Advanced Modern Algebra second edition

Selected Solutions

Chapter 1: Groups I

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1.1. Classical Formulas

Exercise 1.1. Given $M, N \in \mathbb{C}$, prove that there exists $g, h \in \mathbb{C}$ with $g + h = M$ and $gh = N$.

Proof. Consider the quadratic equation $x^2 - Mx + N = 0$ and apply the quadratic formula, we have two roots $r_1 = \frac{-M+\sqrt{M^2-4N}}{2}$ and $r_2 = \frac{-M-\sqrt{M^2-4N}}{2}$. Notice that $r_1 + r_2 = -M$ and $r_1 r_2 = N$. Then we see that $-r_1, -r_2 \in \mathbb{C}$ that satisfies the relation. $\square$

Exercise 1.3. (i) Find the complex roots of $f(x) = x^3 - 3x + 1$.

(ii) Find the complex roots of $f(x) = x^4 - 2x^2 + 8x - 3$.

Exercise 1.4. Show that the quadratic formula does not hold for $f(x) = ax^2 + bx + c$ if we view the coefficients a, b, c as lying in the integers mod 2.

Proof. Take $f(x) = x^2 + x + 1$, applying the quadratic formula, we have $r_1 = \frac{-1+\sqrt{1-4}}{2} = \frac{1+1}{2} = 1 \neq 0$ and $r_2 = \frac{-1-\sqrt{1-4}}{2} = \frac{1-1}{2} = 1 \neq 0$. $\square$

1.2. Permutations

Exercise 1.5. Give an example of functions $f : X \to Y$ and $g : Y \to X$ such that $gf = 1_X$ and $fg \neq 1_Y$.

Proof. Consider $f : \mathbb{Z} \to \mathbb{Z}, g : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = -x, g(x) = |x|$. Then we see that $gf(x) = |-x| = x$, $\forall x \in \mathbb{Z}$ while $fg(1) = -|1| = -1$. $\square$

Exercise 1.6. Prove that the composition of functions is associative: if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, then

$$h(gf) = (hg)f.$$

Proof. $h(gf)(x) = h(g(f(x)) = (hg)f(x)$ □

Exercise 1.7. Prove that the composite of two injections is an injection, and that the composite of two surjections is a surjection. Conclude that the composite of two bijections is a bijection.

Proof. Injection: Suppose we have $f : X \to Y$, $g : Y \to Z$ both injections. Then we see that $\forall a_1, a_2 \in X$, $a_1 = a_2 \implies f(a_1) = f(a_2) \implies g(f(a_1)) = g(f(a_2))$. Hence the composition of two injections is an injection.
Surjection: Suppose we have $f : X \to Y$, $g : Y \to Z$ both surjections. Then we see that $\forall c \in Z, \exists b \in Y$ s.t. $g(b) = c$. Also, since f is surjective, there is $a \in X$ s.t. $f(a) = b$. Hence there is $a \in X$ with $g(f(a)) = c$ for all $c \in Z$.
Bijection: Since bijections are injective and surjective at the same time, compositions of two bijections must also be both injective and surjective at the same time. □

Exercise 1.8 (Pigeonhole Principle).

(i) Let $f : X \to X$ be a function, where X is a finite set. Prove equavalence of the following statements.

   (a) f is an injection.

   (b) f is a bijection.

   (c) f is a surjection.

(ii) Prove that no two of the statements in (i) are equivalent when X is an infinite set.

(iii) Suppose there are 501 pigeons, each sitting in some pigeonhole. If there are only 500 pifeonholes, prove that there is a hole containing more than one pigeon.

Proof.

(i) Since bijective iff surjective and injective, we only need to proof (a) iff (c). Suppose f is injective, then no two elements in X are mapped to the same element, hence $|X| = |f(X)|$ or equavalently, f is surjective.

(ii)

□

Exercise 1.9. Let Y be a subset of a finite set X, and let $f : Y \to X$ be an injection. Prove that there is a permutation $\alpha \in S_X$ with $\alpha|Y = f$.

Exercise 1.10. Find $\mathrm{sgn}(\alpha)$ and α^{-1}, where

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Exercise 1.11. If $\alpha \in S_n$, prove that $\mathrm{sgn}(\alpha^{-1}) = \mathrm{sgn}(\alpha)$.

Exercise 1.12. If $1 \leq r \leq n$, show that there are

$$\frac{1}{r}[n(n-1)...(n-r+1)]$$

r-cycles in S_n.

Hint . There are exactly r cycle notations for any $r-$cycle.

