Mapping the Latent Structure of Economic Bias: A DML-PULS Fusion Framework

Tyler Venner, Team Sharks

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Abstract

This document details a novel, two-stage methodological framework for visualizing the high-dimensional structure of economic bias. The analysis first confronts the challenge of deterministic salary structures (e.g., Rookie, Max, and Minimum contracts) by stratifying the player pool. This allows us to isolate the "Free Market" sub-population whose salaries are statistically negotiated. We first use a Double/Debiased Machine Learning (DML) pipeline on this "Free Market" subset to learn the "price" of bias, isolating the performance-debiased coefficients ($\hat{\gamma}$). We then apply these coefficients to *all* players to construct a "Debiased Attribution Matrix" (L) representing how the entire league is valued under free-market rules. This L matrix serves as the input for a Probabilistic Unfolding of Latent Space (PULS) model. The resulting "Bias Attribution Map" is a 3D visualization that moves beyond a simple list of effects to reveal the geometric structure of the bias itself, clustering factors and identifying the player archetypes most affected by them.

1 Introduction

1.1 The Problem: The "Wall of Coefficients"

Standard econometric analysis for quantifying bias, while powerful, often concludes with a static table of coefficients and p-values. This "wall of coefficients" is effective at answering the binary question of statistical significance but provides little to no intuition about the underlying *system* of bias. It fails to reveal the latent structure or inter-relationships between the factors under study.

*Plain-language: Our DML pipeline gives us a robust table of coefficients. This tells us **if** 'Age' or 'Draft_Number' impacts salary (after accounting for performance). It **cannot** tell us if 'Age' and 'Draft_Number' are just two aspects of the **same underlying bias** (e.g., a "Veteran/Pedigree" bias).*

1.2 The Methodological Gap

This limitation stems from the different objectives of the tools commonly used in econometrics versus those in psychometrics or consumer science.

- **DML (Econometrics):** Excels at causal estimation and confounding control. It answers the question $Y \sim Z \mid X$ ("What is the effect of Z on Y, holding X constant?"). Its output is a low-dimensional list of numbers ($\hat{\gamma}$).
- **PULS (Psychometrics):** Excels at non-linear dimensionality reduction and visualizing latent preference structures. It answers the question "What 2D map best explains the high-dimensional preference matrix L?". Its output is a low-dimensional map.

1.3 A Proposed Synthesis: The Bias Attribution Map

This report details a novel method to bridge this gap by using the outputs of DML as the inputs for PULS. This fusion creates a two-stage framework that leverages the strengths of both disciplines.

- **Stage 1: DML (The "Engine").** We use DML to rigorously isolate and quantify the performance-debiased "price" of each bias factor, resulting in a vector of coefficients, $\hat{\gamma}$.
- **Stage 2: PULS (The "Compass").** We use PULS to take these isolated attributions and visualize their latent structure, creating a map that shows how the factors relate to each other and to the individuals (players) in the dataset.

1.4 Notation

Throughout this report, we will use the following mathematical notation:

- Y: The outcome variable, (log)Salary.
- X: The vector of player performance statistics (confounders).
- Z: The vector of contextual/bias factors (treatments).
- ϵ_Y, ϵ_Z: The vectors of residuals from the DML confounding models.
- $\hat{\gamma}$: The final vector of debiased DML coefficients, one for each factor in Z.
- L: The $n \times m$ "Liking" (Debiased Attribution) Matrix.
- C, P: The coordinate matrices for Players ("Consumers") and Factors ("Products") in the final PULS map.

2 Stage 1: The DML Pipeline (Isolating Debiased Effects)

2.1 Objective: The Frisch-Waugh-Lovell Theorem

The foundational goal of our DML pipeline is to estimate the "pure" effect of our contextual/bias factors (Z) on player salary (Y), after removing the confounding effects of on-court performance (X). This is a modern, machine-learning-based application of the Frisch-Waugh-Lovell theorem. We seek to find the coefficients γ in the final model:

$$Y = \alpha + Z^T\gamma + f(X) + \epsilon$$

The DML "double-residual" method provides a robust way to estimate γ by first "cleaning" both Y and Z of their relationship with X.

2.2 Step 1: The Outcome Model & Residuals

First, we model the outcome (Y, log-salary) as a function of the confounders (X, performance stats) using a flexible machine learning model, $\hat{f}$.

$$Y_i = f(X_i) + \epsilon_{Y,i}$$

We then compute the "outcome residuals," $\hat{\epsilon}_{Y,i}$, which represent the portion of salary unexplained by performance.

$$\hat{\epsilon}_{Y,i} = Y_i - \hat{f}(X_i)$$

Plain-language: $\hat{\epsilon}_{Y,i}$ is the player's "unexplained salary"—the amount they are paid above or below what their on-court performance predicts. This is our "mispricing" metric.

