

1. Math 5470, Midterm 2 by: Tyler Troller

$$\dot{x} = x^2 + y^2 = 1$$

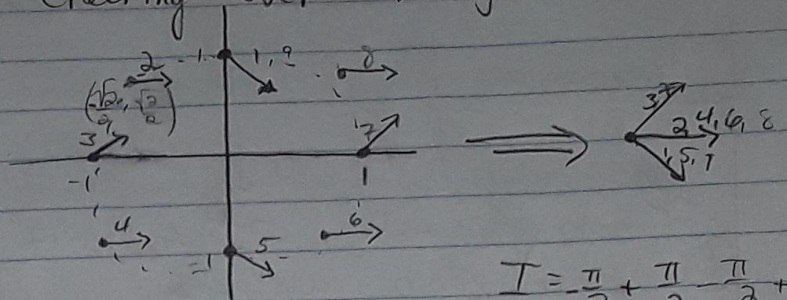
$$\dot{y} = x^2 - y^2$$

Assume Linearization:

$$J = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix};$$

$$J_{(0,0)} = 0$$

checking over: $x^2 + y^2 = 1$



$$I = -\frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2} = 0$$

$$I = 0$$

2: $\dot{x} = x - y$ $(x^*, y^*) = (2, 2), (-2, 2)$

$\dot{y} = x^2 - 4$ Nullclines: $\dot{x} = 0 = x - y \Rightarrow y = x$

$\dot{y} = 0 = x^2 - 4 \Rightarrow x = \pm 2$

Assume Linearization

$$J = \begin{bmatrix} 1 & -1 \\ 2x & 0 \end{bmatrix};$$

$$J_{(2,2)} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}$$

① Since $\gamma > 0$ and $\gamma^2 - 4\Delta < 0$ we have an unstable spiral.

$$\gamma = 1, \Delta = 4, \gamma^2 - 4\Delta = -15 < 0$$

$$(1-\lambda)(-\lambda) + 4 = \lambda^2 - \lambda + 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-16}}{2}$$

② Both Eigenvalues have positive real part: $\frac{1}{2}$

By observations ① & ②, this is a robust case where the linearization is valid.

So, this is a repeller

$$\lambda_1 = \frac{1 + i\sqrt{15}}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} - \frac{i\sqrt{15}}{2} & -1 \\ 4 & -\frac{1}{2} - \frac{i\sqrt{15}}{2} \end{bmatrix}$$

$\lambda_2 =$ does not matter for $\forall \lambda$ the complex case

Midterm 2, cont.

2: Proceed with checking $(-2, -2)$

$$J(-2, -2) = \begin{bmatrix} 1 & -1 \\ -4 & 0 \end{bmatrix}$$

$$\gamma = 1, \Delta = -4, \gamma^2 - 4\Delta = 17$$

① Since $\Delta < 0$, we have a saddle.

$$(1-\lambda)(-\lambda) - 4 = \lambda^2 - \lambda - 4 = 0$$

$$\lambda = \frac{1 \pm \sqrt{17}}{2}$$

② Since both eigenvectors have a positive real part, we have a repeller again. So, by ①, we have an "unstable" saddle.

$$\lambda_1 = \frac{1 + \sqrt{17}}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} - \frac{\sqrt{17}}{2} & -1 \\ -4 & \frac{1}{2} - \frac{\sqrt{17}}{2} \end{bmatrix}; \lambda_2 = \frac{1 - \sqrt{17}}{2} \Rightarrow \begin{bmatrix} \frac{1}{2} + \frac{\sqrt{17}}{2} & -1 \\ -4 & \frac{1}{2} + \frac{\sqrt{17}}{2} \end{bmatrix}$$

