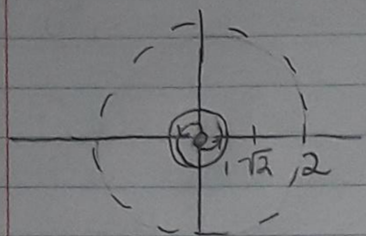
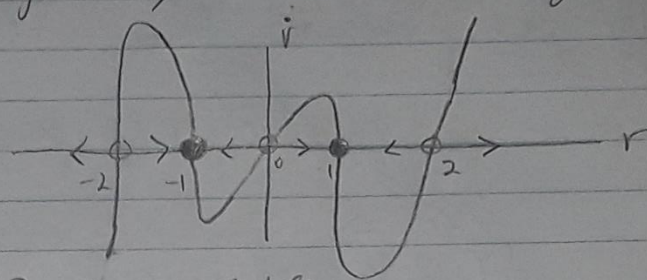
Final Exam ; Chaos Theory ; Tyler Trotter

1: $\dot{r} = r(1-r^2)(4-r^2) = (r-r^3)(4-r^2) = 4r - r^3 - 4r^3 + r^5 = r^5 - 5r^3 + 4r$

$\dot{\theta} = 2 - r^2$

$r^* = 0, \pm 1, \pm 2 \equiv 0, 1, 2$ for $\dot{r}$; $r^* = \pm\sqrt{2} \equiv \sqrt{2}$ for $\dot{\theta}$

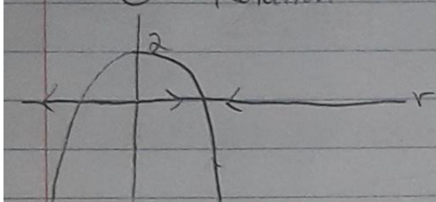




$r = 0, 2$ are unstable

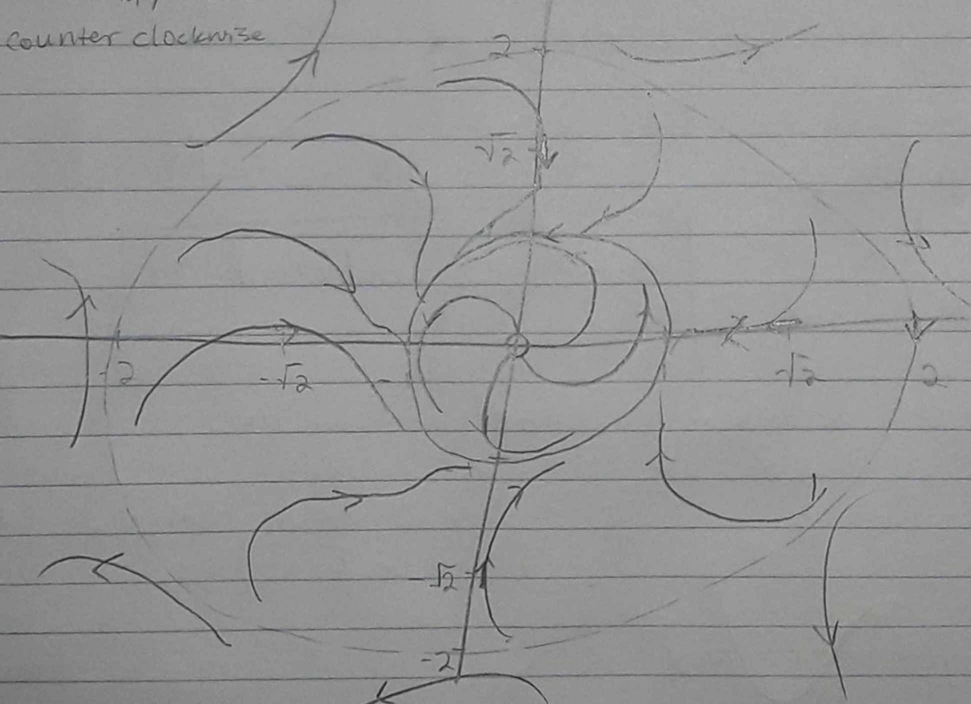
$r = 1$ is stable

$\dot{\theta}$ Rotation



Around $r = \pm\sqrt{2}$, θ does not change, but r does, so there is only temporary radial movement.

for $|r| \leq \sqrt{2}$, movement is counter clockwise



Final Exam

2: (a) $\ddot{x} + \alpha\dot{x} + x - \beta x^3 = F\cos(\omega t)$

b) $\ddot{x} - \alpha\dot{x} + x - \beta x^3 = 0$

c) $\alpha\dot{x} + x - \beta x^3 = F$

$\boxed{\text{No chaos can exist in 1 or 2-D Differential Equations}}$

for each, construct: $y_{n+1} = f(y_n)$, a Poincaré Map

a) let $\dot{x} = y$

$$\dot{y} = \ddot{x} = F\cos(\omega t) - \alpha y + x - \beta x^3$$

$$\dot{y} = 0 \Rightarrow F\cos(\omega t) - \alpha y + x - \beta x^3 = 0$$

$$f(y) = \frac{\beta x^3 - x - F\cos(\omega t)}{\alpha}$$

$$f'(y) = \frac{3\beta}{\alpha}x^2 - \frac{1}{\alpha} - \frac{F}{\alpha}\cos(\omega t) \; ; \; \left|\frac{3\beta}{\alpha}x^2 - \frac{1+F}{\alpha}\right|$$

