Name(2 pts): Tyler Trotter

Math 2210 Exam 4

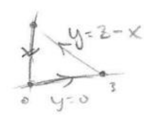
By signing your name below, you agree to follow the Student Code of Conduct regarding "Academic Honesty" as you take this exam. You agree to take this exam without any outside help of any form. (Example: graphing calculator, book, notes, search engines, websites, friends or family etc help is NOT allowed).

Signature: Tyler Trotter

Show all appropriate work and simplify all answers.

1. (16 pts each) Consider the line integral $\int_C (x+y)\,dx - x^2\,dy$, where C consists of the line segments from $(0,0)$ to $(3,0)$, from $(3,0)$ to $(0,3)$ and from $(0,3)$ to $(0,0)$. Verify Green's Theorem by:

(a) Evaluating the line integral using Green's Theorem.



$\frac{\partial Q}{\partial x} = -2x \qquad \frac{\partial P}{\partial y} = 1$

$\ominus \int_0^3 \int_0^{3-x} 2x+1 \; dy\,dx$

$= -\int 2xy + y \Big|_0^{3-x}$

$-\int 2x(3-x) + 3-x$

$6x - 2x^2 + 3 - x = -2x^2 + 5x + 3$

$9 - \frac{45}{2}$

$\frac{18}{2} - \frac{45}{2} = \frac{27}{2}$

$-\int_0^3 -2x^2\,5x + 3 \; dx$

$= -\left(-\frac{2x^3}{3} + \frac{5x^2}{2} + 3x \right|_0^3 = 18 - \frac{45}{2} - 9$

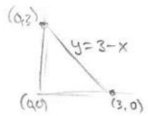
$-\left(-18 + \frac{45}{2} + 9 = \frac{45}{2} - \frac{18}{2} = \boxed{\frac{27}{2}}\right.$

(1. continued) $\int_C (x+y)\,dx - x^2\,dy$, C from $(0,0)$ to $(3,0)$ to $(0,3)$ to $(0,0)$

(b) Evaluating the line integral directly (i.e. use 16.2 techniques).

C_1: $x = 0 + t$; $0 \le t \le 3$
$y = 0$

$$\int_{0}^{3} t - t^2 \, dt \quad (C_1)$$



$$\frac{t^2}{2} - \frac{t^3}{3}\Big|_0^3 = \frac{9}{2} - 9 = -\frac{9}{2}$$

$3 - 4t^2$

$3t$

$x = 2t$

$3 - 2t$

$3t - \frac{4t^3}{3}$

C_2: $x = 3 - t$ $dx = -1\,dt$
$y = 3 - (3-t) = t$ $dy = dt$
$0 \le t \le 3$

$\frac{9}{2} - \frac{27}{6}$

$$\int_0^3 (3-t) + t \overset{(-1\,dt)}{} - (3-t)^2 (dt) \quad (C_2)$$

$-(9 - 6t + t^2)$

$$= \int -3 - 9 + 6t - t^2 \, dt = -t^2 + 6t - 12$$

$(2t+1)(3-t)$

$3 - t + t$

$$= -\frac{t^3}{3} + 3t^2 - 12t \Big|_0^3$$

$-\frac{t^3}{3}$

$$= -9 + 27 - 36 = -18$$

C_3: let $y = 3 - t$ $dy = -1\,dt$
$x = 0$ $0 \le t \le 3$

$$\int_0^3 3 - t \, dt \quad (C_3)$$

$$= 3t - \frac{t^2}{2}\Big|_0^3 = 9 - \frac{9}{2}$$

$3\left(\frac{3}{2}\right) - \frac{4}{3}\left(\frac{3}{2}\right)^3$

$$\boxed{C_1 + C_2 + C_3 = -18}$$

$\frac{4}{3} \cdot \frac{27}{8}$

Hm . . .