2.3 Step 2: The Treatment Models & Residuals

Second, we repeat this process for *each* of our m contextual factors (Z_j). We model each Z_j as a function of the same performance stats X using a set of models, $\hat{h}_j$.

$$Z_{ij} = h_j(X_i) + \epsilon_{Z,ij}$$

We then compute the "treatment residuals," $\hat{\epsilon}_{Z,ij}$, for each player i and factor j.

$$\hat{\epsilon}_{Z,ij} = Z_{ij} - \hat{h}_j(X_i)$$

*Plain-language: $\hat{\epsilon}_{Z,ij}$ is the "unexplained part" of the bias factor. For example, the part of a player's 'Draft_Number' that **cannot** be predicted by their on-court performance stats. This is the "pure" component of the factor we are interested in.*

2.4 Step 3: The Final Debiased OLS & Coefficient Vector

Finally, we run a simple, robust OLS regression on these "double-residuals." We regress the unexplained salary $\hat{\epsilon}_Y$ onto the unexplained bias factors $\hat{\epsilon}_Z$.

$$\hat{\epsilon}_Y = \gamma_0 + \sum_{j=1}^{m} \gamma_j \hat{\epsilon}_{Z,j} + \nu$$

This is the "All-at-Once" model described in our methodology (STA160_Model_Extensions.pdf). The resulting vector of coefficients, $\hat{\gamma}$, is our debiased estimate of the bias effects.

Plain-language: This is the final, debiased regression. The resulting coefficient $\hat{\gamma}_j$ is the "price" for one unit of the "performance-free" bias factor. For example, "For each unit of 'unexplained draft pick,' salary increases by $\hat{\gamma}_{draft}$ log-dollars."

2.5 The Interpretive Dead End

This procedure robustly produces the vector $\hat{\gamma}$. A standard analysis would present this vector in a table, similar to Table 1, and conclude the study.

Table 1: Mock DML Output (Final OLS Coefficients)

Bias Factor (Z_j)	**Coefficient ($\hat{\gamma}_j$)**	**Std. Error**	**p-value**
const	0.012	(0.009)	0.182
Age	0.027	(0.010)	**0.007**
Draft_Number	-0.018	(0.005)	**<0.001**
is_USA	0.005	(0.015)	0.739
Owner_Net_Worth_B	0.019	(0.008)	**0.021**
Stadium_Cost	0.002	(0.007)	0.791
... *(5 more factors)*	...	...	...

This table is the "wall of coefficients." It tells us *which* factors are significant but provides no insight into their relationships. We see that 'Age', 'Draft_Number', and 'Owner_Net_Worth_B' are significant. But are they related? Are ‘Age‘ and ‘Draft_Number‘ part of a single ”pedigree” bias? Are ‘Owner_Net_Worth_B‘ and ‘Stadium_Cost‘ part of a ”team resources” bias? The table cannot answer these questions. This is the interpretive dead end we aim to overcome.

2.6 Methodological Refinement: Data Stratification

Before constructing the attribution matrix L, a critical data-processing step is required. The foundational assumption of our DML model, $Y = f(X)+\epsilon$, is that salary (Y) is a *statistical* function of performance (X) with random variation (ϵ). However, in professional sports, a substantial subset of player salaries are *not* statistically determined; they are fixed by deterministic rules defined in the league’s Collective Bargaining Agreement (CBA).

2.6.1 The Deterministic Salary Problem

Applying a statistical model to deterministically-set salaries constitutes severe model misspecification. Consider a high-performing rookie on a fixed-scale contract: the DML outcome model will produce a large negative residual ($\hat{\epsilon}_Y \ll 0$), not because the player is systematically ”underpaid” by market forces, but because their compensation is dictated by a CBA rule that caps rookie salaries regardless of performance.

If we were to train our DML pipeline on a dataset mixing free-market and deterministic contracts, the learned coefficients $\hat{\gamma}$ would be biased and uninterpretable. The ”price of bias” would be confounded by the presence of salary floors and ceilings.

2.6.2 Deterministic Salary Rules (2024-2025 NBA CBA)

For the 2024-2025 season, we identify four primary contract categories based on a $141M salary cap:

1. **Rookie Scale Contracts:** First-round draft picks with 0–3 Years of Service (YOS) are assigned salaries from a predetermined scale based solely on draft position.
2. **Maximum Contracts:** Elite players’ salaries are capped at a percentage of the league salary cap, determined by YOS:
   - 0–6 YOS: 25% of cap ($35,147,000)
   - 7–9 YOS: 30% of cap ($42,176,400)
   - 10+ YOS: 35% of cap ($49,205,800)
3. **Veteran Minimum Contracts:** Salaries are set at a mandatory floor based on YOS, ranging from $1,157,153 (0 YOS) to $3,303,771 (10+ YOS).
4. **Free-Market Contracts:** All remaining players whose salaries are negotiated within the constraints of supply and demand.

