# M5470/6440 Chaos Theory

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## I One-Dimensional Flows

- Flows on the Line
- Bifurcations
- Flows on the Circle

## II Two-Dimensional Flows

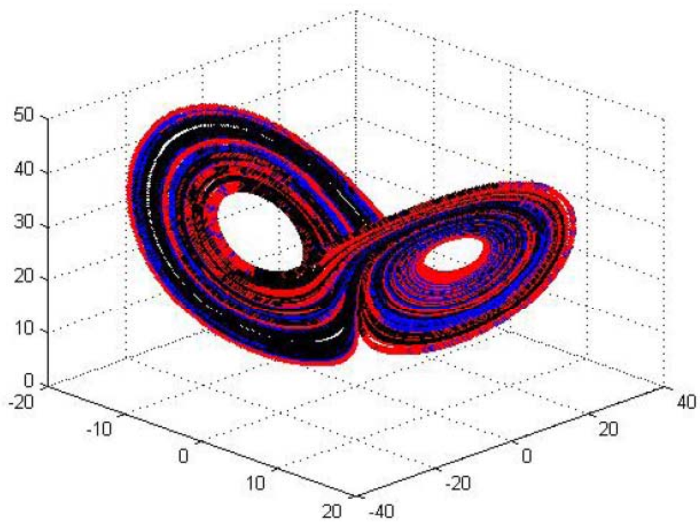
- Linear Systems
- Phase Plane
- Limit Cycles
- Bifurcations

## III Chaos

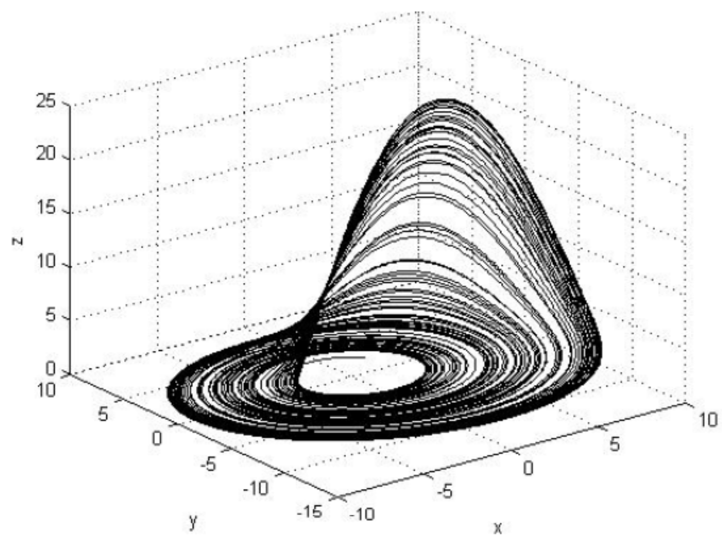
- Lorenz Equations
- One-Dimensional Maps
- Fractals
- Strange Attractors

# Fractals as Attractors of Nonlinear Dynamical Systems

Fractals can be generated as strange attractors of Nonlinear Dynamical Systems, for example, attractor of trajectories of the Lorenz dynamical system, Rossler attractor, attractor of Ueda system.




Lorenz attractor




Rossler attractor

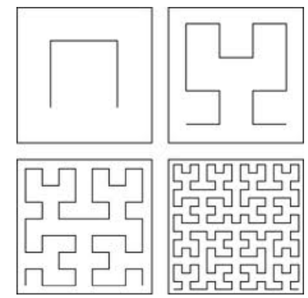
# Iterated Function Systems

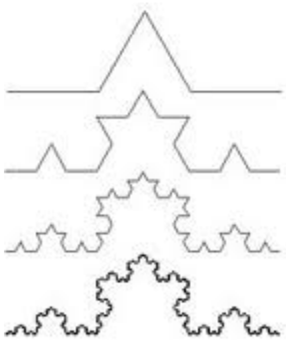
The fractals are constructed using a fixed geometric replacement rule: Cantor set, Sierpinski carpet or gasket, Peano curve, Koch snowflake, Menger sponge.

