Math 5470/6440 Chaos theory

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1D: Phase portrait and fixed points

Today we will discuss:

Pendulum problem. Linearization.

Phase space.

One-dimensional systems. Fixed points and stability.

Population growth model. Logistic equation.

Linear stability analysis.

One-dimensional or first order system is:

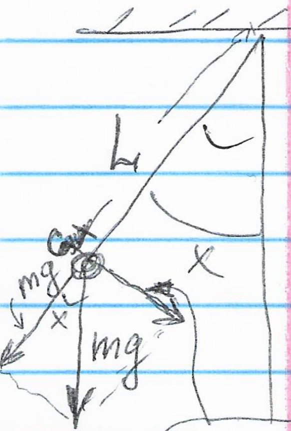
$$\dot{x} = f(x)\,, \qquad x(0) = x_0$$

Here $x(t)$ is real valued function of time t,

f is a smooth, real-valued function of x.

x is position of a particle, $\dot{x}$ is velocity.

Ex: Swinging of pendulum



$F = mg \sin x$

x is angle of the pendulum from the vertical

L - the length of the pendulum

$F = ma$

$-\cancel{m}g \sin x = \cancel{m}a \Rightarrow a = -g \sin x$

$s = Lx, \quad v = \frac{ds}{dt} = L\frac{dx}{dt} \Rightarrow$

$a = L\ddot{x} = -g \sin x$ | g is acceleration due to gravity

$$\boxed{\ddot{x} + \frac{g}{L} \sin x = 0}$$

Then:
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{L} \sin x_1 \end{cases}$$

Equivalent system is nonlinear

Linearization: small angle approximation

$|x| << 1$: $\sin x \simeq x \Rightarrow$ the system is:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{L} x_1 \end{cases}$$

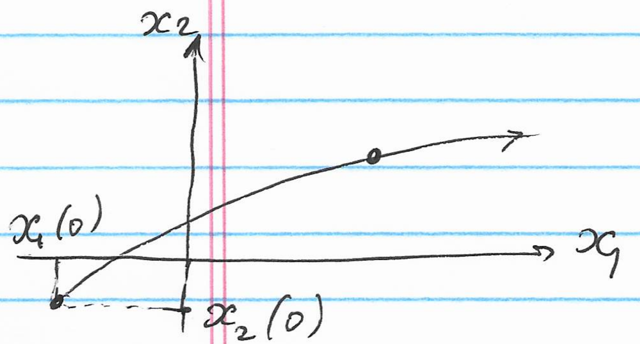
only a part of the problem corresponding to low energy

Linearized problem does not describe high energy solutions - when pendulum whirls over the top (when x is not small)

Consider dependence on time:

$x_1(t) = x(t)$ – position of the pendulum

$x_2(t) = \dot{x}(t)$ – velocity



Consider $(x_1(t), x_2(t))$ - point moving along a curve – – trajectory – in coordinate (phase) space (x_1, x_2) starting from initial point $(x_1(0), x_2(0))$.

Different trajectories for different initial conditions.

2-dimensional system – 2-dim. phasespace

Ex. Write system of odes in canonical form

$$\begin{cases} \ddot{x} + \sin(x-y) = 0 \\ \dot{y} + \frac{1}{2} x = 0 \end{cases}$$

Let $\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = y \end{cases}$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\sin(x_1 - x_3) \\ \dot{x}_3 = -\frac{1}{2} x_1 \end{cases}$$

Linear or nonlinear?

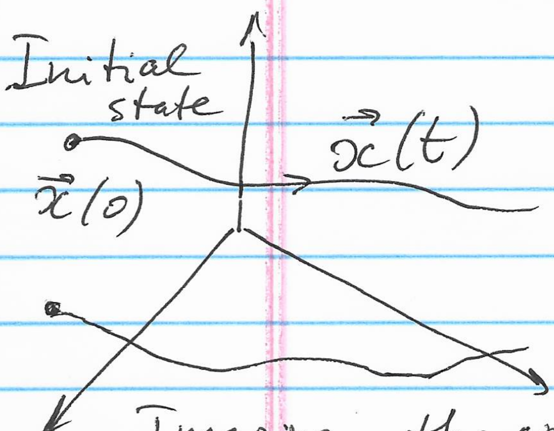
Phase space

Consider $(x_1, x_2, \dots x_n)$ as a vector in n-dimensional phase space : $\begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$

The point $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}$ moving along a curve defines a trajectory in the phase space

Diff eqn can be written as

$$\dot{\vec{x}} = \vec{F}(\vec{x})$$



Given the initial state $\vec{x}(0)$, we need to find (predict) future states

Different initial conditions (states) generate different trajectories

Imagine, the space is filled with fluid

The system

$$\underbrace{\dot{\vec{x}}}_{\text{fluid velocity}} = \vec{F}(\underbrace{\vec{x}}_{\text{given pt. in space}})$$

is called flow in phase space

$\Rightarrow$ the fluid velocity is known at each point.

