Math 5470/6440 Chaos theory

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1D: Linear stability analysis

Today we will discuss:

Linearization and linear stability analysis.

Growth / decay rate and characteristic time scale.

Existence / uniqueness of solution of one-dimensional system.

Impossibility of oscillations in 1D.

Potentials.

One-dimensional or first order system is:

$$\dot{x} = f(x)\,, \qquad x(0) = x_0$$

Here $x(t)$ is real valued function of time t,

f is a smooth, real-valued function of x.

x is position of a particle, $\dot{x}$ is velocity.

2.4 <u>Linear Stability Analysis</u>

? Rate of decay to the stable fixed pt?

To answer, linearize about the fixed pt

$\dot{x} = f(x)$, Let x^* be a fixed pt,

$\eta(t) = x(t) - x^*$ is a small perturbation

Derive diff. eqn for η:

$$\dot{\eta} = \frac{d}{dt}(x(t) - \underset{\substack{\Uparrow \\ \text{const}}}{x^*}) = \frac{d}{dt}x(t) = \dot{x} = f(\underset{x = x^* + \eta}{x})$$

$\dot{\eta} = f(x^* + \eta)$ — Use Taylor's expansion

$$f(x^* + \eta) = \underbrace{f(x^*)}_{\overset{\shortparallel}{0}\ (x^* \text{ is fixed pt})} + \eta f'(x^*) + O(\eta^2) \leftarrow \begin{cases}\text{quadratically} \\ \text{small} \\ \text{terms in } \eta\end{cases}$$

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2)$$

If $f'(x^*) \neq 0$, $O(\eta^2)$ terms are negligible $\Rightarrow$

$\dot{\eta} = \eta f'(x^*)$ | Linear eqn in η, linearized about x^*

$f'(x^*)$ — growth (or decay) rate

(as in $\eta = e^{kt}$, $k = f'(x^*)$, — soln for $\dot{\eta} = k\eta$)

Linearized eqn for η: (if $f'(x^*) \neq 0$)

$$\boxed{\dot{\eta} = \eta f'(x^*)}$$

Approximation as we dropped quadratic terms in η

Soln: $\eta(t) = e^{kt}$ for $k = f'(x^*)$ - growth / decay rate

$f'(x^*) > 0 \Rightarrow \eta$ grows exponentially
x^* is unstable

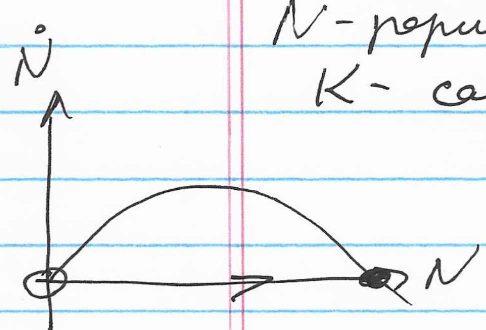
$f'(x^*) < 0 \Rightarrow \eta$ decays exponentially fast
x^* is stable

If $f'(x^*) = 0$ the nonlinear terms are not negligible, nonlinear analysis is required to determine stability

$\dfrac{1}{|f'(x^*)|}$ - characteristic time scale

<u>Ex</u>. <u>Logistic eqn</u> $\dot{N} = rN\left(1 - \frac{N}{K}\right)$

N - population size, $r > 0$ growth rate
K - carrying capacity



$$f(N) = rN\left(1 - \frac{N}{K}\right) = rN - \frac{r}{K}N^2$$

$N^* = 0$, $N^* = K$

$$f'(N) = r - 2r\frac{N}{K}$$

$f'(0) = r > 0$: $N^* = 0$ - unstable point, exponential growth

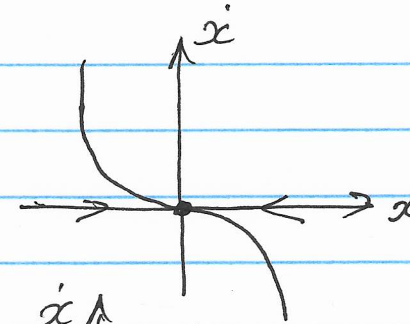
$f'(K) = -r < 0$: $N^* = K$ - stable fixed point with exponentially decaying perturbations

