Advanced Modern Algebra second edition

Selected Solutions

Chapter 1: Groups I

September 29, 2023

1.1. Classical Formulas

Exercise 1.1. Given $M, N \in \mathbb{C}$, prove that there exists $g, h \in \mathbb{C}$ with $g + h = M$ and $gh = N$.

Proof. Consider the quadratic equation $x^2 - Mx + N = 0$ and apply the quadratic formula, we have two roots $r_1 = \frac{-M+\sqrt{M^2-4N}}{2}$ and $r_2 = \frac{-M-\sqrt{M^2-4N}}{2}$. Notice that $r_1 + r_2 = -M$ and $r_1 r_2 = N$. Then we see that $-r_1, -r_2 \in \mathbb{C}$ that satisfies the relation. $\square$

Exercise 1.3. (i) Find the complex roots of $f(x) = x^3 - 3x + 1$.

(ii) Find the complex roots of $f(x) = x^4 - 2x^2 + 8x - 3$.

Exercise 1.4. Show that the quadratic formula does not hold for $f(x) = ax^2 + bx + c$ if we view the coefficients a, b, c as lying in the integers mod 2.

1.2. Permutations

Exercise 1.5. Give an example of functions $f : X \to Y$ and $g : Y \to X$ such that $gf = 1_X$ and $fg \neq 1_Y$.

Proof. Consider $f : \mathbb{Z} \to \mathbb{Z}, g : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = -x, g(x) = |x|$. Then we see that $gf(x) = |-x| = x$, $\forall x \in \mathbb{Z}$ while $fg(1) = -|1| = -1$. $\square$

Exercise 1.6. Prove that the composition of functions is associative: if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, then

$$h(gf) = (hg)f.$$

Proof. $h(gf)(x) = h(g(f(x)) = (gh)f(x)$ and $\square$

Exercise 1.7. Prove that the composite of two injections is an injection, and that the composite of two surjections is a surjection. Conclude that the composite of two bijections is a bijection.

Exercise 1.8 (Pigeonhole Principle). (i) Let $f : X \to X$ be a function, where X is a finite set. Prove equavalence of the following statements.

(a) f is an injection.

(b) f is a bijection.

(c) f is a surjection.

(ii) Prove that no two of the statements in (i) are equivalent when X is an infinite set.

(iii) Suppose there are 501 pigeons, each sitting in some pigeonhole. If there are only 500 pifeonholes, prove that there is a hole containing more than one pigeon.

Exercise 1.9. Let Y be a subset of a finite set X, and let $f : Y \to X$ be an injection. Prove that there is a permutation $\alpha \in S_X$ with $\alpha|Y = f$.

Exercise 1.10. Find $\text{sgn}(\alpha)$ and α^{-1}, where

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Exercise 1.11. If $\alpha \in S_n$, prove that $\text{sgn}(\alpha^{-1}) = \text{sgn}(\alpha)$.

Exercise 1.12. If $1 \leq r \leq n$, show that there are

$$\frac{1}{r}[n(n-1)...(n-r+1)]$$

r-cycles in S_n.

Hint i. f