Introduction to Quantum Information and Quantum Machine Learning

Laboratory - class 4

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Quantum Oracle - Preparation for Grover’s Algorithm

Grover's algorithm is a quantum algorithm designed to run on a quantum computer, presented by Lov K. Grover in 1996 [1] and published in 2001 [2].
The algorithm involves searching a database consisting of N elements in order to find a highlighted element in it. This is a similar problem to "reverse" searching a phone book. In a book containing N data, we want to find the name of the owner of a given number.

Computational complexity
While the number of steps required to solve the problem using the classical algorithm is of the order of *O(N)* Grover's quantum algorithm only needs about O($\sqrt{N}$) steps, so it allows for a quadratic acceleration of the program execution time.

The algorithm involves searching for a given element in an unsorted N-element set. The search problem comes down to determining, by means of unitary transformations, the appropriate index defining a given element in the set.

[1] L. K. Grover. *A fast quantum mechanical algorithm for database search*. In STOC ’96: Proceedings p. 212-219, New York, NY, USA (1996)
[2] Grover L.K.: *From Schrödinger’s equation to quantum search algorithm*, American Journal of Physics, 69(7): 769-777 (2001)

1. XOR Oracle and Phase Oracle

XOR Oracle

The problem of N data in an unstructured database

Clasiccal maps (function) $f\colon \{0,1\}^n \to \{0,1\}$

$$f(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}$$

$a = ?$

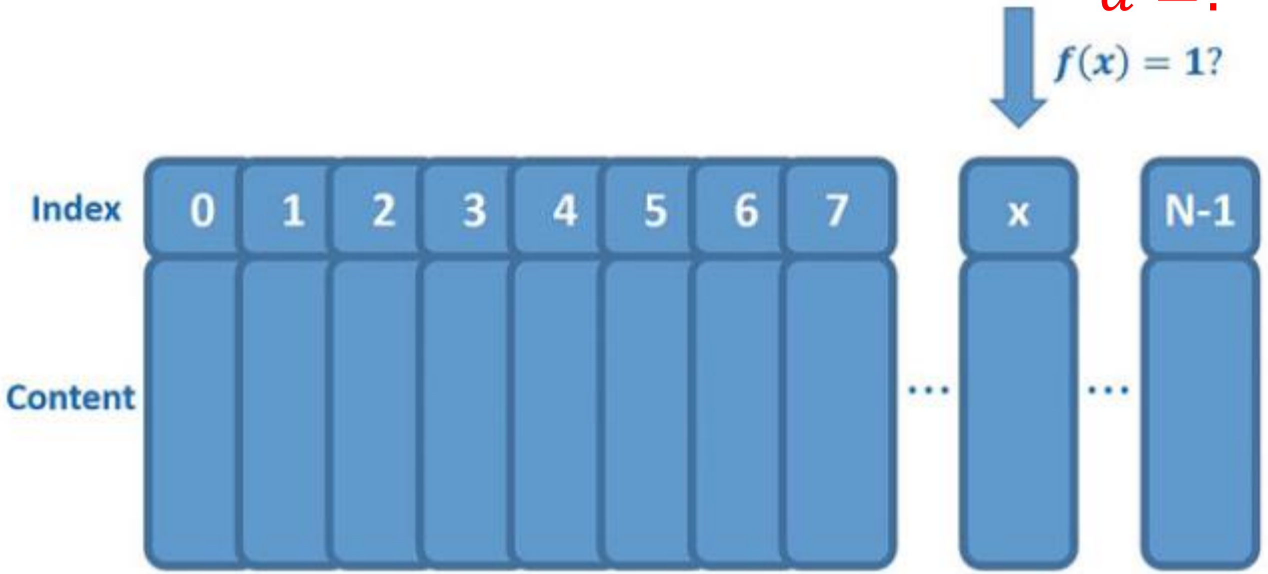


Illustration of a searching process in a database with N entries.
A function f (x) s used to determine if the target is located at entry x.

Computational Complexity

Classic search

→ Average $\mathcal{O}\left(\frac{N}{2}\right) \sim \mathcal{O}(N)$

Grover's quantum algorithm

→ Average $\mathcal{O}(\sqrt{N})$

Grover's algorithm provides a quadratic speedup over the classical one

XOR Oracle

$$U_f\ (|y\rangle \otimes |x\rangle) = |y \oplus f(x)\rangle \otimes |x\rangle$$

$\boldsymbol{a = 10}$

$\|i\rangle$	$\|y\rangle\|x\rangle$	$U_f\ (\|y\rangle\otimes\|x\rangle)$
$\|0\rangle$	$\|0\ 00\rangle$	$\|0\ 00\rangle$
$\|1\rangle$	$\|0\ 01\rangle$	…
$\|2\rangle$	$\|0\ \mathbf{10}\rangle$	$\|1\ \mathbf{10}\rangle$
$\|3\rangle$	$\|0\ 11\rangle$	…
…	…	…
$\|n-1\rangle$	$\|1\ 11\rangle$	$\|1\ 11\rangle$

