u,v) = \langle u+v,\ 2-3u,\ 1+u-v\rangle \quad 0 \le u \le 2;\ -1 \le v \le 1$

$$S = \iint_D |r_u \times r_v|\, dA = \iint_D \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \quad \text{where } z = g(x,y)$$

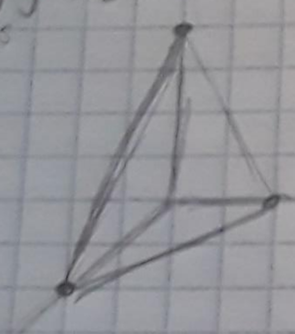
$$\begin{array}{cccc} & i & j & k \\ r_u = & \langle 1, & -3, & 1\rangle \\ r_v = & \langle 1, & 0, & -1\rangle \end{array}$$

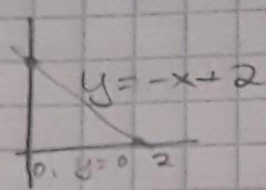
$$\langle 3, 2, 3\rangle \qquad = \langle 3-0,\ -(-1-1),\ 0+3\rangle$$

$$\iint \sqrt{9+4+9}\, dA = \int_0^2\int_{-1}^1 \sqrt{22}\, dv\, du$$

$$\int_0^2 2\sqrt{22}\, du = \boxed{4\sqrt{22}}$$

7 #10 $\quad \iint_S xz\, dS \qquad S$ is $2x+2y+z=4$ in the first octant






$z = 4-2x-2y$

$z_x = -2$

$z_y = -2$

$$\int_0^2\int_0^{-x+2} x(4-2x-2y)\sqrt{(-2)^2+(-2)^2+1}\; dy\, dx$$

$$\int_0^2 \sqrt{3}\int_0^{-x+2} 4x - 2x^2 - 2xy\; dy$$

$$= \int_0^2 4xy - 2x^2y - xy^2 \Big|_0^{-x+2} \qquad -x(x^2-4x+4)$$

$$\int 4x(-x+2) - 2x^2(-x+2) - x(-x+2)^2$$

$$\cancel{-4x^2} + 8x \cancel{+2x^3} - 4x^2 - \cancel{x^3} + \cancel{4x^2} - 4x$$

$$\int x^3 - 4x^2 + 4x$$

$$\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2 \Big|_0^2$$

$$3\left(4 - \frac{32}{3} + 8\right) = \boxed{4}$$

16.7 # 22 $r(u,v) = \langle u\cos(v), u\sin(v), v \rangle$ $0 \le u \le 1$, $0 \le v \le \pi$

$F(x,y,z) = \langle z, y, x \rangle$

$r_u = \langle \cos(v), \sin(v), 0 \rangle$

$r_v = \langle -u\sin(v), u\cos(v), 1 \rangle$ $= \langle \sin v, -\cos v, u \rangle$

$\langle v, u\sin(v), u\cos(v) \rangle \cdot \langle \sin(v), -\cos(v), u \rangle$

$\int_0^{\pi}\int_0^1 v\sin v - u\cos(v)\sin(v) + u^2\cos(v) \, du \, dv$

$vu\sin(v) - \frac{u^2\cos(v)}{2} + \frac{u^3\cos(v)}{3} \Big|_0^1$

$v\sin(v) - \frac{\cos(v)}{2} + \frac{\cos(v)}{3}$

$v\sin(v) + \frac{2\cos(v) - 3\cos(v)}{6} = \int v\sin(v) - \frac{\cos(v)}{6}$

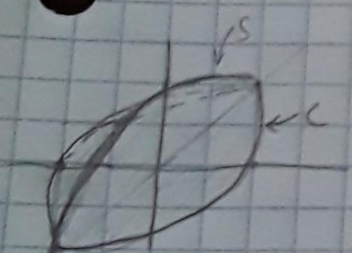
let $u = v$ $dv = \sin$
$du = dv$ $v = -\cos$

$-v\cos(v) \Big|_0^{\pi} = [-\pi(-1)] - [0]$

$= \boxed{\pi}$

16.8 # 2

Stokes' $\int_C F \cdot dr = \iint_S \operatorname{curl} F \cdot dS$




$F(x,y,z) = \langle x^2 \sin z, y^2, xy \rangle$

$S = z = 1 - x^2 - y^2$ (irrelevant)