Karl Weierstrass (1872): Nondifferentiable function

Georg Cantor (1883): Cantor set

Giuseppe Peano (1890), David Hilbert (1891): Space filling curves



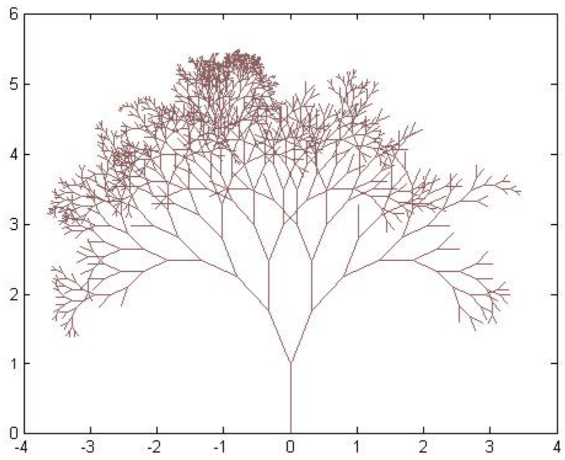


Helge Von Koch (1904): Koch snowflake

Waclaw Sierpinski (1915): Sierpinski triangle and carpet

# Random Fractals

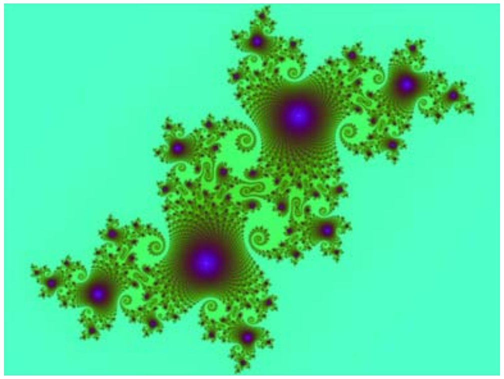
Random fractals can be generated by stochastic rather than deterministic processes, for example, trajectories of the Brownian motion, fractal landscapes and random trees.



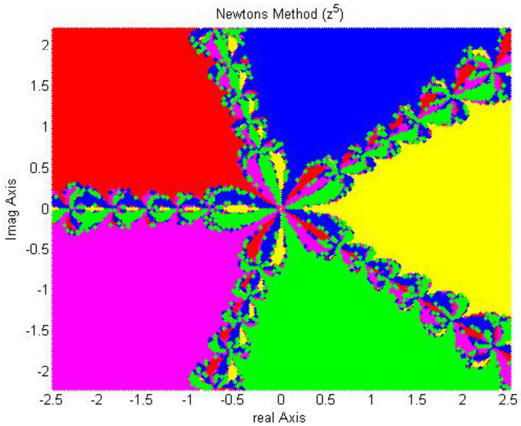
# Escape-time fractals

Escape-time fractals — These are based on sensitive dependence of the trajectories on the starting point or on initial conditions.

Examples of this type are the Julia and Mandelbrot sets (Gaston Julia, Pierre Fatou, Benoit Mandelbrot), and Newton fractal.



Julia set




Newton fractal

# Benoit Mandelbrot, A Life in Many Dimensions (2015)

- **Contents:**
- Introduction — Benoit Mandelbrot: Nor Does Lightning Travel in a Straight Line *(M Frame)*
- Fractals in Mathematics — Chapters by Michael Barnsley, Julien Barral, Kenneth Falconer, Hillel Furstenberg, Stephane Jaffard, Michael Lapidus, Jacques Peyriere & Murad Taqqu
- Fractals in Physics — Chapters by Amon Aharony, Bernard Sapoval, Michael Shlesinger, Katepalli Sreenivasan & Bruce West
- Fractals in Computer Science — Chapters by Henry Kaufman & Ken Musgrave
- Fractals in Engineering — Chapters by Nathan Cohen & Marc-Olivier Coppens
- Fractals in Finance — Chapters by Martin Shubik & Nassim Taleb
- Fractals in Art — Chapters by Javier Barrallo, Ron Eglash & Rhonda Roland Shearer
- Fractals in History — Chapter by John Gaddis
- Fractals in Architecture — Chapter by Emer O'Daly
- Fractals in Physiology — Chapter by Ewald Weibel
- Fractals in Education — Chapters by Harlan Brothers & Nial Neger
- Fractals in Music — Chapter by Charles Wuorinen
- Fractals in Film — Chapter by Nigel Lesmoir-Gordon
- Fractals in Comedy — Chapter by DemetrMartin

