

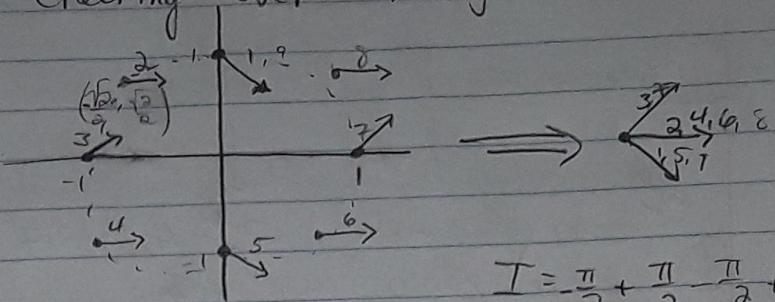
1: Math 5470, Midterm 2 by: Tyler Trotter
Checking over $x^2 + y^2 = 1$

$$\begin{aligned}\dot{x} &= x^2 + y^2 = 1 \\ \dot{y} &= x^2 - y^2\end{aligned}$$

Assume Linearization:

$$J = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \end{bmatrix};$$

$$J_{(0,0)} = 0$$



$$I = -\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} = 0$$

$$[I = 0]$$

2: $\dot{x} = x - y$ $(x^*, y^*) = (2, 2), (-2, -2)$
 $\dot{y} = x^2 - 4$ Nullclines: $\dot{x} = 0 = x - y \Rightarrow y = x$
 $\dot{y} = 0 = x^2 - 4 \Rightarrow x = \pm 2$

Assume Linearization:

$$J = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix}; \quad J_{(2,2)} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$$

(1) Since $\lambda_1, \lambda_2 > 0$ and $\text{Tr } J < 0$, we have an unstable spiral.

$$(1-\lambda)(-2\lambda) + 4 = \lambda^2 - \lambda + 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1 - 16}}{2}$$

Both Eigenvalues have positive real part. $\frac{1}{2}$

By observations (1) & (2), this is a robust case where the linearization is valid.

$$\lambda_1 = \frac{1 + i\sqrt{15}}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} & -1 \\ 4 & -\frac{1}{2} - \frac{i\sqrt{15}}{2} \end{bmatrix}$$

λ_2 does not matter for v_2 . the complex case

2: Midterm 2, cont.
Proceed with checking $(-2, -2)$

$$J(-2, -2) = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$$

$$\gamma = 1, \Delta = -4, \gamma^2 - 4\Delta = 17$$

① Since $\Delta < 0$, we have a saddle.

$$(1-\lambda)(-\lambda) - 4 = \lambda^2 - 2\lambda - 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{17}}{2}$$

② Since both eigenvectors have a positive real part, we have a repeller again. So, by ①, we have an "unstable" saddle.

$$\lambda_1 = \frac{1 + \sqrt{17}}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{17}}{2} & -1 \\ -4 & \frac{1}{2} + \frac{\sqrt{17}}{2} \end{bmatrix}, \lambda_2 = \frac{1 - \sqrt{17}}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{17}}{2} & -1 \\ -4 & \frac{1}{2} - \frac{\sqrt{17}}{2} \end{bmatrix}$$

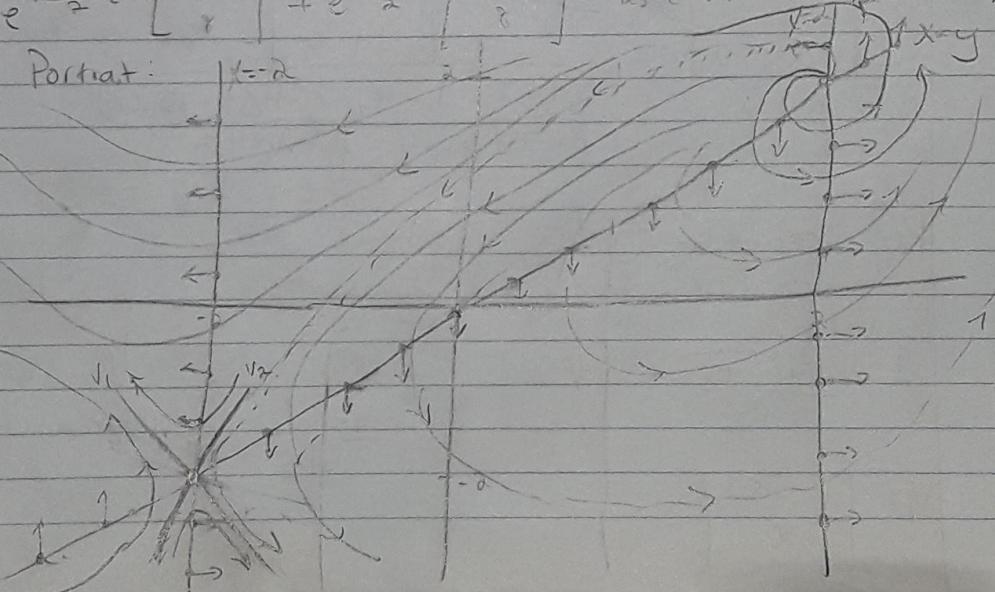
$$v_1 = \begin{bmatrix} -1 - \sqrt{17} \\ 8 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 + \sqrt{17} \\ 8 \end{bmatrix}$$

$$x(t) = e^{\frac{1+\sqrt{17}}{2}t} \begin{bmatrix} -1 - \sqrt{17} \\ 8 \end{bmatrix} + e^{\frac{1-\sqrt{17}}{2}t} \begin{bmatrix} -1 + \sqrt{17} \\ 8 \end{bmatrix}$$

as $t \rightarrow \infty$ we approach v_1 .

