Math 5470, Midterm 1 by Tyler Trotter

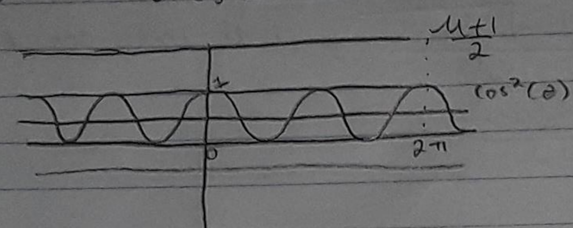
1: $\dot{\theta} = \mu - \cos(2\theta)$

$\dot{\theta} = \mu - (2\cos^2(\theta) - 1) = \mu - 2\cos^2(\theta) + 1$

Step 1: Find the fixed points in terms of μ:

$\mu + 1 - 2\cos^2\theta = 0$

$\mu + 1 = 2\cos^2\theta$; $\frac{\mu+1}{2} = \cos^2(\theta)$



$\frac{\mu+1}{2}$ is tangent to $\cos^2\theta$ when $\frac{\mu+1}{2} = 0$ and $\frac{\mu+1}{2} = 1$

$\mu_c = -1$ and $\mu_c = 1$

For $\mu > 1$ or $\mu < -1$, we have no critical points x^*.

$\mu - \cos(\theta) > 0$

For $\mu = 1$, we have $1 + 1 - 2\cos^2(\theta)$; $2 - 2\cos^2(\theta) = 0$; $\cos^2(\theta) = 1$

Then $\theta^* = n\pi$ for $n \in \mathbb{Z}$

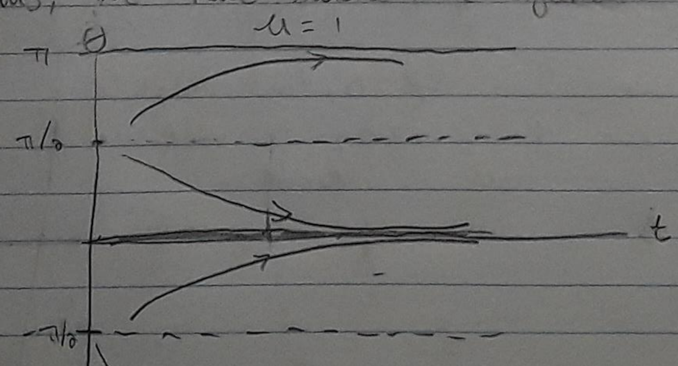
For $-1 < \mu < 1$, we have $\mu + 1 - 2\cos^2(\theta)$; $\cos^2(\theta) = \frac{\mu+1}{2}$

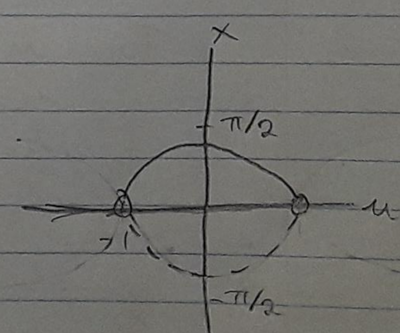
For $\mu = -1$, we have $-1 + 1 - 2\cos^2(\theta)$; $-2\cos^2(\theta) = 0$

Then $\theta^* = q\frac{\pi}{2}$, $q \in \mathbb{Z}$

Graphically, you can see two critical points spawn from the bifurcations of $\theta^* = n\pi$, $\mu = 1$ and $\theta^* = q\frac{\pi}{2}$, $\mu = -1$.

Thus, we have saddle-node bifurcations:





2: $\dot{x} = x(1+\mu x + x^2)$

First, attempt to find critical points when $\dot{x}=0$.

$$x(1+\mu x+x^2)=0 \;;\quad x^*=0$$

$$x=0,\quad 1+\mu x+x^2=0$$

Find roots: $\dfrac{-\mu \pm \sqrt{\mu^2-4}}{2}$; real roots occur when $\mu>2$, $\mu<-2$

So, $x^* = 0,\ \dfrac{-\mu \pm \sqrt{\mu^2-4}}{2}$

$$\dot{x} = x+\mu x^2+x^3 \;;\quad \frac{d}{dt}(\dot{x}) = 1+2\mu x+3x^2$$

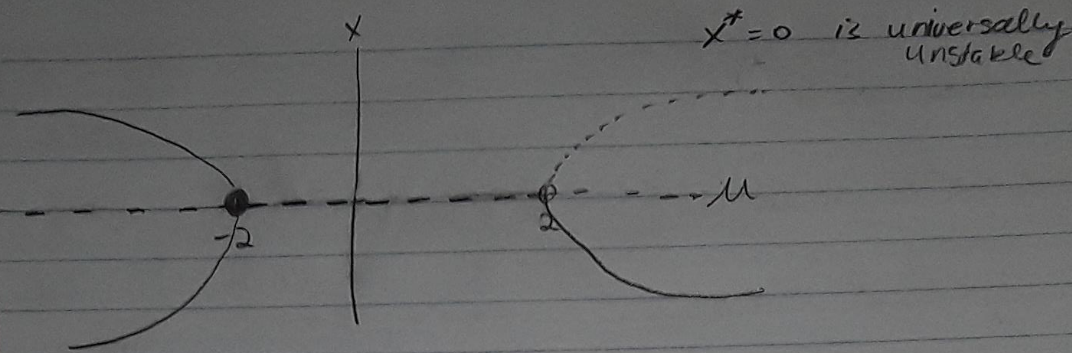
$\ddot{x}(0) = 1 > 0$, since $\ddot{x}(0)>0$, we have an unstable fixed point for all μ.