2.6.3 Our Solution: The ”Learn vs. Apply” Framework

Given this institutional reality, we stratify the full dataset (D_{ALL}, containing all n players) into four mutually exclusive subsets: D_{Rookie}, D_{Max}, D_{Min}, and D_{FM} (Free Market). Let n_{FM} denote the number of players in D_{FM}. Our methodology then proceeds in four distinct stages:

Stage 1: Stratify. We programmatically assign each player a categorical variable `Contract_Type` $\in$ {Rookie, Max, Min, FM} using the CBA rules above. All players not classified as Rookie, Max, or Min are assigned to D_{FM} by default.

Stage 2: Learn the Market Price (DML on D_{FM} only). We train the *entire* DML pipeline—the outcome model $\hat{f}_{FM}$, the treatment models $\{\hat{h}_{j,FM}\}_{j=1}^{m}$, and the final OLS regression—*exclusively on the n_{FM} players in D_{FM}*. This restriction is essential for obtaining an unbiased estimate of the "free-market price" of bias. The output of this stage is:

- The trained models: $\hat{f}_{FM}$ and $\{\hat{h}_{j,FM}\}_{j=1}^{m}$
- The coefficient vector: $\hat{\gamma} \in \mathbb{R}^m$

Stage 3: Apply the Market Price to All Players. We now use the models trained in Stage 2 to compute treatment residuals for *all n players in D_{ALL}*, regardless of contract type. For any player i and factor j:

$$\hat{\epsilon}_{Z,ij} = Z_{ij} - \hat{h}_{j,FM}(X_i)$$

Here, Z_{ij} is player i's actual value for bias factor j, and $\hat{h}_{j,FM}(X_i)$ is the free-market model's prediction for what that value "should be" based on their performance. We then construct the complete $n \times m$ Debiased Attribution Matrix by applying the fixed coefficient vector $\hat{\gamma}$ to all players:

$$L_{ij} = \hat{\gamma}_j \cdot \hat{\epsilon}_{Z,ij} \quad \text{for all } i \in D_{ALL}$$

Stage 4: Visualize the Entire League (PULS). The complete matrix L (after min-max scaling to L') is provided as input to the PULS algorithm, producing a map containing all n players.

2.6.4 Interpreting the Attribution Matrix Across Contract Types

The matrix L has fundamentally different interpretations depending on a player's contract structure:

- **For Free-Market Players ($i \in D_{FM}$):** The entry L_{ij} represents the *actual* contribution of bias factor j to player i's observed salary, as predicted by the DML model. This is a *factual* attribution.
- **For Deterministic-Contract Players ($i \in D_{Rookie} \cup D_{Max} \cup D_{Min}$):** The entry L_{ij} represents a *counterfactual* attribution: the contribution bias factor j *would make* to the player's salary if their compensation were determined by free-market negotiations rather than CBA constraints.

This counterfactual interpretation is the key methodological insight. The PULS map will reveal the latent "bias profile" of the *entire league*, showing which players would be most advantaged or disadvantaged by bias in an unconstrained market, *regardless of their actual contract status*. This allows us to ask questions like: "Which rookies are protected by the salary scale from negative bias?" or "Which max-contract players would earn even more in a truly free market?"

2.6.5 The Transportability Assumption

This "Learn vs. Apply" approach rests on a critical assumption known in causal inference as *transportability* [**?**]: the "price of bias" learned from free-market players ($\hat{\gamma}$) would apply to other players if their salaries were negotiated rather than determined by CBA rules.