$-C_1 = C_3$; C_2 = Green's theorem — I can't seem to replicate solutions

2. Consider the vector field $\mathbf{F} = \langle 3x^2y + e^x \cos z, x^3 - 2, -e^x \sin z - 2z \rangle$.

(a) (4 pts) Use the curl to show $\mathbf{F}$ is conservative.

$$\text{curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y + e^x\cos z & x^3 - 2 & -e^x \sin z - 2z \end{vmatrix}$$

$$= \langle 0 - 0, -(-e^x \sin z - \overset{0}{e^x \sin z}), 3x^2 - \overset{0}{3x^2} \rangle$$

$$= \boxed{0, \text{ thus conservative}}$$

(b) (8 pts) Find a function f such that $\mathbf{F} = \nabla f$.

$$f = \int P\,dx = \int 3x^2y + e^x \cos z \, dx = x^3y + e^x \cos z + g(y, z)$$

$$Q = f_y = x^3 + g_y(y, z) = x^3 - 2 \quad ; \quad g_y(x, y) = -2$$

$$f = \int Q\,dy = x^3y - 2y + h(z)$$

$$R = f_z = h'(z) = -e^x \sin z - 2z$$

$$f = \int h'(z)\,dz = e^x \cos z - z^2 + C$$

$$\boxed{f = x^3y + e^x \cos z - 2y - z^2 + C, \text{ where } C = 0}$$

(c) (6 pts) Evaluate $\int \mathbf{F} \cdot d\mathbf{r}$ using *FTOLI*, where $\mathbf{r}(t) = \langle t^2, 2t + 1, t^2 - t \rangle$, $0 \le t \le 1$.

$$f(r(1)) - f(r(0)) = f(1, 3, 0) - f(0, 1, 0)$$

$$f(1, 3, 0) = (1)(3) + e^{(1)} \cos(0) - 2(3) - 0 = 3 + e - 6 = -3 + e$$

$$f(0, 1, 0) = 0(1) + 1 - 2(1) - 0 = -1$$

$$= \boxed{-4 + e}$$

3. (14 pts) Find the equation of the tangent plane to the surface given by $x = u^2 - 2v^2$, $y = 3u - v$, $z = 2u + 3v$ when $u = 2$ and $v = -1$.

$X = u^2 - 2v^2$ $\quad X = 4 - 2 = 2$ $\quad (x_0, y_0, z_0) = (2, 7, 1)$

$y = 3u - v$ $\quad y = 3(2) - (-1) = 7$

$Z = 2u + 3v$ $\quad z = 2(2) + 3(-1) = 1$

$r(u,v) = \langle u^2 - 2v^2, 3u - v, 2u + 3v \rangle$

$r_u = \langle \overset{4}{2u}, 3, 2 \rangle$

$\times\ r_v = \langle \overset{-8}{-4v}, -1, 3 \rangle$ $\quad = \langle 9 + 2, -(12 + 16), -4 + 24 \rangle$

$n = \langle 11, -28, 20 \rangle$

$11(x - 2) - 28(y - 7) + 20(z - 1) = 0$

$11x - 22 - 28y + 196 + 20z - 20 = 0$

$\boxed{11x - 28y + 20z - 154 = 0}$

v

4. (16 pts) Find the surface area of the portion of $\mathbf{r}(u,v) = \langle \overset{x}{3u+v}, \overset{y}{2u-v}, \overset{z}{u+2v} \rangle$ that lies over the region in the uv-plane inside $u^2+v^2=4$ with $v \geq 0$.

$$r_u = \langle 3, 2, 1 \rangle$$
$$\times\ r_v = \langle 1, -1, 2 \rangle$$
$$= \langle 4+1, -(6-1), -3-2 \rangle$$
$$= |\langle 5, -5, -5 \rangle| = \sqrt{25+25+25}$$

$$\iint 5\sqrt{3}\ dA$$ * Convert to polar

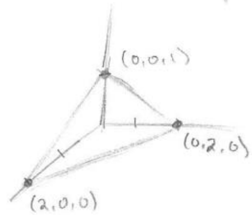
$$5\sqrt{3} \int_0^{2\pi} \int_0^{2} r\ dr\ d\theta \ ;\ 5\sqrt{3} \int_0^{2\pi} \frac{r^2}{2} \Big|_0^2 = 8\pi \cdot 5\sqrt{3} = \boxed{40\pi\sqrt{3}}$$