$$\ddot{x}\left(\frac{-\mu+\sqrt{\mu^2-4}}{2}\right) = 1+2\mu\left(\frac{-\mu+\sqrt{\mu^2-4}}{2}\right)+3\left(\frac{\mu^2}{4}+(\mu\sqrt{\mu^2-4})+\frac{\mu^2-4}{4}\right)$$

For $\mu>2$, we have: $1-\mu^2+\mu\sqrt{\mu^2-4}+\frac{3\mu^2}{4}+3\mu+3\sqrt{\mu^2}$ [illegible]
which is greater than 0, thus unstable
For $\mu<2$, we have less than 0, thus stable
For $\mu=2$, we have a bifurcation

$$\ddot{x}\left(\frac{-\mu-\sqrt{\mu^2-4}}{2}\right)$$

For $\mu>2$, we have $\ddot{x}>0$, thus stable
$\mu<-2$, we have $\ddot{x}<0$; thus unstable
$\mu=2$, we have a bifurcation



3: $\dot{x} = 3 - re^{-x^2}$

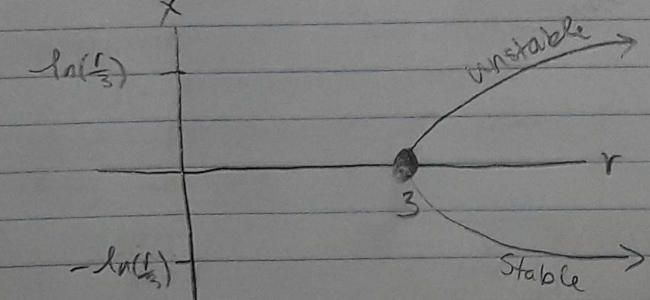
$\dot{x} = 0$, $3 - re^{-x^2} = 0$, $re^{-x^2} = 3$, $e^{-x^2} = \frac{3}{r} =$

$\Rightarrow e^{x^2} = \frac{r}{3}$, $\ln(e^{x^2}) = \ln(\frac{r}{3})$, $x^2 = \ln(\frac{r}{3})$

Since $x^2 \geq 0$, $r \geq 3$, so $x^* = \pm \ln(\frac{r}{3})$

A bifurcation occurs when $r = 3$ since no x values yield $\dot{x} = 0$ for $r < 3$. When $r > 3$, we split into two fixed points at $\pm \ln(\frac{r}{3})$.

Saddle-node bifurcation

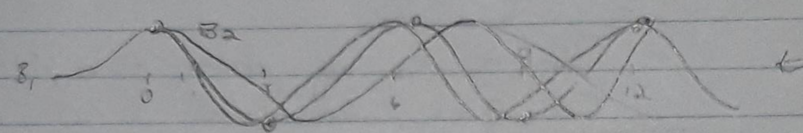


4: Bell 1: $\dot{\theta}_1 = \omega_1 = \frac{\pi}{T_1} = \frac{\pi}{3}$, Following Ex. 4.2.1

Bell 2: $\dot{\theta}_2 = \omega_2 = \frac{\pi}{T_2} = \frac{\pi}{4}$

$\phi = \theta_1 - \theta_2$, $\dot{\phi} = \dot{\theta}_1 - \dot{\theta}_2 = \omega_1 - \omega_2$

$T_{synch} = \frac{\pi}{\omega_1 - \omega_2} = \left(\frac{1}{T_1} - \frac{1}{T_2}\right)^{-1} = \left(\frac{1}{3} - \frac{1}{4}\right)^{-1} = \boxed{12 \text{ seconds}}$



5: $\dot{x} = ax + x^3 - x = x(a-1+x^2)$

$\dot{x}=0$, $x(a-1+x^2)=0$; $x^*=0$ is a fixed point

$x^2-1+a=0$; $x^2=-a+1$, Since $x^2 \geq 0$, a must be less than 1. $x = \pm\sqrt{-a+1}$

See: $-a+1 \geq 0$; $-a \geq -1$; $a \leq 1$

$\ddot{x} = a + 3x^2 - 1$

$\ddot{x}(0) = a-1$

For $a=1$, we have a bifurcation as $\ddot{x}=0$

For $a<1$, we have $\ddot{x}<0$, thus stable

$\ddot{x}(\sqrt{-a+1})$

For $a=1$, we have the 0 case

For $a<1$, we have $(a-1)+3(-a+1) = -2a+2$ which is greater than 0. So, unstable

$\ddot{x}(-\sqrt{-a+1})$

For $a=1$, we have the 0 case

For $a<1$, we have $(a-1)+3(-a+1) = -2a+2$ which is greater than 0. So, unstable

Graphically, when $a=0$, we have x^3-x which is stable at $x=0$. $x^*=0$ is universally stable and then we split into a subcritical pitchfork of 3 fixed points where $\pm\sqrt{-a+1}$ are universally unstable

$f(x) = -\frac{dV}{dt} = ax + x^3 - x$; By Integration, we get:

$V(x) = -\frac{a}{2}x^2 + \frac{x^2}{2} - \frac{x^4}{4}$.

$a=0$, we get: $a<0$ $a>0$