Dynamical Systems – systems that evolve in time:
settle to steady state
continue in periodic fashion
change their behavior unpredictably

mid-1600s: Newton – laws of motion – classical mechanics

late 1800s: Poincare introduced chaos – when deterministic system depends on initial conditions

1-st half of 1900s: nonlinear oscillators: radio, radar, laser

1963: Lorentz discovered chaotic motion of strange attractor
L. studied convection rolls in the atmosphere $\to$ chaos & unpredictability of weather

Chaos: (*) Solutions never become steady or periodic state – they oscillate in an irregular, aperiodic fashion

(**) Slightly different initial conditions lead to totally different behavior $\Rightarrow$

The system is unpredictable!

1970-s Chaos has structure!

Ruelle & Takens: turbulence in fluids

Feigenbaum: completely different systems become chaotic the same way!

1980-s Mandelbrot: Fractals

Non-differentiable fns - Weierstrass fns (monsters)

Cantor's devil's stairs

Julia sets - 1920s

1990s Chaos and fractals - everywhere:

Chaotic behavior in chemical reactions, electronic circuits, mechanical oscillators, semiconductors, biological systems, ...

Two types of dynamical systems:

(1) differential equations - describe systems continuous in time

(2) iterated maps (difference eqns) $x_{n+1} = f(x_n)$ - systems with discrete time

Diff. eqns: Ordinary and partial

Ex: Damped harmonic oscillator

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

ODE involves ordinary derivatives $\frac{dx}{dt}, \frac{d^2x}{dt^2}$

$x = x(t)$ - displacement

$t$ - time, independent variable

pde: Ex - the heat eqn: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$u = u(x,t)$ - temperature

$t, x$ - independent variables (time, space)

Systems of odes:

$\dot{x}_i = \frac{dx_i}{dt}$

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2, \dots x_n) \\ \dot{x}_2 = f_2(x_1, x_2, \dots x_n) \\ \dots \\ \dot{x}_n = f_n(x_1, \dots x_n) \end{cases}$$

$x_i$, $i = 1, 2, \dots n$, could be concentrations of chemicals, positions of planets, populations of birds, ...

Canonical Form: all eqns have only 1-st derivatives
(not all systems can be written in canonical form,
ex: $\sin \dot{x} = \cos x$ )

We will consider odes in canonical form.

Back to damped oscillator: $m\ddot{x} + b\dot{x} + kx = 0$

Bring to canonical form: Let $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases}$

Then: $\dot{x}_1 = \dot{x} = x_2 \Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = -\frac{b}{m}\dot{x} - \frac{k}{m}x$

$= -\frac{b}{m}x_2 - \frac{k}{m}x_1$

$\Rightarrow$ Equivalent system:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 \end{cases}$$

- Linear system in canonical form

System is linear as rhs is linear wrt $x_i$.

Autonomous and nonautonomous systems

↑ do not include dependence on time

↑ time dependent

Harmonic oscillator that we considered is autonomous

Ex. of non-autonomous system: forced harmonic oscillator

$$m\ddot{x} + b\dot{x} + kx = F\cos t$$

As before, $x_1 = x$
$x_2 = \dot{x}$ and now, $x_3 = t$

Equivalent system is
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{m}(-kx_1 - bx_2 + F\cos x_3) \\ \dot{x}_3 = 1 \end{cases}$$

The phase space is 3-dimensional

Generally, $(n+1)$-dimensional system and
$(n+1)$-dim. phase space