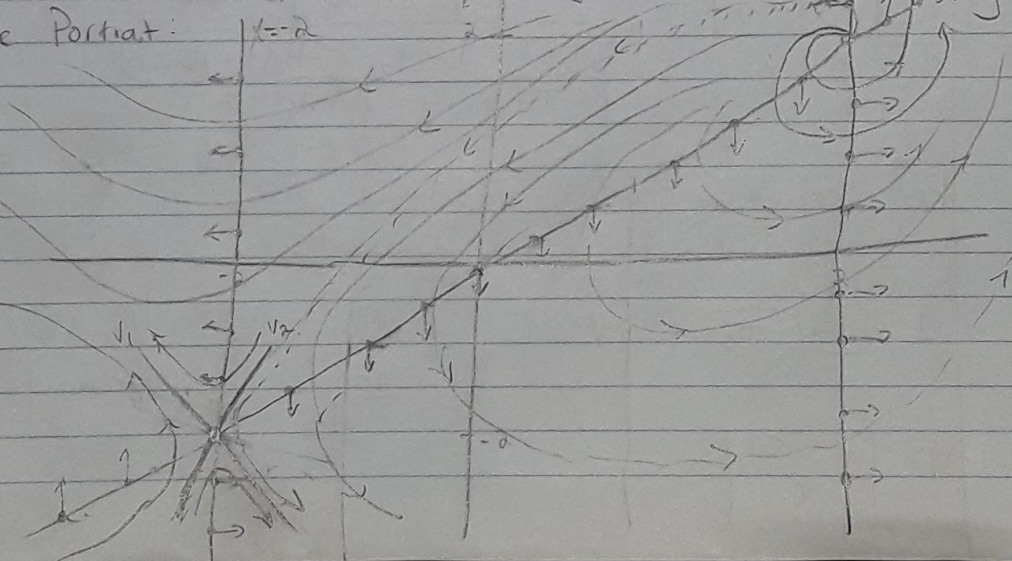
$$V_1 = \begin{bmatrix} -1 - \sqrt{17} \\ 8 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -1 + \sqrt{17} \\ 8 \end{bmatrix}$$

$$X(t) = e^{\frac{1+\sqrt{17}}{2}t} \begin{bmatrix} -1 - \sqrt{17} \\ 8 \end{bmatrix} + e^{\frac{1-\sqrt{17}}{2}t} \begin{bmatrix} -1 + \sqrt{17} \\ 8 \end{bmatrix}$$

as $t \rightarrow \infty$ we approach V_1

Phase Portrait:



Midterm 2

$$3. \dot{r} = \sin r$$

$$\dot{\theta} = 1$$

$$r^* = \pi$$

$$x(t) = r(t) \cos(\theta(t)), \quad x = r \cos \theta, \quad y = r \sin \theta, \quad r = \sqrt{x^2 + y^2}$$

$$\dot{x} = \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta}$$

$$\sin(r) \cos \theta - r \sin \theta \quad x(t) = \sin \theta \cos \theta$$

$$= \pi \sin \theta \cos \theta$$

$$\dot{x} = \dot{r} \cos \theta - r \sin \theta \cdot \dot{\theta}$$

$$\dot{r} = \sin r, \quad \dot{x} = \sin(r) \cos \theta - r \sin \theta$$

$$\dot{\theta} = 1$$

$$\dot{x} = \sin(r) \cos \theta - y$$

$$\dot{x} r = x \sin(r) - r y$$

$$\dot{x} = \frac{x \sin(\sqrt{x^2 + y^2}) - y \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$$

$$\dot{x} = \frac{x \sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} - y \rightarrow -y$$

$$y = r \sin \theta; \quad \dot{y} = \dot{r} \sin \theta + r \cos \theta \dot{\theta}$$

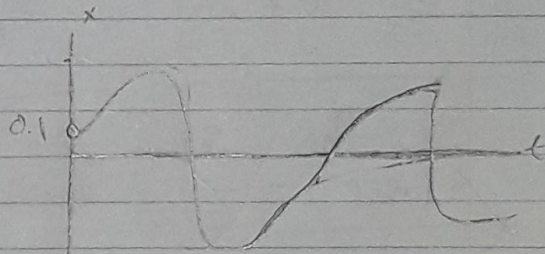
$$\dot{y} = \sin(r) \sin \theta + x$$

$$\dot{y} r = \sin(r) y + x r$$

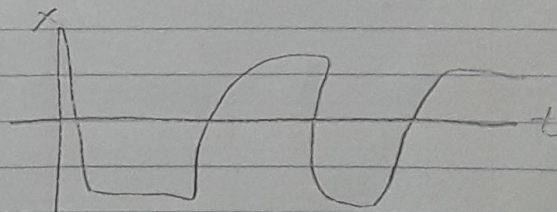
$$\dot{y} = \frac{y \sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} + x$$

