Advanced Modern Algebra second edition

Selected Solutions

Chapter 1: Groups I

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1.1. Classical Formulas

Exercise 1.1. Given $M, N \in \mathbb{C}$, prove that there exists $g, h \in \mathbb{C}$ with $g + h = M$ and $gh = N$.

Proof. Consider the quadratic equation $x^2 - Mx + N = 0$ and apply the quadratic formula, we have two roots $r_1 = \frac{-M+\sqrt{M^2-4N}}{2}$ and $r_2 = \frac{-M-\sqrt{M^2-4N}}{2}$. Notice that $r_1 + r_2 = -M$ and $r_1 r_2 = N$. Then we see that $-r_1, -r_2 \in \mathbb{C}$ that satisfies the relation. $\square$

Exercise 1.3. (i) Find the complex roots of $f(x) = x^3 - 3x + 1$.

(ii) Find the complex roots of $f(x) = x^4 - 2x^2 + 8x - 3$.

Exercise 1.4. Show that the quadratic formula does not hold for $f(x) = ax^2 + bx + c$ if we view the coefficients a, b, c as lying in the integers mod 2.

1.2. Permutations

Exercise 1.5. Give an example of functions $f : X \to Y$ and $g : Y \to X$ such that $gf = 1_X$ and $fg \neq 1_Y$.

Proof. Consider $f : \mathbb{Z} \to \mathbb{Z}, g : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = -x, g(x) = |x|$. Then we see that $gf(x) = |-x| = x$, $\forall x \in \mathbb{Z}$ while $fg(1) = -|1| = -1$. $\square$

Exercise 1.6. Prove that the composition of functions is associative: if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, then

$$h(gf) = (hg)f.$$

Proof. $h(gf)(x) = h(g(f(x)) = (gh)f(x)$ and $\square$

Exercise 1.7. Prove that the composite of two injections is an injection, and that the composite of two surjections is a surjection. Conclude that the composite of two bijections is a bijection.