<u>Characteristic time scale</u>: $\dfrac{1}{|f'(N^*)|} = \dfrac{1}{r}$

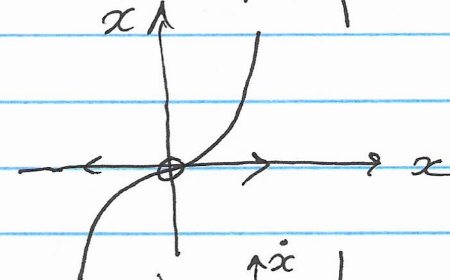
If $f'(x^*) = 0$, linear stability analysis does not work

Ex. (a) $\dot{x} = -x^3$

$x^* = 0$, $f'(x^*) = -3(x^*)^2 = 0$



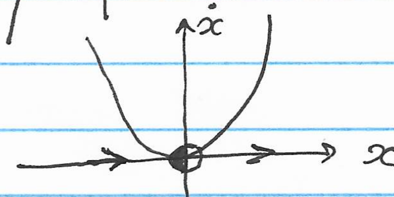
(b) $\dot{x} = x^3$, $x^* = 0$, $f'(x^*) = 0$



(c) $\dot{x} = x^2$, $x^* = 0$, $f'(x^*) = 0$

$x^* = 0$ is half-stable fixed pt.

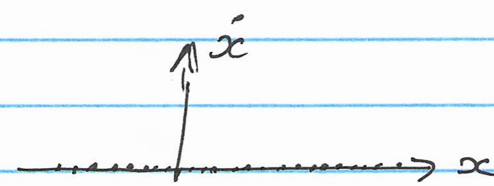
$\Rightarrow$ attracting from left,
repelling on the right



half-stable

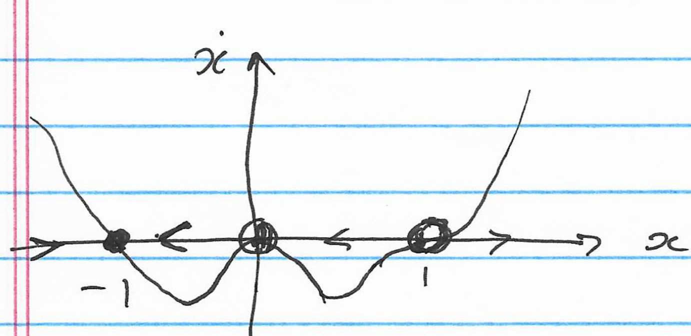
(d) $\dot{x} = 0$

$x^* = ?$



Any pt in $(-\infty, \infty)$ is stable fixed pt — perturbations neither grow nor decay

(e) $\dot{x} = (x+1)x^2(x-1)^3$



$x_1^* = -1$ - stable

$x_2^* = 0$ - half-stable

$x_3^* = 1$ - unstable

Existence and Uniqueness: If $f(x)$ is smooth enough, $\exists!$ solution.

IVP: $\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$

Suppose $f(x), f'(x)$ are continuous on open interval R, (or f is cont. differentiable on R), and $x_0 \in R$. Then:

the IVP has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

(*) Does the solution exist for all times? - No

Ex. $\dot{x} = 1 + x^2$, $x(0) = x_0 = 0$

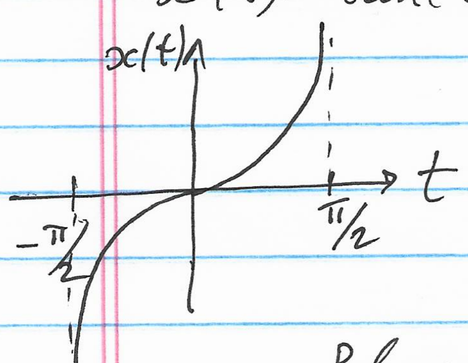
$f(x) = 1 + x^2$ - cont., $f'(x)$ - cont. $\forall x \in R$

$\Rightarrow$ Thm says: $\exists!$ soln $\forall x_0$.

