Introduction to Quantum Information and Quantum Machine Learning

Laboratory - class 6

Dr Gustaw Szawioła, docent PUT
Dr Sci. Eng. Przemysław Głowacki

Phase estimation
(using IQFT)

Phase Estimation

We're concerned with an eigenvalue problem, namely an equation of the form $A\,x = \lambda\,x$
where $A \in \mathbb{C}^{2^m \times 2^m}, x \in \mathbb{C}^{2^m}$ and $\lambda \in \mathbb{C}$.

Note that we write the dimension as 2^m for convenience, since *m* qubits imply a state space of size 2^m.
In the quantum case, we're only going to be concerned with unitary operators, which we normally write as *U*.
Since these operators satisfy $U^\dagger U=I$, any eigenvalue has magnitude one. Since $|\lambda|=1$, we can write it without loss of generality as $\lambda = 2^{2\pi i\phi}$, where $\mathbf{0 \leq \phi \leq 1}$ is called the ***phase***.

Let's introduce some useful notation that is common in quantum algorithms. The phase ϕ is going to be between zero and one, so we can write it as a decimal in binary notation as follows:

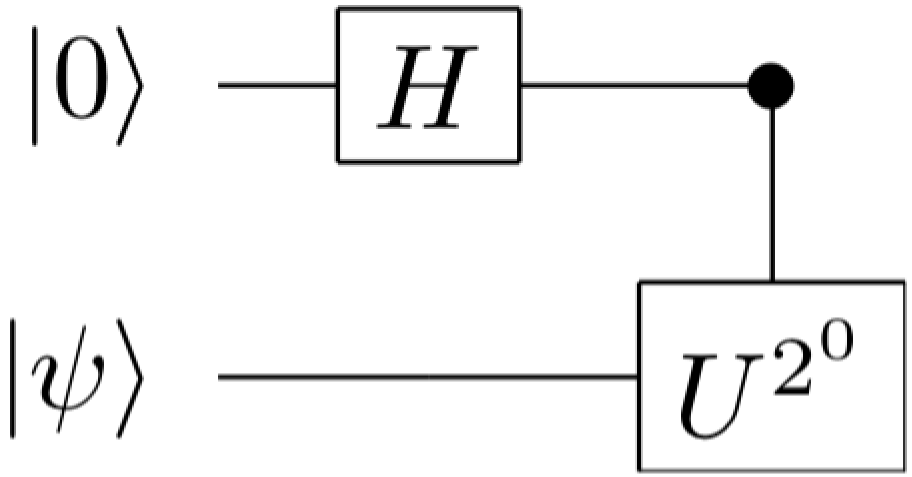
$$\boldsymbol{\phi = 0.\phi_1\phi_2 \ldots \phi_n}$$ where each ϕ_i is either zero or one.

$$\boldsymbol{\phi = 0.\phi_1\phi_2 \ldots \phi_n \Leftrightarrow \phi = \sum_{k=1}^{n} \phi_n\, 2^{-k}}$$

The numer 0.5 in decimal is 0.1 in binary $0.1=(1)\cdot 2^{-1}=1/2=0.5$

The numer 0.75 in decimal is 0.11 in binary
$0.11=(1)\cdot 2^{-1}+(1)\cdot 2^{-2}=1/2 + 1/4=0.75$

Action of Controlling an Operator on its Eigenstate



Here we let U be a unitary operator and $\langle\psi|$ an eigenstate with eigenvalue $\lambda = e^{2\pi i\, 0.\phi_1}$

$$|+\rangle = \hat{H}|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \hat{H}|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