Exercise 1.13. (i) If α is an $r-$cycle, show that $\alpha^r = (1)$.

Hint . If $\alpha = (i_0...i_{r-1})$, show that $\alpha^k(i_0) = i_j$, where $k = qr + j$ and $0 \leq j < r$.

(ii) If α is an $r-$cycle, show taht r is the smallerst positive integer k such that $\alpha^k = (1)$.

Exercise 1.14. Show that an $r-$cycle is an even permutation if and only if r is odd.

Exercise 1.15. (i) Let $\alpha = \beta\delta$ be a factorization of a permutation α into disjoint permutations. If β moves i, prove that $\alpha^k(i) = \beta^k(i)$ for all $k \geq 1$.

(ii) Let β and γ be cucles both of which move i. If $\beta^k(i) = \gamma^k(i)$ for all $k \geq 1$, prove that $\beta = \gamma$.

Exercise 1.16. Given $X = 1, 2, ..., n$, let us call a permutation τ of X an ***adjacency*** if it is a transposition of the form $(i\ i+1)$ for $i < n$.

(i) Prove that every permutation in S_n, for $n \geq 2$, is a product of adjacencies.

(ii) If $i < j$, prove that $(i\ j)$ is a product of an odd number of adjacencies.

Hint . Use induction on $j - i$.

Exercise 1.17. (i) Prove, for $n \geq 2$, that every $\alpha \in S_n$ is a product of transpositions each of whose factors moves n.

Hint . If $i < j < n$, then $(j\ n)(i\ j)(j\ n) = (i\ n)$, by Lemma 1.7, so that $(i\ j) = (j\ n)(i\ n)(j\ n)$.

(ii) Why doesn't part (i) prove that a 15-puzzle with even starting position α which fixes $\square$ can be solved?

Exercise 1.18. Define $f : 0, 1, 2, ..., 10 \to 0, 1, 2, ..., 10$ by

$$f(n) = \text{the remainder after dividing } 4n^2 - 3n^7 \text{ by } 11.$$

(i) Show that f is a permutation.

(ii) Compute the parity of f.

(iii) Compute the inverse of f.

Exercise 1.19. If α is an $r-$cucle and $1 < k < r$, is α^k an $r-$cycle?

Exercise 1.20. (i) Prove that if α and β are (not necessarily disjoint) permutations that commute, then $(\alpha\beta)^k = \alpha^k\beta^k$ for all $k \geq 1$.

Hint . First show that $\beta\alpha^k = \alpha^k\beta$ by induction on k.

(ii) Given an example of two permutations α and β for which $(\alpha\beta)^2 \neq \alpha^2\beta^2$.

Exercise 1.21. (i) Prove, for all i, that $\alpha \in S_n$ moves i if and only if α^{-1} moves i.

(ii) Prove that if $\alpha, \beta \in S_n$ are disjoint and if $\alpha\beta = (1)$, then $\alpha = (1)$ and $\beta = (1)$.

Exercise 1.22. Prove that the number of even permutations in S_n is $\frac{1}{2}n!$.

Hint . Let $\tau = (1\,2)$, and define $f : A_n \to O_n$, where A_n is the set of all even permutations in S_n and O_n is the set of all odd permutations, by $f : \alpha \to \tau\alpha$. Show that f is a bijection, so that $|A_n| = |O_n|$ and, hence, $|A_n = \frac{1}{2}n!$.

Exercise 1.23. (i) How many permutations in S_5 commute with $\alpha = (1\ 2\ 3)$, and how many $textifeven$ permutations in S_5 commute with α?

Hint . Of the six permutations in S_5 commuting with α, only three are even.

(ii) Same question for $(1\,2)(3\,4)$.

Hint . Of the eight permutations in S_4 commuting with $(1\,2)(3\,4)$, only four are even.

Exercise 1.24. Given an example of $\alpha, \beta, \gamma \in S_5$, with $\alpha \neq (1)$, such that $\alpha\beta = \beta\alpha, \alpha\gamma = \gamma\alpha$, and $\beta\gamma \neq \gamma\beta$.

Exercise 1.25. If $n \geq 3$, prove that if $\alpha \in S_n$ commutes with every $\beta \in S_n$, thwn $\alpha = (1)$.

Exercise 1.26. If $\alpha = \beta_1 ... \beta_m$ is a product of disjoint cycles and δ is disjoint from α, show that $\beta_1^{e_1} ... \beta_m^{e_m}\delta$ commutes with α, where $e_j \geq 0$ for all j.