Phase Oracle

$$U_f|x\rangle = (-1)^{f(x)}|x\rangle$$

$\boldsymbol{a = 010}$

$\|i\rangle$	$\|x\rangle$	$U_f\|x\rangle$
$\|0\rangle$	$\|0\ 00\rangle$	$\|0\ 00\rangle$
$\|1\rangle$	$\|0\ 01\rangle$	…
$\|2\rangle$	$\|\mathbf{0\ 10}\rangle$	$-\|\mathbf{0\ 10}\rangle$
$\|3\rangle$	$\|0\ 11\rangle$	…
…	…	…
$\|n-1\rangle$	$\|1\ 11\rangle$	$\|1\ 11\rangle$

Create the Matrix of U_f operation

XOR Oracle

$$\begin{array}{cc} & j\ column \\ & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ i\ row \quad \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7 \end{array} & \left[\begin{array}{cccccccc} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right] \end{array}$$

quantum gate

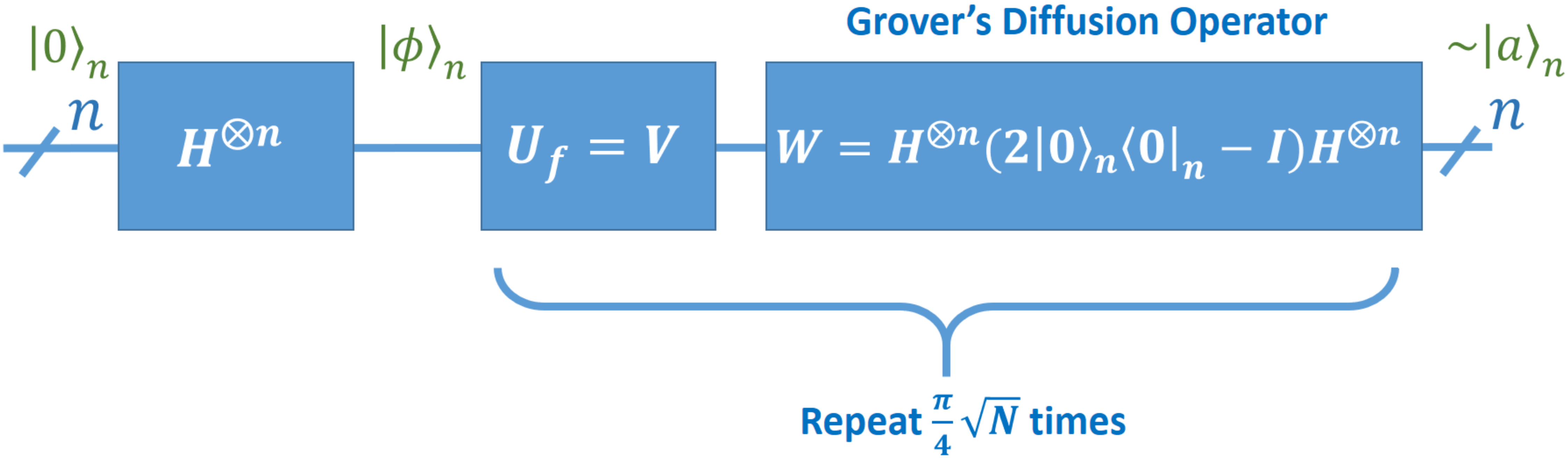
Phase Oracle

$$\begin{array}{cc} & j\ column \\ & \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ i\ row \quad \begin{array}{c} 0\\1\\2\\3\\4\\5\\6\\7 \end{array} & \left[\begin{array}{cccccccc} \\ \\ \\ \\ \\ \\ \\ \\ \end{array} \right] \end{array}$$

quantum gate

2. Grover’s Diffusion operator

Implementation of the Grover's diffusion operator



Implementation of the Grover's diffusion operator

- $|0\rangle_{n\,n}\langle 0| = (\quad)$
- $\hat{\mathbb{I}} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$
- $2\,|0\rangle_{n\,n}\langle 0| - \hat{\mathbb{I}} = 2(\quad) - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = (\quad)$

Implementation of the Grover's diffusion operator

- $\hat{X}^{\otimes n}\left(2\,|0\rangle_{n\,n}\langle 0| - \hat{\mathbb{I}}\right)\hat{X}^{\otimes n} = 2\,\hat{X}^{\otimes n}|0\rangle_{n\,n}\langle 0|\hat{X}^{\otimes n} - \hat{X}^{\otimes n}\hat{\mathbb{I}}\hat{X}^{\otimes n} = 2|1\rangle_{n\,n}\langle 1| - \underbrace{\hat{X}^{\otimes n}\hat{X}^{\otimes n}}_{=\hat{\mathbb{I}}} =$