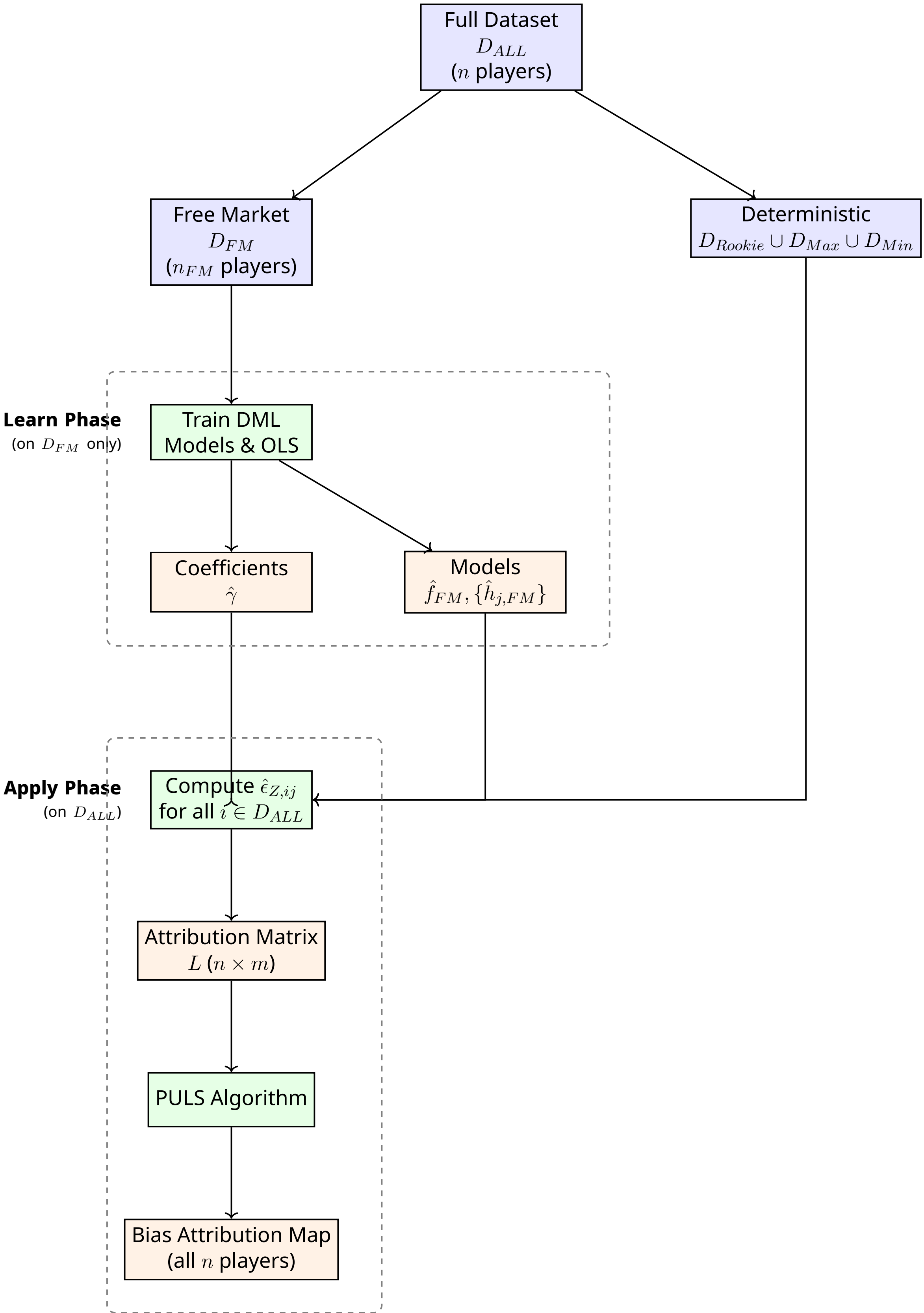

Figure 1: Learn vs. Apply: Data flow in the stratified DML-PULS framework. The "price of bias" ($\hat{\gamma}$) is learned from free-market players but applied to the entire league to construct the attribution matrix.

Formally, we assume the structural relationship:

$$Y_i = \alpha + Z_i^T \gamma + f(X_i) + \epsilon_i$$

holds for *all* players, but is only *observable* (i.e., estimable from data) for the free-market subset. The CBA rules introduce *censoring* or *truncation* of the outcome Y for rookies, max players, and minimum-salary players, but do not change the underlying relationship between performance-adjusted bias factors and team valuation.

This assumption is reasonable if we believe that the factors teams value when unconstrained (e.g., draft pedigree, age, market size) represent stable organizational preferences, even though the *level* of salary is mechanically constrained by the CBA for certain groups.

Diagnostic Check. As a validation step, we assess whether the models ($\hat{f}_{FM}$ and $\{\hat{h}_{j,FM}\}$) trained on D_{FM} generalize to other contract types by examining:

1. **Covariate Overlap:** We use the Kolmogorov-Smirnov test to compare the distribution of each performance statistic X_k between D_{FM} and the other subsets. Significant differences ($p < 0.05$) indicate potential extrapolation issues.
2. **Residual Stability:** We examine the distribution of $\hat{\epsilon}_{Z,ij}$ across contract types. Large systematic shifts would suggest the treatment models do not generalize well.