5. (18 pts) Use Stoke's Theorem to compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where
$\mathbf{F}(x,y,z) = (x^3 - z^2)\mathbf{i} + (\sqrt{y} - x^2)\mathbf{j} + z^5\mathbf{k}$ and C is the positively oriented triangle with vertices $(2,0,0)$, $(0,2,0)$, $(0,0,1)$.

$$\int_C F \cdot dr = \iint_S \text{curl } F \cdot dS$$

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 - z^2 & \sqrt{y} - x^2 & z^5 \end{vmatrix} = \langle 0-0, -(0+2z), -2x-0\rangle$$

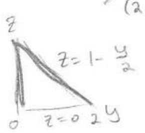
$$= \langle 0, -2z, -2x \rangle$$

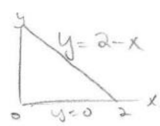


$\langle -2, 0, 1\rangle$
$\times \langle -2, 2, 0\rangle$ $= \langle -2, -2, -4\rangle$

$= \langle 1, 1, 2\rangle$

$(x-2) + y + 2z = 0$





$x + y + 2z = 2$

$z = -\frac{x}{2} - \frac{y}{2} + 1$

$F(x,y,g(x,y)) = \langle x^3 - (-\frac{x}{2} - \frac{y}{2} + 1)^2, \sqrt{y} - x^2, (-\frac{x}{2} - \frac{y}{2} + 1)^5 \rangle$

let $z = g(x,y)$; $\iint_S \text{curl } \vec{F} \cdot dS = \iint_A -P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \, dA$

$$\int_0^2 \int_0^{2-x} (0)(-\tfrac{1}{2}) - (\cancel{2}(-\tfrac{x}{2} - \tfrac{y}{2} + 1))(-\tfrac{1}{2}) - 2x \, dA$$

$0 + \frac{x}{2} + \frac{y}{2} - 1 - 2x = \frac{x}{2} - \frac{4x}{2} = -\frac{3x}{2}$

$$\frac{1}{2}\int_0^2 \int_0^{2-x} -3x + y - 2 \, dy\, dx = -3xy + \frac{y^2}{2} - 2y \Big|_0^{2-x}$$

cont.
in box

$\boxed{\text{cont.}}$

$-3x(2-x) + \frac{(2-x)^2}{2} - 2(2-x)$

$-6x + 3x^2 + 2 - 2x + \frac{x^2}{2} - 4 + 2x$

$\int \frac{7x^2}{2} - 6x - 2 \, dx$

$\frac{1}{2}\left(\frac{7x^3}{6} - 3x^2 - 2x \Big|_0^2\right.$

$\frac{14}{3} - 6 - 2$

$= \boxed{-\frac{10}{3}}$

Formulas

Fundamental Theorem of Line Integrals: If $\mathbf{F}$ is conservative,
$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

Green's Theorem: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$

Curl: $\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$ **Divergence:** $\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$

Surface Area:

1) $A(S) = \iint\limits_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$

2) $A(S) = \iint\limits_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$

Surface Integral of a Scalar Function (Flux):

1) $\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$

2) $\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}\,dA$ where $z = g(x,y)$

Surface Integral of a Vector Function:

1) $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_S \mathbf{F} \cdot \mathbf{n}\,dS$ with $\mathbf{n}$ a unit normal vector to S

2) $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA$

3) $\iint\limits_S \mathbf{F} \cdot d\mathbf{S} = \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$ where $z = g(x,y)$ oriented upward

Stokes' Theorem:

$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_S \mathrm{curl}\mathbf{F} \cdot d\mathbf{S}$