Phase Portrat:



Midterm 2

3. $r = \sin r$

$\theta = 1$

$$x(t) = r(t) \cos(\theta(t)), \quad x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}$$

$x = r \cos \theta - r \sin \theta$

$$\dot{x} = \sin(r) \cos \theta - r \sin \theta \cdot \dot{\theta} = \sin \theta \cos \theta$$

$$\dot{x} = r \cos \theta - r \sin \theta \cdot \dot{\theta}$$

$r = \sin r$

$\dot{r} = \sin(r) \cos \theta - r \sin \theta$

$\dot{\theta} = 1$

$$\dot{x} = \sin(r) \cos \theta - y$$

$$xr = x \sin(r) - ry$$

$$\dot{x} = \frac{x \sin(\sqrt{x^2+y^2}) - y}{\sqrt{x^2+y^2}}$$

$$\boxed{\dot{x} = \frac{x \sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} - y} \rightarrow -y$$

$$y = r \sin \theta ; \quad \dot{y} = r \sin \theta + r \cos \theta \dot{\theta}$$

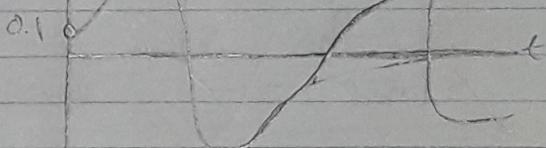
$$y = \sin(r) \sin(\theta) + x$$

$$yr = \sin(r) y + xr$$

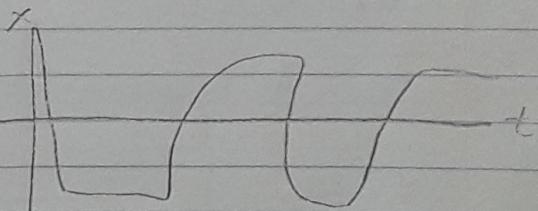
$$\dot{y} = y \sin(\sqrt{x^2+y^2}) + x$$

$$\boxed{\dot{y} = \frac{y \sin(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} + x} \rightarrow x$$

a) $x(0) = 0.1, y(0) = \pi$



b) $x(0) = 2\pi - 0.1, y(0) = 0$



Midterm 2

4: $\dot{x} = y + 2xy$
 $\dot{y} = x + x^2 - y^2$

$$(x^*, y^*) = (0, 0), (-1, 0)$$

Nullclines: $\dot{x} = 0 = y + 2xy \Rightarrow y = -\frac{1}{2}x$, $y = 0$

$$\dot{y} = 0 = x + x^2 - y^2$$

Assume linearization.

$$J = \begin{bmatrix} 2y & 1+2x \\ 1+2x & -2y \end{bmatrix}; J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\gamma = 0, \lambda = -1, \gamma^2 - 4\lambda = 4$$

Check: $f_y = 1+2x$

$$\lambda^2 - 1 = 0, \lambda = \pm 1 \Rightarrow \text{Saddle}$$

Gradient: $g_x = 1+2x$

$$\dot{x} = -\frac{\partial V}{\partial x};$$

$$\int y + 2xy \, dx = xy + x^2y + C(y)$$

$$\frac{\partial}{\partial y} (xy + x^2y + C(y)) = x + x^2 + C'(y)$$

$$C'(y) = -\frac{y^3}{3}$$

$$V(x,y) = -xy - x^2y + \frac{y^3}{3}$$

$$\left[-\frac{\partial V}{\partial x} = y + 2xy \underset{= \dot{x}}{\overset{-x}{\longrightarrow}} \quad -\frac{\partial V}{\partial y} = x + x^2 - y^2 \underset{= \dot{y}}{\overset{y}{\longrightarrow}} \right]$$

So, this system can be written as $-\nabla V$. By Theorem 7.2.1, closed orbits are impossible in this system. \blacksquare

Midterm 2

S a) Find a conserved quantity for:

$$\ddot{x} = x - x^2$$

$$\dot{y} = y$$

$$\ddot{y} = x - x^2$$

$$J = \begin{bmatrix} 0 & 1 \\ 1-x & 0 \end{bmatrix}$$

b)

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \gamma = 0, \Delta = -1, \gamma^2 - 4\Delta = 4$$

$t \rightarrow \infty, \rightarrow e^t$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

$$J_{(1,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \gamma = 0, \Delta = 1, \gamma^2 - 4\Delta = -4$$

So, the origin is a saddle!