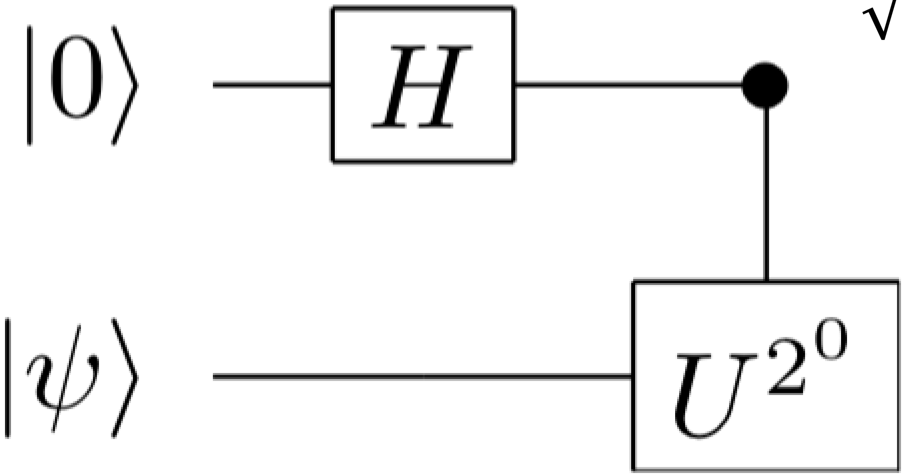
H gate on first qubit

$$|+\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle)$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle U^{2^0}|\psi\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.\phi_1}\,|1\rangle\right)\otimes|\psi\rangle$$

Action of Controlling an Operator on its Eigenstate

$$\frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle U^{2^0}|\psi\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.\phi_1}\,|1\rangle\right)\otimes|\psi\rangle$$



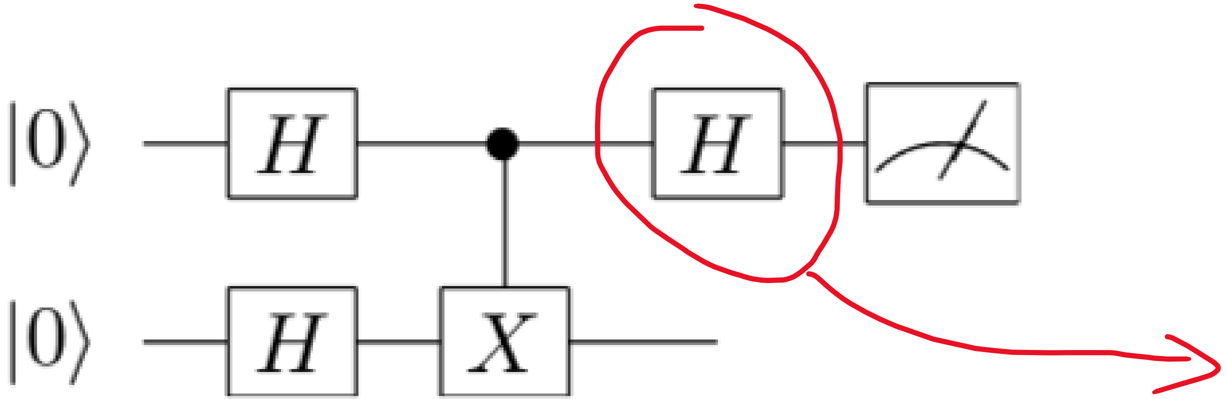
The second qubit register $|\psi\rangle$ hasn’t change.
$|\psi\rangle$ is an eigenstate of U^{2^0}.
Simply:

$$(|0\rangle + |1\rangle) \longrightarrow \left(|0\rangle + e^{2\pi i\, 0.\phi_1}\,|1\rangle\right)$$

$$\hat{H}\left(|0\rangle + e^{2\pi i\, 0.\phi_1}\,|1\rangle\right) = \frac{1}{\sqrt{2}}\left[\left(1 + e^{2\pi i\, 0.\phi_1}\right)|0\rangle + \left(1 - e^{2\pi i\, 0.\phi_1}\right)|1\rangle\right]$$

In case $\phi_1 = 0, e^{2\pi i\, 0.\phi_1}$=1, hence the state is exacly $|0\rangle$.