$C = x^2 + y^2 = 1$ ; $r = 1$

$r(t) = \langle r\cos t, r\sin t \rangle = \langle \cos t, \sin t, 0 \rangle$

$F(r(t)) = \langle \cos^2(t)\sin(0), \sin^2(t), \cos(t)\sin(t) \rangle$

$= \langle 0, \sin^2(t), \cos(t)\sin(t) \rangle$

$\frac{dr}{dt} = \langle -\sin(t), \cos(t), 0 \rangle$

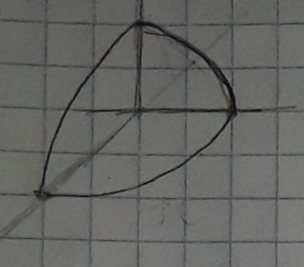
$\int_{0\,C}^{2\pi} F(r(t)) \cdot r'(t) \, dt = \int_0^{2\pi} \langle 0, \sin^2(t), \cos(t)\sin(t) \rangle \cdot \langle -\sin t, \cos(t)$

$= \int \sin^2(t)\cos(t)$

$= \left( \frac{\sin^3(t)}{3} \right) \Big|_0^{2\pi} = \boxed{0}$

.8 #9 Evaluate $\int_C F \cdot dr = \iint_S \text{curl} F \cdot dS$

$F(x,y,z) = \langle xy, yz, zx \rangle$, C: $z = 1 - x^2 - y^2$ in octant I



$$\text{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \langle 0 - y, -(z - 0), 0 - x \rangle$$

$F_{new} = \langle -y, -z, -x \rangle$

$z = g(x,y)$

$$\iint_D y(-2x) + z(-2y) - x \, dA$$

$x^2 + y^2 = 1$

$y = \sqrt{1 - x^2}$

$$= \iint -2xy - 2yz - x$$

$$= \iint -2xy - 2y(1 - x^2 - y^2) - x \, dA$$

$$-2xy - 2y + 2x^2y + 2y^3 - x \, dA$$

→ Convert to polar $-2(r\cos\theta)(r\sin\theta) - 2(r\sin\theta)$

$$+ 2(r\cos\theta)^2(r\sin\theta) + 2(r\sin\theta)^3 - (r\cos\theta)$$

$$2\sin\theta \, (- r\cos\theta - r + r^2\cos^2\theta + r^2\sin^2\theta - \frac{r\cos\theta}{2}$$

$$r^2 - r - \frac{3r\cos\theta}{2}$$

$$\frac{r^3}{3} - \frac{r^2}{2} - \frac{3r^2\cos\theta}{4} \Big|_0^1$$

$$\int -\frac{1}{6} - \frac{3}{4}\cos\theta$$

$$-\frac{\theta}{6} - \frac{3\sin\theta}{4} \Big|_0^{\pi/2}$$

$$\left(-\frac{\pi}{12}\right) - \left(-\frac{3}{4}\right)$$

$$= \boxed{\frac{9 - \pi}{12}}$$

16.9 # 11 $\quad F(x,y,z) = \langle 2x^3 + y^3,\ y^3 + z^3,\ 3y^2 z \rangle$

$$\iint F \cdot dS = \iiint_V \operatorname{div} F \cdot dV$$

$$\operatorname{div} F = 6x^2 + 3y^2 + 3y^2 = 6x^2 + 6y^2 \text{ or } 6(x^2+y^2)$$

$$6\iiint x^2+y^2 \quad r \cdot dz\, dr\, d\theta$$

$$6\int_0^{2\pi}\int_0^1\int_0^{1-r^2} r^2 \cdot r \, dz\, dr\, d\theta$$

$$= r^3 z \Big|_0^{1-r^2} \qquad = r^3(1-r^2)$$

$$6\int (r^3 - r^5$$

$$= 6\left(\frac{r^4}{4} - \frac{r^6}{6}\right|_0^1$$

$$= 6\left(\frac{3}{12} - \frac{2}{6}\right)$$

$$\int 6\left(\frac{1}{12}\right) d\theta = \frac{12\pi}{12} = \boxed{\pi}$$