$$\dot{y} = \frac{y \sin(\sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} + x \rightarrow x$$

$$a) x(0) = 0.1, y(0) = \pi$$



$$b) x(0) = 2\pi - 0.1, y(0) = 0$$



Midterm 2

4: $\dot{x} = y + 2xy$
 $\dot{y} = x + x^2 - y^2$

$(x^*, y^*) = (0, 0), (-1, 0)$

Nullclines: $\dot{x} = 0 = y + 2xy$, $y = -2xy$, $y = 0$
 $\dot{y} = 0 = x + x^2 - y^2$

Assume linearization.

$J = \begin{bmatrix} 2y & 1+2x \\ 1+2x & -2y \end{bmatrix}$; $J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\lambda = 0$, $\Delta = -1$, $\lambda^2 - 4\Delta = 4$

$\lambda^2 - 1 = 0$, $\lambda = \pm 1 \Rightarrow$ Saddle

Check: $f_y = 1 + 2x$

Gradient: $g_x = 1 + 2x$

$\dot{x} = -\frac{\partial V}{\partial x}$;

$\int y + 2xy dx = xy + x^2y + C(y)$

$\frac{\partial}{\partial y} (xy + x^2y + C(y)) = x + x^2 + C'(y)$
 $C'(y) = -\frac{y^3}{3}$

$V(x,y) = -xy - x^2y + \frac{y^3}{3}$

$\left[\begin{array}{l} -\frac{\partial V}{\partial x} = y + 2xy \\ \qquad \qquad \qquad = \dot{x} \end{array} \quad , \quad \begin{array}{l} -\frac{\partial V}{\partial y} = x + x^2 - y^2 \\ \qquad \qquad \qquad = \dot{y} \end{array} \right]$

So, this system can be written as $-\nabla V$. By Theorem 7.2.1, closed orbits are impossible in this system. \square

Midterm 2

5. Find a conserved quantity for:

$$\ddot{x} = x - x^2$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\lambda_1 = 1$$

$$\lambda_2 = -1$$

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$t \rightarrow \infty, \rightarrow e^t$$

$$\dot{x} = y$$

$$\dot{y} = x - x^2$$

$$(x^*, y^*) = (0, 0), (1, 0)$$

$$J = \begin{bmatrix} 0 & 1 \\ 1-2x & 0 \end{bmatrix}$$

b)

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\gamma = 0, \Delta = -1, \gamma^2 - 4\Delta = 4$$

So, the origin is a saddle!

$$J_{(1,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\gamma = 0, \Delta = 1, \gamma^2 - 4\Delta = -4$$

So, this is a center!

Since a conserved quantity exists, the linearization is accurate. The trajectories are closed curves for (1,0).