Let's integrate: $\frac{dx}{dt} = 1 + x^2 \Rightarrow \int \frac{dx}{1+x^2} = \int dt \Rightarrow$

$\tan^{-1}(x) = t + C$, as $x(0) = 0 \Rightarrow C = 0$

$\Rightarrow$ $x(t) = \tan(t)$



- $x(t)$ exists only on interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. There is no other soln outside $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $x(0) = 0$

when $t \to \pm\frac{\pi}{2}$, $x(t) \to \pm\infty$

Blow up: soln reaches ∞ at finite time!

(*) What if f is not cont. differentiable?

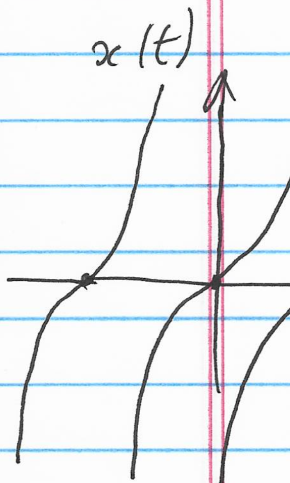
Ex. $\dot{x} = x^{2/3}$ $\qquad f(x) = x^{2/3}$ $\qquad$ We will show that solution is not unique.

Use separation of variables:

$$\int \frac{dx}{x^{2/3}} = \int dt \qquad \left(\text{Use } \int u^n du = \frac{1}{n+1} u^{n+1} + C \right)$$

$$\int x^{-2/3} dx = 3x^{1/3} + C \Rightarrow 3x^{1/3} + C = t$$

$$x^{1/3} = \frac{1}{3}(t-c) \Rightarrow x = \frac{1}{3^3}(t-C)^3 = \frac{1}{27}(t-c)^3$$



Let $x(0)=0 \Rightarrow x(t) = \frac{1}{27}t^3$
$C=0$

If $x(t_0) = 0 \Rightarrow C = t_0$

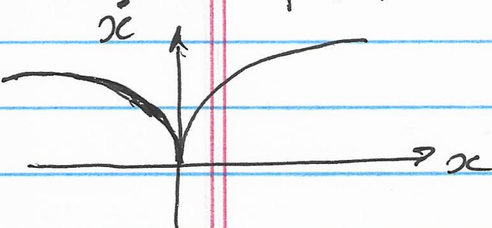
$$x(t) = \frac{1}{27}(t-t_0)^3$$

Graphs are horizontal translations of $\frac{1}{27}t^3$

Missing solution:
$x(t) \equiv 0$

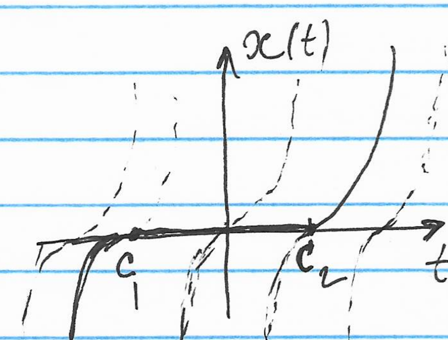
$x(t) = 0$ – singular solution which exists but cannot be constructed using separation of variables

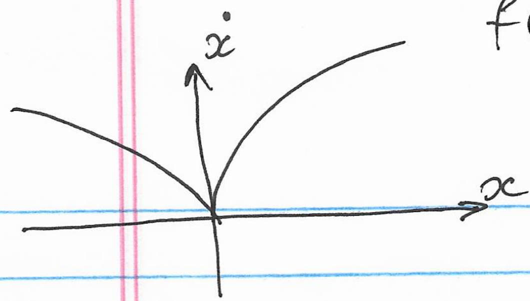
Why? $f'(x)$ is not cont. at $x=0$; $f(x) = x^{2/3}$ is not cont. differentiable at $x=0$



$\Rightarrow$ there are ∞ – many solns:

$$x(t) = \begin{cases} \frac{1}{27}(t-C_1), & -\infty < t < C_1 \\ 0, & C_1 \le t \le C_2 \\ \frac{1}{27}(t-C_2), & t > C_2 \end{cases}$$