If substantial violations are detected, a more conservative approach would restrict the final PULS map to free-market players only, sacrificing generality for certainty.

3 Stage 2: The PULS Framework (Mapping Latent Structure)

3.1 Objective: The Consumer-Product Inverse Problem

The PULS model is a non-linear dimensionality reduction technique designed to solve an inverse problem. As detailed in the original PULS technical report (main.pdf), its primary objective is:

Given a "liking" matrix L (of size n Consumers $\times$ m Products), find the d-dimensional (e.g., $d = 2$) coordinate maps for Consumers (C) and Products (P) such that the geometric distance between a consumer C_i and a product P_j on the map corresponds to the observed liking L_{ij}.

3.2 The Core Insight: Product-Product Proximity

While PULS models both consumers and products, the key feature for our analysis is the resulting geometry of the product map, P. The algorithm's optimization objective places products on the map based on their "liking" profiles. This leads to a fundamental insight: *products that are "liked" by the same consumers will be placed near each other on the map.*

This emergent property—the clustering of products based on shared preference patterns—is the exact mechanism we need to reveal the latent structure of our bias factors.

Plain-language: In PULS, "products" that are "liked" by the same "consumers" will be placed near each other on the map. This reveals the latent structure of the "products." We will leverage this to see which "bias factors" are "liked" by the same "players."

4 The DML-PULS Fusion: Constructing the Bias Attribution Map

4.1 The Core Analogy

The bridge between the two methodologies is a novel analogy. We re-frame the output of the DML model as the input for the PULS model.

- **PULS Consumers (i) $\rightarrow$ NBA Players ($i = 1...n$)**
- **PULS Products (j) $\rightarrow$ Bias Factors ($j = 1...m$)**

4.2 Defining the "Liking" Matrix (L): The Debiased Attribution

Our goal is to create an $n \times m$ matrix L where the entry L_{ij} represents the total amount of (performance-adjusted) unexplained salary for player i that can be attributed to bias factor j. Critically, this matrix is constructed for *all n players in D_{ALL}*, including those on deterministic contracts, using the free-market pricing rules established in Section 2.6.

From the DML final stage (Section 2.3), the linear predictor for unexplained salary is:

$$\hat{\epsilon}_{Y,i} = \gamma_0 + \sum_{j=1}^{m} \hat{\gamma}_j \hat{\epsilon}_{Z,ij}$$

For players in the free-market subset ($i \in D_{FM}$), this equation describes their *actual* salary decomposition. For players on deterministic contracts, it describes a *counterfactual* salary structure (see Section 2.6.4).

The term $\hat{\gamma}_j \hat{\epsilon}_{Z,ij}$ is precisely the additive contribution of factor j to player i's (actual or counterfactual) unexplained salary. This is the quantity we visualize. We therefore define the **Debiased Attribution Matrix** as:

$$L_{ij} = \hat{\gamma}_j \cdot \hat{\epsilon}_{Z,ij} \quad \text{for all } i \in D_{ALL},\ j = 1, \ldots, m$$

where:

- $\hat{\gamma}_j$ is the free-market "price" learned exclusively from D_{FM} (a fixed constant),
- $\hat{\epsilon}_{Z,ij} = Z_{ij} - \hat{h}_{j,FM}(X_i)$ is computed for all players using the treatment model trained on D_{FM}.

4.3 A Naive (and Flawed) Approach: $L = \hat{\gamma} \cdot Z$

The simplest way to construct L would be to multiply the DML coefficients by the raw Z values.

$$L_{ij} = \hat{\gamma}_j \cdot Z_{ij}$$

Plain-language: This is a "Step 1" mistake, analogous to the naive deterministic model in the PULS report. We could just multiply the coefficient for age ($\hat{\gamma}_{age}$) by the player's age (Z_{age}) to get their "age attribution."

4.4 Why This Naive Approach Fails (Diagnostic)

This approach is both theoretically and statistically flawed, and would lead to a meaningless map.

- **Theoretical Failure:** It ignores the core premise of the DML model. The coefficient $\hat{\gamma}_j$ is not the "price" for the raw factor Z_j; it is the "price" for the *performance-debiased residual*, $\hat{\epsilon}_{Z,j}$. Using the raw Z_j would ignore the entire double-residual procedure.
- **Statistical Failure (Incomparable Scales):** This is the critical, practical pitfall. The Z matrix has columns with wildly different units (e.g., Age in *years*, 'Draft_Number' in *picks*, 'Stadium_Cost' in *millions of dollars*). The variance of the 'Stadium_Cost' column in L would be orders of magnitude larger than the 'Age' column, not because it is a more powerful bias, but simply because its units are larger. This would completely distort the PULS map, as the scaling of the L matrix would be dominated by these arbitrary units.