The phase space is n-dimensional $\Rightarrow$ the system is n-dimensional (or n-th order)

For pendulum, $n = 2$

One-dimensional or first-order systems

<u>n = 1</u> $\dot{x} = f(x)$

$x(t)$ is real-valued function of t

f is smooth, real-valued function of x

? What is the dimension of the phase space?

Can interpret diff. eqns as vector fields

Ex: $\begin{cases} \dot{x} = \sin x \\ x(0) = x_0 \end{cases}$ (Can integrate: $\int dt = \int \frac{dx}{\sin x}$

Soln: $t = \ln \left| \dfrac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right|$

Graphically: x - position of pt. on real line

$\dot{x}$ - velocity of this pt.

$\dot{x} = \sin x$ represent vector field on the line

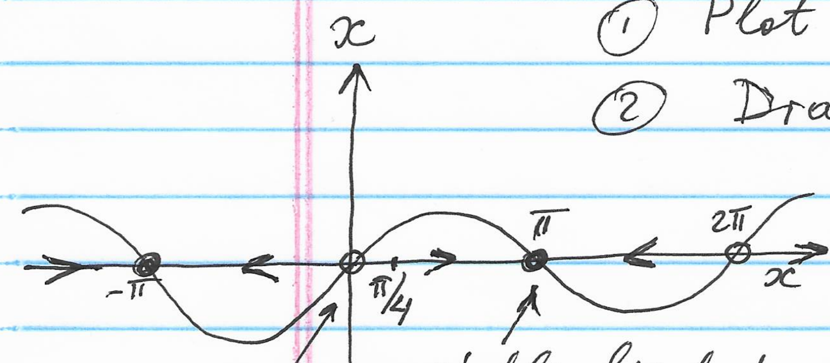
① Plot $\dot{x}$ versus x

② Draw arrows to show velocity vector

$\dot{x} > 0$ - arrow to the right

$\dot{x} = 0$ - fixed points

$\dot{x} < 0$ - arrow to the left

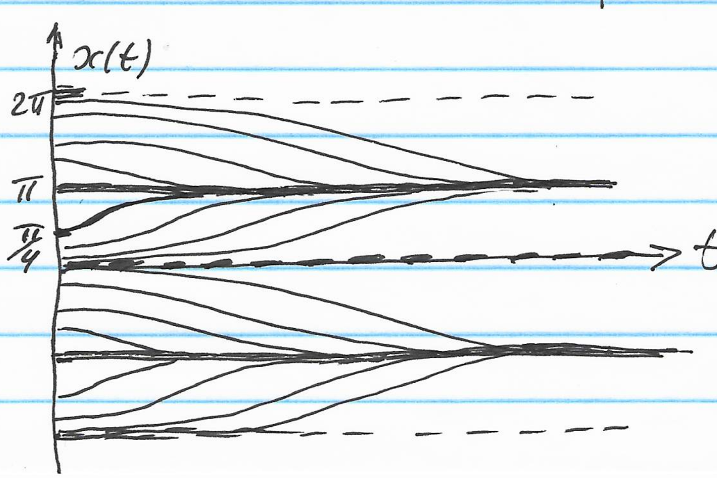


unstable fixed pt, repeller, source

stable fixed pt attractor, sink

$\dot{x} = 0$

quillibrium pts

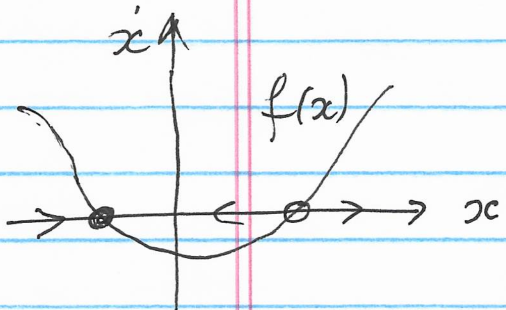


Particle starting at $\frac{\pi}{4}$ approaches a stable fixed point at $x = \pi$.