$$\ddot{x} - x + x^2 = 0$$

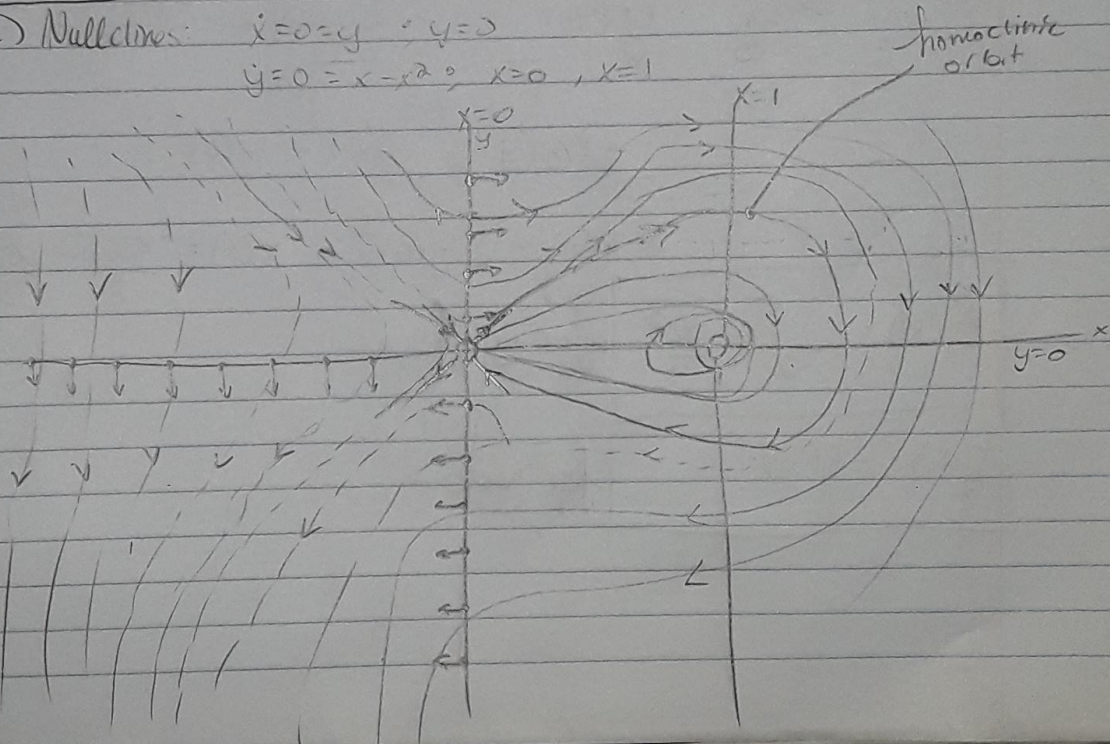
$$\ddot{x} \cdot \dot{x} - x \cdot \dot{x} + x^2 \cdot \dot{x} = 0$$

$$\int (\ddot{x} \dot{x} - x \dot{x} + x^2 \dot{x}) dx = \frac{\dot{x}^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} + C$$

$$E = \frac{1}{2} \dot{x}^2 - \frac{1}{2} x^2 + \frac{1}{3} x^3$$

c) Nullclines: $\dot{x} = 0 \Rightarrow y = 0$

$$\dot{y} = 0 \Rightarrow x - x^2 = 0, x = 0, x = 1$$



$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^2\end{aligned}$$

5: d) $\lim_{t \rightarrow \infty} F(t) = (0, 0)$

So, $\lim_{t \rightarrow \infty} x(t) = 0$

Simultaneously

$\lim_{t \rightarrow \infty} y(t) = 0$

$\lim_{t \rightarrow \infty} \frac{dx}{dt} = y = 0$; So, $\lim_{t \rightarrow \infty} \dot{x} = 0$

$\lim_{t \rightarrow \infty} \frac{dy}{dt} =$

$\lim_{t \rightarrow \infty} \int_0^t \dot{x} dt = \int_0^t y dt$ $\int_0^t x \Big|_0^t = \frac{y^2}{2} \Big|_0^t$ $\lim_{t \rightarrow \infty} x(t) = x(0)$

$\lim_{t \rightarrow \infty} \int_0^t \dot{y} dt = \int_0^t x - x^2 dt$ $\int_0^t y \Big|_0^t = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^t$

$\lim_{t \rightarrow \infty} y(t) = -y(0) = \frac{x(t)^2 - x(0)^2}{2} - \frac{x(t)^3 - x(0)^3}{3}$

$-x(0) = \frac{1}{2} [y(t)^2 - y(0)^2]$

$x(0) = \frac{1}{2} y(0)^2 = x(0) = \frac{x(0)^2}{2} - \frac{x(0)^3}{3}$

$f_1(x) = \frac{x^2}{2} - \frac{x^3}{3}$	for $y=0$
$f_2(x) = -f_1(x)$	for $y \neq 0$