$f(x)$ is not cont. differentiable

$f(x) = x^{2/3}$ – the IVP has ∞-many solutions / Thm does not apply

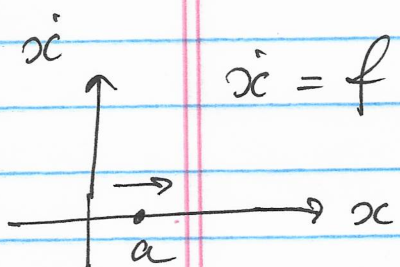
$x^* = 0$ is extremely unstable fixed pt.

$$f'(x) = (x^{2/3})' = \frac{2}{3} x^{-1/3} = \frac{2}{3} \frac{1}{x^{1/3}}$$

$$f'(0) = \frac{2}{3} \frac{1}{x^{1/3}} \Big|_{x=0} = \infty \Rightarrow \text{growth rate} = \infty$$

Characteristic time $= \dfrac{1}{|f'(0)|} = 0$ – time scale

Impossibility of oscillations in 1D



$\dot{x} = f(x)$

If for some point $x = a$, particle passes it in positive direction, velocity $\dot{x} > 0$.

To return back, particle would need to pass this point in negative direction and to have velocity $\dot{x} < 0$.

But particle cannot have both velocity $\dot{x} > 0$ and $\dot{x} < 0$ at the same point as $f(x)$ is single-valued fn.

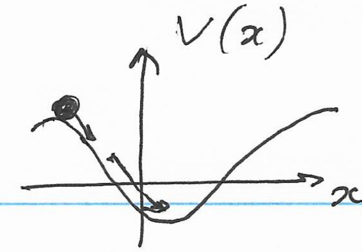
$\Rightarrow$ In 1D points move monotonically, never reverse direction

$\Rightarrow$ No periodic solutions!

? How about harmonic oscillator: $m\ddot{x} + kx = 0$?

2.7 Potentials

$\dot{x} = f(x)$, $f(x) = -\dfrac{dV}{dx}$



Particle sliding down the walls of potential well.

- always downhill: to show this calculate

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt} = \left(\frac{dV}{dx}\right)(\dot{x}) = \frac{dV}{dx}\cdot f(x) = -\left(\frac{dV}{dx}\right)^2 \leq 0$$

$\Rightarrow$ Potential $V(t)$ decreases along the trajectory

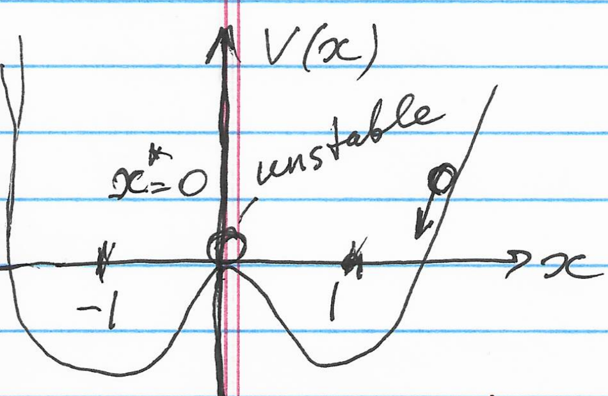
$\Rightarrow$ particle always moves toward lower potential.

$\dfrac{dV}{dx} = 0 \iff \dot{x} = 0$ - equilibria points

Min of $V(x)$ correspond to stable equilibria (fixed pts)

Max of $V(x)$ — " — unstable fixed pts.

Ex. $\dot{x} = x - x^3$ Graph the potential and find fixed pts.



$f(x) = x - x^3$ - Find $V(x)$

$-\dfrac{dV}{dx} = x - x^3 \Rightarrow$

$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$

put $C = 0$ (it doesn't matter)

$x^* = -1$ or $x^* = 1$ | stable fixed pts

Double-well potential

Local minima: $x = \pm 1$ - stable

Local max: $x = 0$ - unstable