Phase estimation on Pauli-$\hat{X}$.



The eigenvalues of $\hat{X}$ are -1 and 1 with eigenvectors $|-\rangle$ and $|+\rangle$

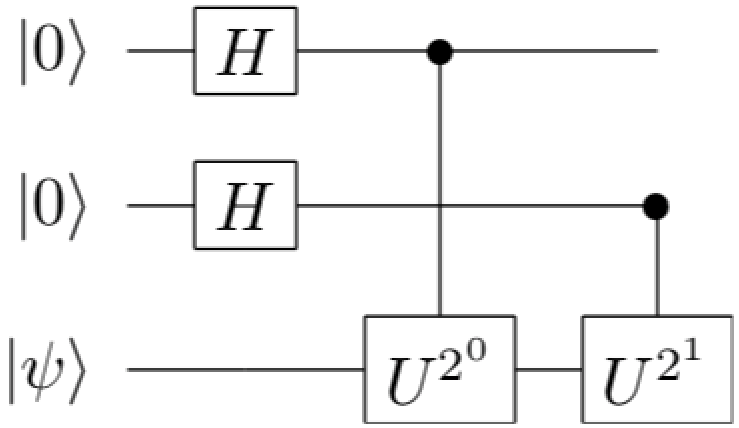
$$\hat{H}\,|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|+\rangle \otimes |+\rangle = |00\rangle + |01\rangle + |10\rangle + |11\rangle$$

$$|0\rangle$$

In case $\phi_1 = 0, \lambda = e^{2\pi i\, 0.\phi_1} = 1,$

Controlling Higher Powers



The final algorithm introduces n qubits in the top register and implements $C(U^{2^k})$ between the k-th qubit and the bottom register.
Example n=2
$|\psi\rangle$ is an m-qubit state and U is m-qubit unitary.

(q0= $|0\rangle$) $\otimes$ H $\otimes$ U^{2^0} $\implies$ $\left(|0\rangle + e^{2\pi i\, 0.\phi_1\phi_2}|1\rangle\right)\otimes(|0\rangle + |1\rangle)\otimes|\psi\rangle$

the relative phase in the first qubit now has two digits because we assumed that $\phi = 0.\phi_1\phi_2$ consists of two digits.

(q1= $|0\rangle$) $\otimes$ H $\otimes$ U^{2^1} $\implies U^{2^1}|\psi\rangle = e^{2\pi i\, 0.\phi_1\phi_2}|\psi\rangle$

Note: $2\phi = 2\cdot 0.\phi_1\phi_2 = 2(\phi_1 2^{-1} + \phi_2 2^{-2}) = \phi_1 + 2\phi_2 2^{-1} = \phi_1.\phi_2$

The effect is that the decimal moves one place to the right.

$e^{2\pi i\,(2\phi)} = e^{2\pi i\,(\phi_1 + 0.\phi_2)} = e^{2\pi i \phi_1}\, e^{2\pi i 0.\phi_2} = e^{2\pi i 0.\phi_2}$ in general $e^{2\pi i(2^k\phi)} = e^{2\pi i 0.\phi_k\phi_{k+1}\dots}$

State after applying the $C(U^{2^1})$ to the wavefunction is:

$$\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.\phi_1\phi_2}\,|1\rangle\right)\otimes\frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i\, 0.\phi_2}\,|1\rangle\right)\otimes|\psi\rangle \qquad (*)$$

The quantum Fourier transform is a unitary change of basis with the following effect:

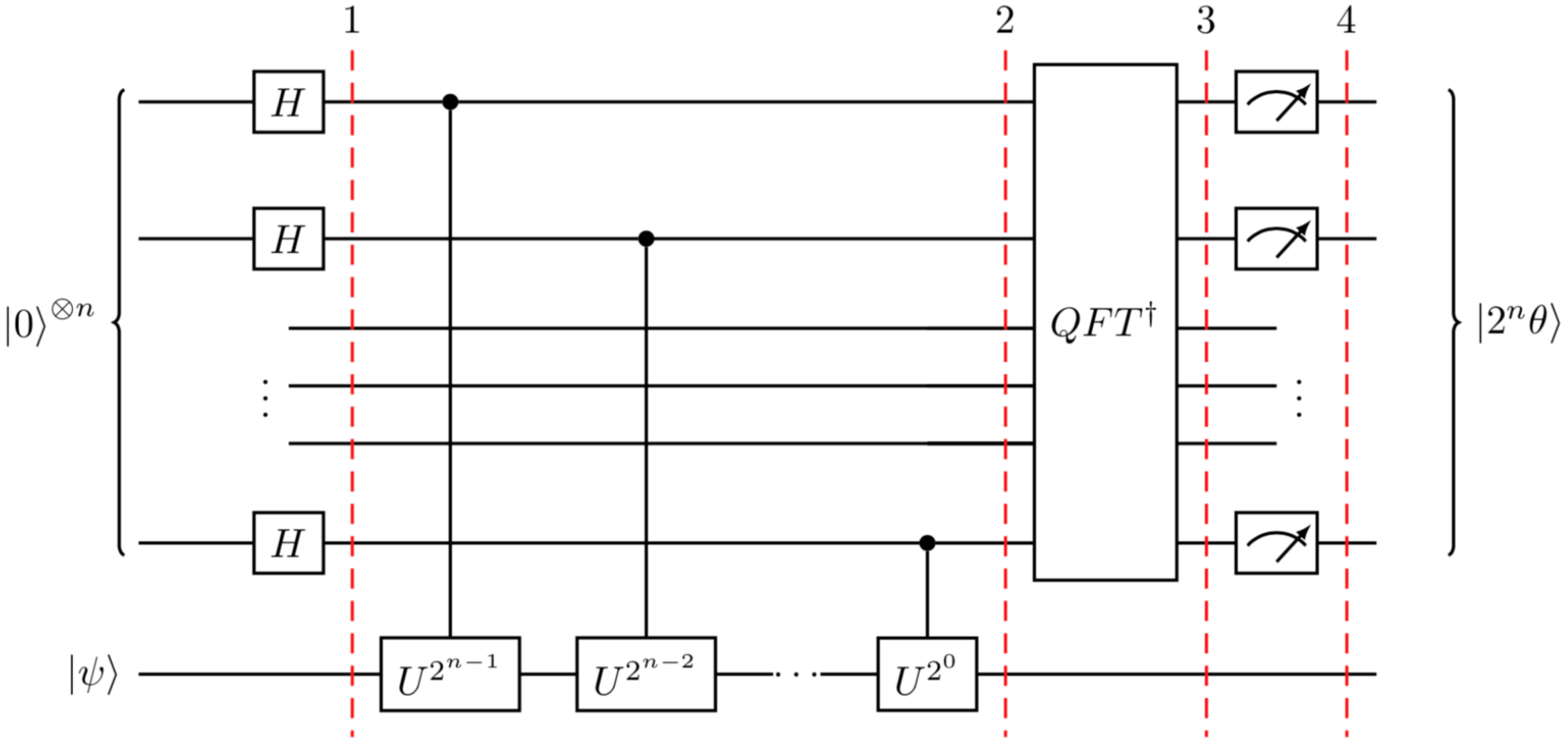
$$QFT(|\phi_1\rangle|\phi_2\rangle \dots |\phi_n\rangle) = 2^{-n/2}\left(|0\rangle + e^{2\pi i\, 0.\phi_1}\,|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i\, 0.\phi_1\phi_2}\,|1\rangle\right) \otimes \dots \otimes \left(|0\rangle + e^{2\pi i\, 0.\phi_1\phi_2\dots\phi_n}\,|1\rangle\right)$$

The state on the right looks exactly like what we end up with from phase estimation!