$$\ddot{x} - x + x^2 = 0$$

Since a conserved quantity exists, the linearization is accurate. The trajectories are closed curves for $(1,0)$

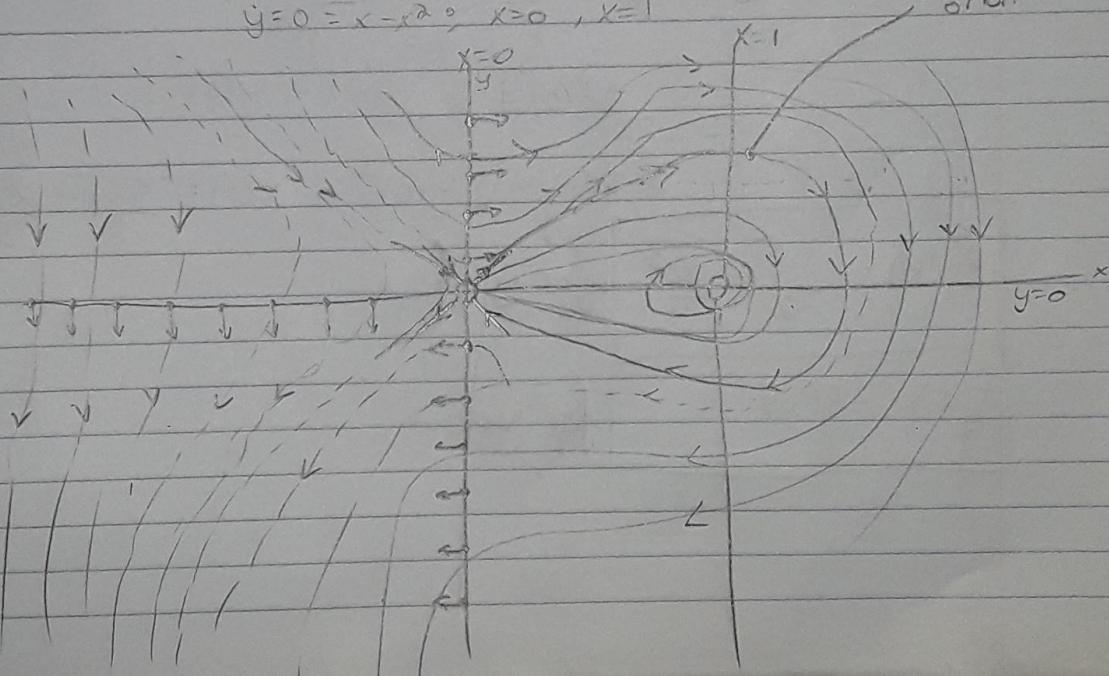
$$\ddot{x} \cdot \dot{x} - x \cdot \dot{x} + x^2 \cdot \dot{x} = 0 \quad ; \quad \int (\dot{x} \ddot{x} - x \dot{x} + x^2 \dot{x}) dt = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} + C$$

$$E = \frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2 + \frac{1}{3} x^3$$

c) Nullclines: $\dot{x} = 0 = y$; $y = 0$

$$y = 0 = x - x^2 \quad ; \quad x = 0, x = 1$$

homoclinic orbit



$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^2\end{aligned}$$

$\text{S: d)} \lim_{t \rightarrow \infty} F(t) = (0, 0)$

So, $\lim_{t \rightarrow \infty} x(t) = 0$

Simultaneously

$\lim_{t \rightarrow \infty} y(t) = 0$

$\lim_{t \rightarrow \infty} \frac{dx}{dt} = y = 0 \Rightarrow \text{So, } \lim_{t \rightarrow \infty} \dot{x} = 0$

$\lim_{t \rightarrow \infty} \frac{dy}{dt} =$

$\lim_{t \rightarrow \infty} \int_0^t \dot{x} dt = \int_0^t y dt \quad ; \lim_{t \rightarrow \infty} x|_0^t = y^2 |_0^t \quad ; \lim_{t \rightarrow \infty} x(t) - x(0) \stackrel{?}{=} 0$

$\lim_{t \rightarrow \infty} \int_0^t y dt = \int_0^t x - x^2 dt \quad ; \lim_{t \rightarrow \infty} y|_0^t = \left[\frac{x^2}{2} - \frac{x^3}{3} \right] |_0^t$

$\lim_{t \rightarrow \infty} y(t) \stackrel{?}{=} -y(0) = \frac{x(t)^2 - x(0)^2}{2} - \frac{x(t)^3 - x(0)^3}{3}$

$-x(0) = \frac{1}{2} [y(t)^2 - y(0)^2]$

$x(0) = \frac{1}{2} y(0)^2 \Rightarrow x(0) = \frac{x(0)^2}{2} - \frac{x(0)^3}{3}$

$f_1(x) = \frac{x^2}{2} - \frac{x^3}{3} \quad \left| \begin{array}{l} \text{for } y=0 \\ \text{for } y \neq 0 \end{array} \right.$

$\frac{f_1(x)}{2} = -f_1(x) \quad \left| \begin{array}{l} \text{for } y=0 \\ \text{for } y \neq 0 \end{array} \right.$