4.5 The Principled Solution: The ϵ_Z-Attribution Matrix

The theoretically and statistically robust solution is to define the L matrix using the two key outputs from the DML pipeline: the coefficients ($\hat{\gamma}$) and the treatment residuals ($\hat{\epsilon}_Z$).

We define the **Debiased Attribution Matrix** L as the element-wise product:

$$L_{ij} = \hat{\gamma}_j \cdot \hat{\epsilon}_{Z,ij}$$

4.6 Why This Model is Superior (Plain-language)

This definition of L creates a sound foundation for the PULS analysis.

- **Theoretically Sound:** L_{ij} is now the "Total Debiased Salary Attribution." It is the "price" of the bias ($\hat{\gamma}_j$) multiplied by the "amount" of performance-free bias the player has ($\hat{\epsilon}_{Z,ij}$). This is the precise quantity we want to visualize.
- **Statistically Robust:** The $\hat{\epsilon}_{Z,j}$ columns are all on a *comparable*, unit-less scale. This claim requires a minor qualification: the 'StandardScaler' in the treatment models (from 'treatment_models.py') ensures the *input* Z_j variables have $Var(Z_j) = 1$, so the *residual* variance, $Var(\hat{\epsilon}_{Z,j}) = Var(Z_j - \hat{h}_j(X))$, will be ≤ 1. This residual variance will vary between factors, depending on how much each factor is predictable by performance X. This is not a flaw; it is an *informative* feature, as the remaining variance reflects the factor's orthogonality to performance. This state is vastly superior to the naive approach, as the variance differences are meaningful, not artifacts of arbitrary units.

4.7 Data Scaling for PULS

The resulting L matrix will contain both positive values (e.g., overpayment attributed to 'Age') and negative values (e.g., underpayment attributed to 'Draft_Number'). The PULS algorithm, however, expects a "liking" score, typically scaled from 0 to 1.

We therefore apply a single, global min-max scaling transformation to the entire L matrix to create our final input, L'.

$$L'_{ij} = \frac{L_{ij} - \min(L)}{\max(L) - \min(L)}$$

This transformation maps the full range of attributions onto a [0, 1] scale. The "liking" scale for PULS is now:

- **Liking = 0.0:** The maximum *underpayment* attribution.
- **Liking = 1.0:** The maximum *overpayment* attribution.

A value of 0.5 would represent zero attribution (where $L_{ij} = 0$).

4.8 A Note on the PULS "Preference" Metaphor

It is important to clarify the interpretation of "preference" in this context. The PULS model was originally developed for consumer science, where "liking" reflects a true psychological preference with properties like satiety.

In our DML-PULS framework, the "liking" (i.e., the scaled attribution L') is a more abstract, geometric quantity. The model does not assume a player *psychologically prefers* a bias factor. Rather, PULS is used here as a powerful geometric clustering algorithm. The "ideal point" is simply the location in the latent space that best represents the player's total bias profile, and "distance" is a non-linear measure of similarity. The interpretation is one of **geometric clustering** rather than **literal preference**.

5 Interpreting the Bias Attribution Map

The final 2D (3D) map generated by PULS provides a rich, visual answer to the questions that the "wall of coefficients" (Table 1) could not. The interpretation is based on the proximity of players (C) and factors (P).

5.1 Player Clusters (The "Consumers")

The map is populated by n player points. Players are positioned on the map based on their "bias profile" (their row in the L' matrix).

- We expect to see clusters of players who are "mispriced" in similar ways. For example, a cluster of high draft picks, a cluster of international players, or a cluster of veterans.

Interpreting Clusters by Contract Type. A particularly insightful analysis involves color-coding the n player points on the map by their `Contract_Type`. This visualization will reveal:

- **Free-Market Clustering:** Players in D_{FM} whose salaries are actually determined by the bias factors should cluster near the corresponding factor points (e.g., older players near the 'Age' factor).
- **Rookie Isolation or Alignment:** Rookie-scale players (D_{Rookie}) will reveal which bias factors they *would* be exposed to in a free market. If rookies cluster far from all factors, it suggests the CBA successfully "protects" them from market biases. Conversely, if they align with negative-attribution factors (e.g., low draft picks near a negatively-valued 'Draft_Number' factor), it identifies which rookies are most disadvantaged by the salary scale.
- **Max-Contract Suppression:** Max-contract players (D_{Max}) with high positive attributions reveal whose market value exceeds their capped salary. A star player located near highly-valued factors (e.g., 'All_Star_Selections', 'Market_Size') but earning "only" $49M indicates suppressed bias effects.