2.2 Fixed Points and Stability

Phase Portrait

$\dot{x} = f(x)$



Imagine fluid flowing along the real line with local velocity $f(x)$

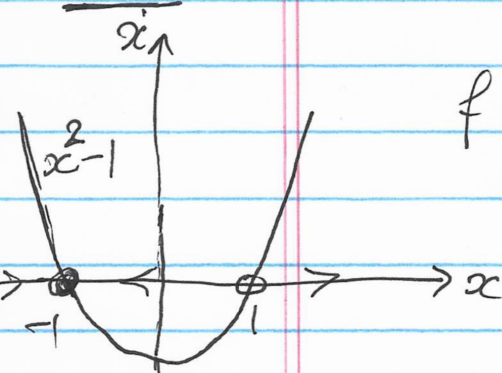
If $f(x) > 0$ – flow is to the right
$f(x) < 0$ – flow is to the left

Fixed pts x^*: $f(x^*) = 0$ – velocity is zero

$f'(x^*) < 0$: x^* is stable equilibrium
disturbances die out in time

$f'(x) > 0$: x^* is unstable equilibrium
disturbances grow in time

Ex. $\dot{x} = x^2 - 1$ Find fixed points and classify their stability.

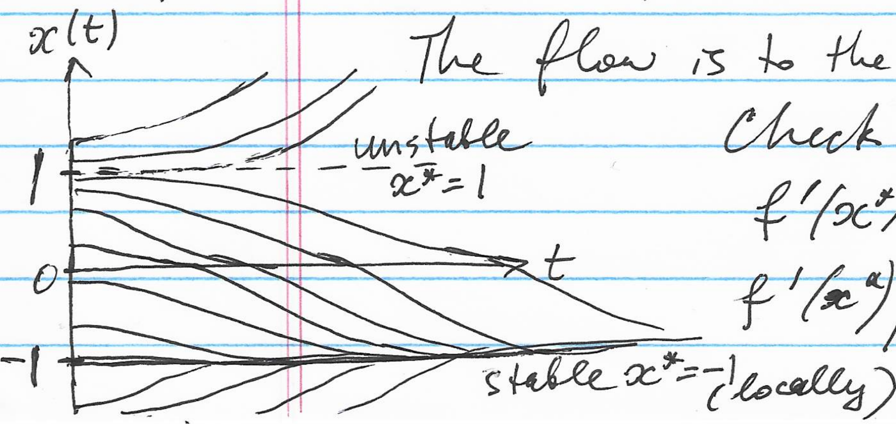


$f(x) = x^2 - 1$ To find x^*, solve $f(x^*) = 0$

$x^2 = 1 \Rightarrow x^* = \pm 1$

The flow is to the right where

$f(x) > 0$: $x^2 - 1 > 0$ or $|x| > 1$



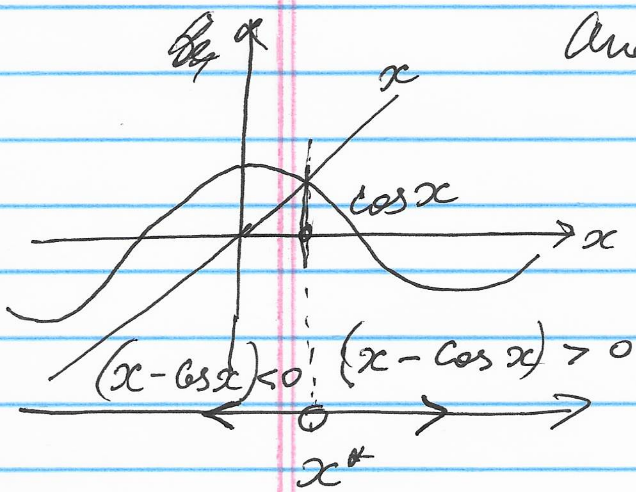
The flow is to the left where $x^2 - 1 < 0$ ($|x| < 1$)

Check $f'(x^*) = 2x^*$

$f'(x^*)\big|_{x^*=-1} = -2 < 0$ – stable
– local disturbances die out

$f'(x^*)\big|_{x^*=1} = 2 > 0$ – unstable
– disturbances grow

Ex: $\dot{x} = x - \cos x$ − find fixed points



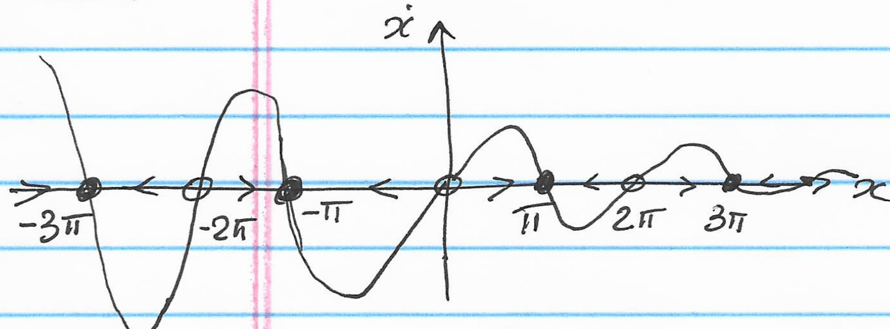
Analize separately:

$x = \cos x$ has ! solution

x^* s.t. $f(x^*) = 0$

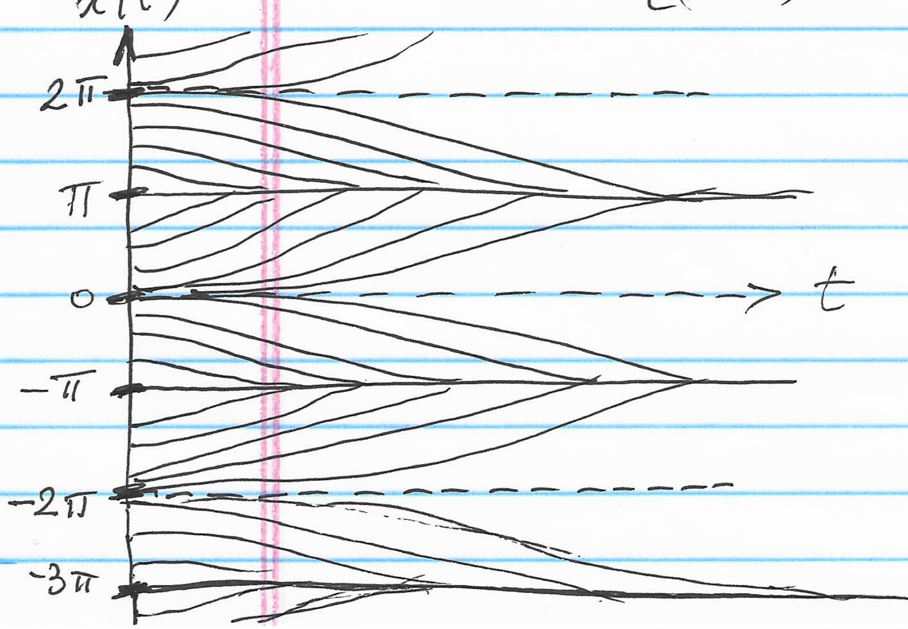
x^* is unstable fixed pt.

Ex: $\dot{x} = e^{-x} \sin x$ Find fixed points and check their stability. Graph $x(t)$ as a function of time



Fixed pts: $\sin x^* = 0 \Rightarrow x^* = \pi k,\ k \in \mathbb{Z}$

$$x^* = \begin{cases} 2n\pi & \text{− unstable} \quad (k = 2n \text{ − even}) \\ (2n+1)\pi & \text{− stable} \quad (k = 2n+1 \text{ − odd}) \end{cases}$$



Population Growth Model

$N(t)$ is the population at time t, $N \geq 0$

$r > 0$ is the growth rate

The simplest model:

$$\dot{N} = rN \quad , \quad N(0) = N_0$$ – population size at $t=0$

The model predicts exponential growth:

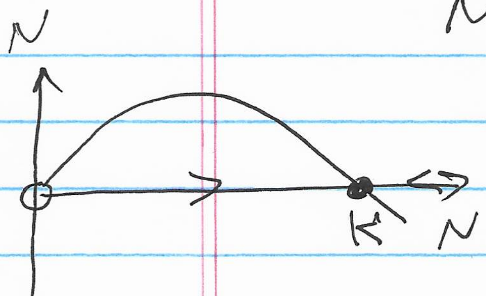
$$N(t) = N_0 e^{rt}$$

Introduce K – carrying capacity

$$r = r(N) = r_0\left(1 - \frac{N}{K}\right)$$

The logistic equation

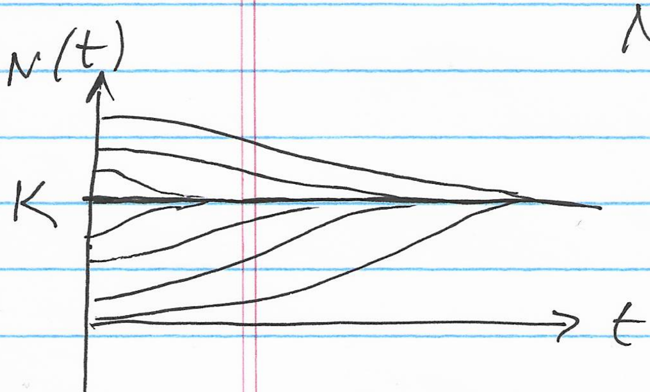
$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$



Fixed pts: $N^* = 0$ – unstable

Small population grows exponentially fast

$N^* = K$ – stable (globally stable)



The population is always approaching its carrying capacity