$\lambda = \lim_{n\to\infty} \sum \ln|f'(y)| \leq 0$ for some values of α, β, ω, F. Chaos exists. Namely when $|3\beta x^2 - (1+F)| < 1$

b) $\dot{x} = y$

$$\dot{y} = \ddot{x} = -\alpha y - x + \beta x^3$$

$$\dot{y} = 0 \Rightarrow \alpha y + x - \beta x^3 = 0$$

$$f(y) = \frac{\beta x^3 - 1}{\alpha} \qquad f'(y) = \frac{3\beta}{\alpha}x^2 \qquad |f'(y)| > 0$$

$$y_{n+1} = \frac{\beta x^3 - 1}{\alpha}$$

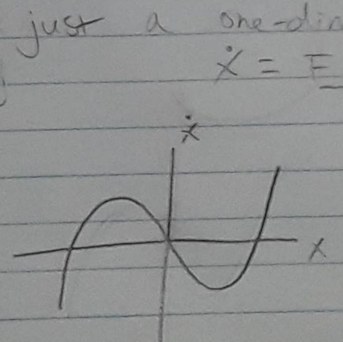
$\lambda = \lim_{n\to\infty} \sum \ln|f'(y)| > 0$ everywhere
No Chaos
2-D System

c) Is just a one-dimensional system on a line:

$$\dot{x} = \frac{F + \beta x^3 - x}{\alpha}$$

which will never exhibit Chaotic behavior for any α, β, ω, F.

$\approx$



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3 a) $\dot{x} = -2xe^{x^2+y^2}$ $\quad$ $(x^*, y^*) = (0,0)$ is the only fixed point since $e^{x^2+y^2} \neq 0$.

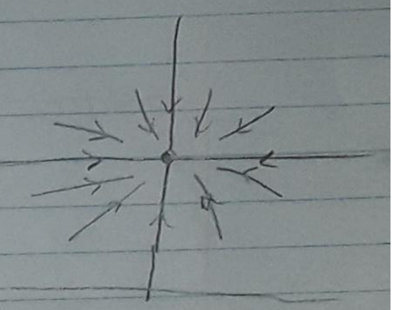
$\dot{y} = -2ye^{x^2+y^2}$

$\dot{x} = 0 = -2xe^{x^2+y^2}$; $xe^{x^2+y^2} = 0$; $x=0$, $y=0$ Similarly $y=0$

Consider $V = e^{x^2+y^2}$; $-\nabla V = \langle -2xe^{x^2+y^2}, -2ye^{x^2+y^2} \rangle$

Since $-\nabla V = \langle \dot{x}, \dot{y} \rangle$, our system is a gradient system and no closed orbits can exist by Theorem 7.2.1.

$x=0$ is a nullcline, $y=0$ is a nullcline



$$J = \begin{bmatrix} -2e^{x^2+y^2} - 4x^2e^{x^2+y^2} & -4xye^{x^2+y^2} \\ -4xye^{x^2+y^2} & -2e^{x^2+y^2} - 4y^2e^{x^2+y^2} \end{bmatrix}$$

$$J_{(0,0)} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$\lambda = -2$ mult. 2 $\Rightarrow$ is an attracting star node by Example 5.2.5

b) $\dot{x} = x^2e^{-x}$ $\quad$ Actually has no fixed points since $\dot{x} = 0$ when $x=0$ and when $x=0$, $\dot{y} \neq 0$ for any real y.