- **Minimum-Salary Diagnostic:** Veteran-minimum players (D_{Min}) clustered near negative-attribution factors indicate players whom the salary floor prevents from being further devalued by bias.

This contract-type overlay transforms the map from a purely descriptive tool into a diagnostic instrument for evaluating the equity and efficiency of the CBA's salary rules.

5.2 Factor Points (The "Products")

The map also contains the $m = 10$ bias factor points. These are the "products" that the players are "consuming."

5.3 The Core Insight: Player-Factor Proximity

Based on the PULS optimization, players will be "pulled" toward the factors that are the primary drivers of their unexplained salary.

- A player who is significantly "overpaid" due to their 'Age' (a high $L'_{i,age}$ score) will be located geometrically close to the 'Age' factor point on the map.
- A player who is "underpaid" due to their 'Draft_Number' (a low $L'_{i,draft}$ score) will be located far away from the 'Draft_Number' point.

5.4 The "So What" Insight: Factor-Factor Proximity

This is the main interpretive payoff of the entire framework. The relative positions of the 10 *factor points* reveal the latent structure of the bias itself.

- **Case 1 (Clustering):** If the 'Age' and 'Draft_Number' factor points are close *to each other* on the map, it implies they are "liked" by the same players. This is powerful visual evidence that they are not two independent biases, but are likely components of a **single, underlying bias structure** (e.g., a "Veteran/Pedigree" bias).
- **Case 2 (Separation):** If "Player-Specific" factors (like 'Age') and "Team-Specific" factors (like 'Owner_Net_Worth_B') are in different, distant quadrants of the map, it provides immediate visual proof of two **distinct categories of bias** operating in the league.

5.4.1 A Necessary Caveat on Interpretation

This interpretation—that proximity implies a shared latent structure—is powerful but requires a caveat. Proximity on the map occurs if the attribution profiles are similar ($L'_{ij} \approx L'_{ik}$ for most i). This can be caused by:

- (a) True structural similarity (our desired interpretation);
- (b) Correlated residuals, where $\hat{\epsilon}_{Z,j}$ and $\hat{\epsilon}_{Z,k}$ happen to be correlated for reasons unrelated to a shared bias.

Therefore, factor proximity *suggests*, but does not definitively *prove*, a shared underlying mechanism. A complementary analysis, such as examining the direct correlation matrix of the $\hat{\epsilon}_Z$ residuals, is a recommended step to support this interpretation.

5.5 The Case of Non-Significant Factors (The "Boring" Middle)

The map also correctly handles the non-significant factors from Table 1, such as 'is_USA'.

- A factor j with a coefficient $\hat{\gamma}_j \approx 0$ will have its entire column in the L matrix as $L_{ij} \approx 0$.
- After the min-max scaling, this column L'_{ij} will be a single constant value for all players (e.g., $L'_{ij} \approx 0.33$, as noted in Section 4.7).
- PULS will correctly place this "product" at the geometric center of the map, as it is "equally-liked" (or "equally-boring") to all players and has no differentiating power.
- **This is a feature, not a bug.** The map automatically creates a visual hierarchy. The most important, structure-defining bias factors will be pulled to the periphery by the player clusters they influence, while the non-significant factors will remain "stuck" in the center.

6 Validation & Advanced Analysis

6.1 Attribute Fitting (from PULS main.pdf)

The resulting map is not just a visualization; it is a new, low-dimensional representation of the players (C). We can validate this new space by using the "Property Fitting" method described in the PULS report. After the map is built, we can regress *other* player variables (e.g., Player Position, Team Market Size, or even raw performance stats from X) onto the player coordinates C. This allows us to see if external factors, which were not used to build the map, can explain the player-bias clusters.

6.2 Stability Analysis

A critical question is whether the "Factor-Factor" geometry (the main insight from Section 5.4) is robust or an artifact of random noise. To ensure this structure is stable, we can run a non-parametric bootstrap procedure:

1. **Resample:** Create a new "liked" matrix L^* by resampling the n players (rows of L') with replacement.
2. **Re-run PULS:** Fit the PULS model on L^* and save the $m = 10$ coordinates of the factor map P^*.
3. **Repeat:** Repeat this process 100-1000 times.
4. **Plot:** Plot the 100-1000 coordinate points for each of the 10 factors.

This procedure will result in "confidence ellipses" for our 10 factor points. If the ellipses for 'Age' and 'Draft_Number' are small and overlapping, it provides strong statistical evidence that their proximity is a stable feature of the data, not a chance occurrence.