We need to apply then to read out the information is the inverse Fourier transform $\mathrm{QFT}^{\dagger}$.

$$QFT^{\dagger}\left(|0\rangle + e^{2\pi i\, 0.\phi_1}\,|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i\, 0.\phi_1\phi_2}\,|1\rangle\right) \otimes |\psi\rangle = |\psi_2\rangle \otimes |\psi_1\rangle \otimes |\psi\rangle$$

General Phase Estimation Algorithm

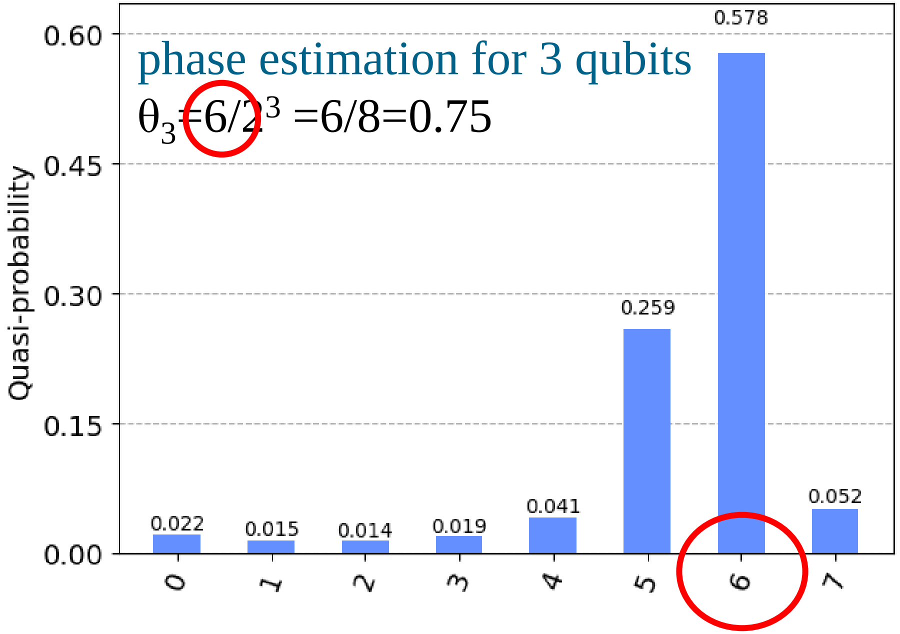


Qiskit Implementation

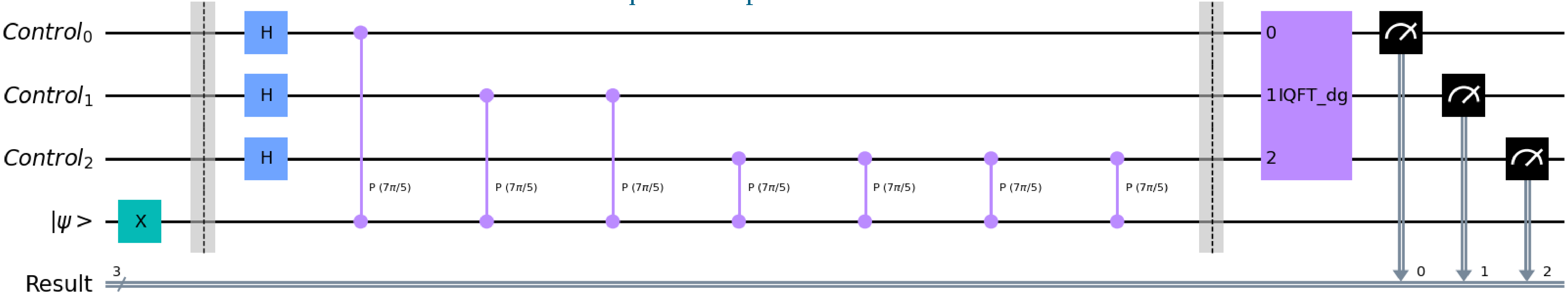
The example:

1) use the file *Lab6.ipynb*, where the recurrency circuit for the phase estimation is defined
2) Theta θ angle is set as 0.7

$\lambda = 2^{2\pi i\theta}$, θ is the phase, n – number of qubits, using the file *Lab6.ipynb* we obtained the highlighted state in decimal notation $d_{h.}$ For 3 qubits system is $d_h = 6$. $\theta_3 = 6/2^3 = 6/8 = 0.75$



Circuit for n=3 qubits for phase estimation



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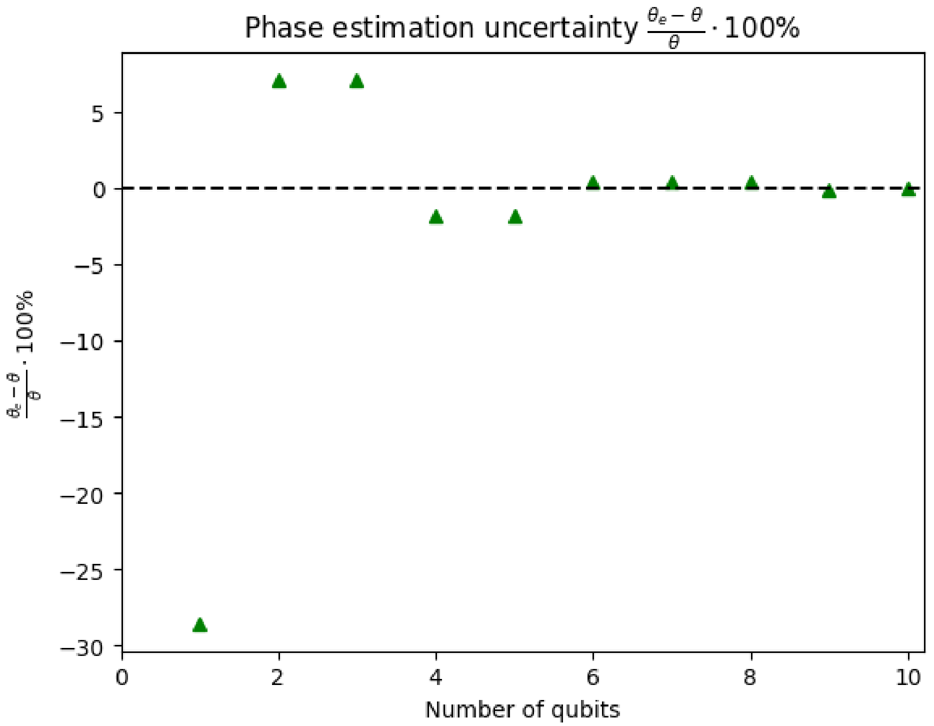
3) We make the phase estimation from n=1 to n=10.

n	d_h	θ_e *phase estimation*
1	1	$1/2^1 = 0.5$
2	3	$3/2^2 = 0.75$
3	6	$6/2^3 = 0.75$
4	11	$11/2^4 = 0.6875$
5	22	$22/2^5 = 0.6875$
6	45	$45/2^6 = 0.703125$
7	90	$90/2^7 = 0.703125$
8	179	$179/2^8 = 0.69921875$



Qiskit Implementation - TASKS

Tasks list:

1) Use the file *Lab6.ipynb*, where the recurrency circuit for the phase estimation is defined

2) Make a phase estimation from n=1 to n=10.

2a) Group 9.45, Theta θ angle =0.33

2b) Group 11.45, Theta θ angle =0.66

3) Present the results:

3a) In table

3b) As a graph for Y axis use $100*(\theta_e - \theta)/\theta$ [%]

The End