$\dot{y} = 1 - x^2 + y^2$

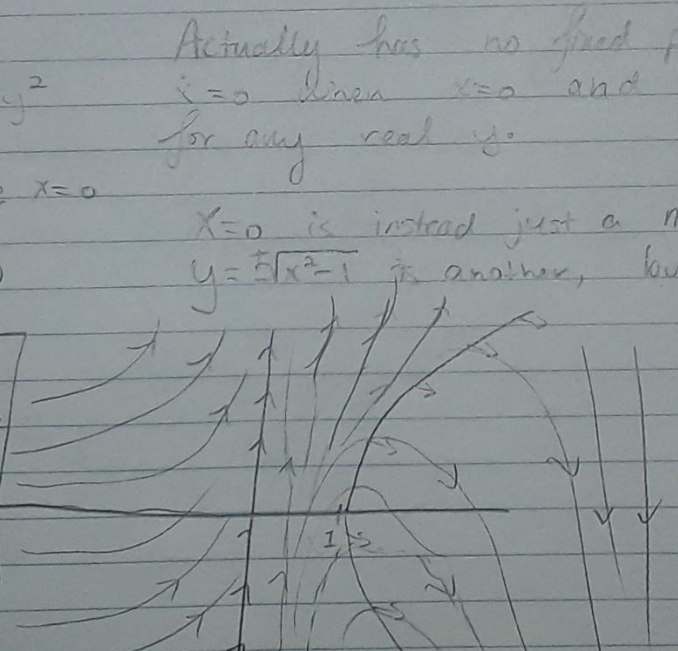
$\dot{x} = 0 = \frac{x^2}{e^x}$; $x=0$

$x=0$ is instead just a nullcline

$\dot{y} = 1 + y^2 \neq 0$

$y = \pm\sqrt{x^2-1}$ is another, but there's no intersec

$\boxed{\text{Since no fixed points exist, the system cannot have periodic orbits.}}$



$f(x^*) = x^*$

$rx^* + x^{*2} = x^*$

$x^{*2} = x^*(1-r)$

Final Exam

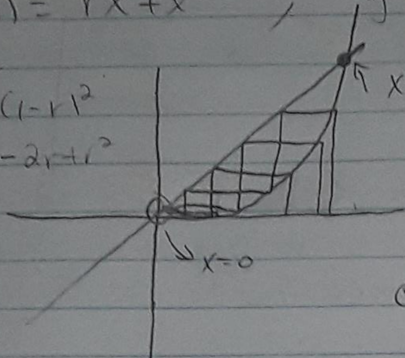
t: $x_{n+1} = rx_n + x_n^2$; $x_n(r + x_n)$; $\boxed{x_n^* = 0, 1-r}$

$f(x) = rx + x^2$; $f'(x) = r + 2x <$

$r(1-r) + (1-r)^2$

$r - r^2 + 1 - 2r + r^2$

$1 - r$



$f'(0) = r = 0$ when $r = 0$

$f'(-r) = r - 2r = -r = 0$ when $r = 0$

So our super stable fixed point occurs when $\boxed{r = 0}$

Consider $f^2(x) = f(f(x)) = r(rx + x^2) + (rx + x^2)^2$

$= r^2x + rx^2 + r^2x^2 + 2rx^3 + x^4$

$f^{2'}(x) = r^2 + 2rx + 2r^2x + 6rx^2 + 4x^3$

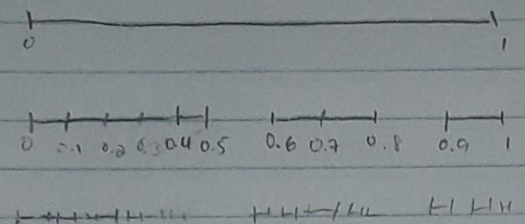
$= f'(f(x)) \cdot f'(x) = 0$

$p = f(q)$, $q = f(p)$

$f(f(p)) = p$

Final Exam

5: Consider a subset of $[0,1]$ consisting of real numbers that can be written without the digits 5 & 8 appearing anywhere in their decimal expansion.



8 Self-Similar intervals each 1/10 original size

a) Yes ; $\boxed{d = \dfrac{\ln(8)}{\ln(10)}}$ $\qquad 8 \quad \frac{1}{5} \cdot \frac{1}{10}$

b)

$\frac{1}{5}$

$\frac{1}{5} + 8\left(\frac{1}{50}\right) +64\left(\frac{1}{250}\right)$

$N(\varepsilon) = 8^n$

$\varepsilon = (1/10)^n$

$$d_{Box} = \lim_{\varepsilon \to 0} \frac{\ln(8^n)}{\ln(10^n)} = \frac{n \cdot \ln(8)}{n \ln(10)} = \frac{\ln(\varepsilon)}{\ln(10)}$$

c) The resultant length per iteration n is $\frac{4}{5}$ the original.

So, $\lim_{n \to \infty} \left(\frac{4}{5}\right)^n = 0 = \mu(S)$

Also, sum up the lengths of deleted segments: $\frac{1}{5} + 8\left(\frac{1}{50}\right) + 64\left(\frac{1}{250}\right)$

$$\frac{1}{10} \sum \left(\frac{8}{5}\right)^k = \sum \frac{1}{2 \cdot 5} \frac{2^{3k}}{5^k} = \sum \frac{2^{2k}}{5^{k+1}} = \frac{1}{5} \sum \frac{4^k}{5^k} = \frac{1}{5} \frac{5}{5-4}$$
$$= \boxed{1}$$

d) This set is topologically equivalent to the Cantor set and is uncountable. For every neighborhood of $x \in S$, $\exists\, x_0 \in S$. Also, S is totally disconnected between any two points in $[0,1]$, there is a number with 5 or 8 in its decimal expansion. Since it is top. equivalent to a Cantor Set, it is uncountably large