6.3 Advanced Validation: Permutation Testing

While bootstrapping (Section 6.2) assesses the *stability* of the factor coordinates, a permutation test can be used to assess the *statistical significance* of the map's structure. For example, to test if 'Age' and 'Draft_Number' are "significantly closer" than expected by chance, one could:

1. Calculate the observed Euclidean distance $d_{obs} = ||P_{Age} - P_{Draft}||$.
2. Create a null distribution by repeatedly shuffling the rows of the L' matrix (i.e., breaking the player-profile relationship), re-running PULS, and calculating the distance d_{null} from the resulting maps.
3. Compare d_{obs} to the null distribution to obtain a p-value.

This provides a formal hypothesis test for the geometric relationships revealed in the map.

6.4 Limitations and Boundary Conditions

While the DML-PULS fusion framework is theoretically sound and empirically powerful, several limitations must be acknowledged:

6.4.1 Limitation 1: The Transportability Assumption

The validity of applying free-market pricing rules ($\hat{\gamma}$) to deterministic-contract players rests on the assumption that bias coefficients are *structural* (i.e., reflect stable team preferences) rather than *context-dependent*. If teams systematically value factors differently when negotiating with rookies versus free agents—for reasons beyond the CBA constraints—the counterfactual attributions in L may be biased.

Mitigation. The diagnostic checks in Section 2.6.5 (covariate overlap tests and residual stability analysis) provide empirical evidence for or against this assumption. Additionally, a sensitivity analysis restricting the PULS map to D_{FM} only can establish a "conservative baseline" interpretation.

6.4.2 Limitation 2: Model Extrapolation Risk

The treatment models $\{\hat{h}_{j,FM}\}$ are trained on the performance statistics (X) of free-market players. If rookies or minimum-salary players have systematically different X distributions (e.g., lower stats, shorter tenure), predictions $\hat{h}_{j,FM}(X_i)$ for these groups constitute *extrapolation*, which may be unreliable.

Mitigation. Modern ensemble methods (e.g., Random Forests, XGBoost) generalize reasonably well under moderate extrapolation. The Kolmogorov-Smirnov tests recommended in Section 2.6.5 quantify the severity of this issue. If substantial non-overlap is detected, treatment residuals for out-of-distribution players should be interpreted with greater uncertainty.

6.4.3 Limitation 3: Geometric Interpretation Ambiguity

As noted in Section 4.8, the PULS model is being used as a *geometric clustering algorithm* rather than a true preference model. While factor proximity on the map suggests shared latent structure, it does not *prove* it. Alternative explanations (e.g., incidental correlation between $\hat{\epsilon}_{Z,j}$ and $\hat{\epsilon}_{Z,k}$) must be ruled out via complementary analyses, such as examining the correlation matrix of $\hat{\epsilon}_Z$ directly or using the permutation tests described in Section 6.3.

6.4.4 Limitation 4: Single-Season Cross-Sectional Data

Our analysis uses a single season's data, which limits causal inference. Bias coefficients ($\hat{\gamma}$) may be season-specific, affected by transient market conditions, league-wide injuries, or rule changes. A more robust analysis would pool multiple seasons or use panel methods to estimate time-averaged bias effects.

Future Extension. A longitudinal DML-PULS framework, where $\hat{\gamma}_t$ is estimated for each season t and the resulting bias maps are compared over time, would reveal whether bias structures are stable or evolving.

7 Conclusion & Future Work

7.1 Summary of Contribution

This framework successfully fuses a rigorous econometric tool (DML) with a powerful psychometric visualization tool (PULS). The primary contributions of this methodology are:

- **A Novel Bridge:** We formally define the ”Debiased Attribution Matrix” ($L = \hat{\gamma} \cdot \hat{\epsilon}_Z$) as the theoretically-sound bridge connecting the two frameworks.
- **A New Visualization:** The resulting ”Bias Attribution Map” allows, for the first time, a visual exploration of the *latent structure* of economic bias. It moves analysis beyond a simple ”wall of coefficients” to a map of relationships.

7.2 Future Work

The ”Bias Attribution Map” serves as a powerful new analytical object, opening several avenues for future research.

- **Prescriptive Analytics (from PULS main.pdf):** The map can be used for ”what-if” scenarios. We can mathematically model a ”new player” (e.g., a hypothetical rookie) and see which existing bias factor they are closest to, predicting how they might be mispriced by the market.
- **Application to Other Domains:** This DML-PULS fusion framework is fully general. It can be applied to any domain where a ”treatment effect” can be estimated and attributed, such as mapping the latent structure of biases in:
  - Real estate (e.g., mapping property features to price residuals).
  - Hiring (e.g., mapping candidate attributes to salary offers).
  - Finance (e.g., mapping company attributes to stock mispricing).