$= 2|1\rangle_{n\,n}\langle 1| - \hat{\mathbb{I}}$

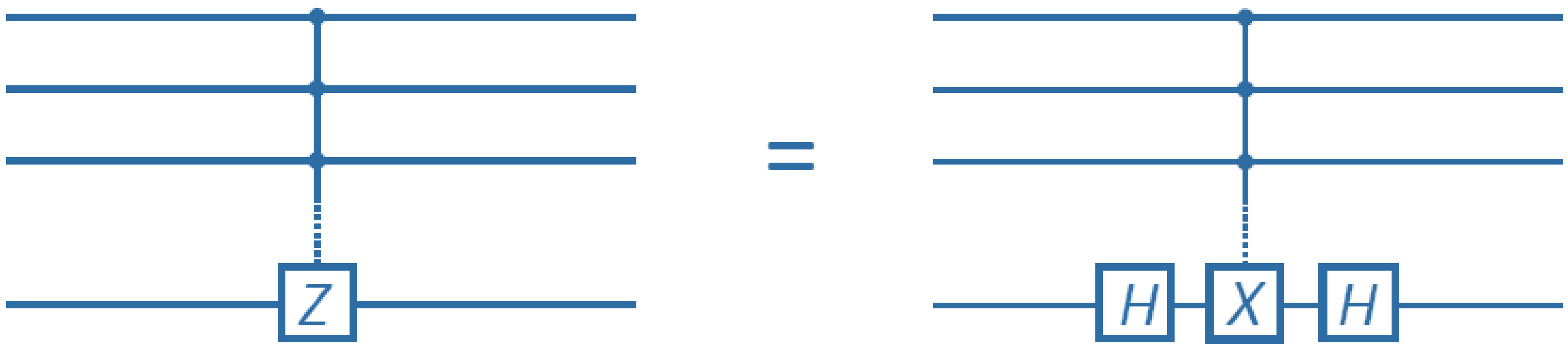
gdzie $|1\rangle_n = \underbrace{|11\cdots 1\rangle}_{n}$

- $2|1\rangle_{n\,n}\langle 1| - \hat{\mathbb{I}} = 2(\quad) - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = -(\quad) = -\left(\hat{\mathbb{I}} - 2|1\rangle_{n\,n}\langle 1|\right)$

Implementation of the Grover's diffusion operator

- $(\hat{\mathbb{I}} - 2|1\rangle_{n\,n}\langle 1|)$

$=$



$$C_{n-1}\hat{Z} = (\hat{H}\otimes\hat{\mathbb{I}})C_{n-1}\hat{X}(\hat{H}\otimes\hat{\mathbb{I}})$$

$$\hat{Z} = \widehat{H}\,\hat{X}\,\widehat{H}\,,\ \hat{X}^{\otimes n}\,\hat{X}^{\otimes n} = \hat{\mathbb{I}}$$

$$\widehat{W} = -\widehat{H}^{\otimes n}\hat{X}^{\otimes n}\,C_{n-1}\hat{Z} = \hat{X}^{\otimes n}(\widehat{H}\otimes\hat{\mathbb{I}})C_{n-1}\hat{X}(\widehat{H}\otimes\hat{\mathbb{I}})\hat{X}^{\otimes n}\widehat{H}^{\otimes n}$$

Tasks:

$$f(x) = \begin{cases} 1, & x = a \\ 0, & x \neq a \end{cases}$$

1. Create the Matrix of U_f operation:
   1.1 as XOR Oracle (**Group @9:45 take a=00, Group @11:45 take a=01)**
   1.2 as Phase Oracle (**Group @9:45 take a=011, Group @11:45 a=100)**
2. From the Matrix of U_f operation create quantum gate

For n =3

3. Calculate the explicit matrix form of operator (lecture 6, frame 35): $|0\rangle_{n\,n}\langle 0|$
4. Calculate the explicit matrix form of operator (lecture 6, frame 35-36): $2\,|0\rangle_{n\,n}\langle 0| - \hat{\mathbb{I}}$
5. Prove that (lecture 6, frame 34): $\hat{X}^{\otimes n}\left(2\,|0\rangle_{n\,n}\langle 0| - \hat{\mathbb{I}}\right)\hat{X}^{\otimes n} = 2|1\rangle_{n\,n}\langle 1| - \hat{\mathbb{I}}$
6. Calculate the explicit matrix form of operator: W Grover's diffusion operator (lecture 6, frame 33-43)
7. Using the Qiskit template (file lab4_template.ipynb) create quantum gates representing U_f and W. (Tip use X, H and CNOT gates to create W)

A1 Appendix

XOR Oracle

$U_f\,(|y\rangle \otimes |x\rangle) = |y \oplus f(x)\rangle \otimes |x\rangle$ $\quad \boldsymbol{a = 10}$

$\|i\rangle$	$\|y\rangle\|x\rangle$	$U_f\,(\|y\rangle\otimes\|x\rangle)$
$\|0\rangle$	$\|0\ 00\rangle$	$\|0\ 00\rangle$
$\|1\rangle$	$\|0\ 01\rangle$	$\|0\ 01\rangle$
$\|2\rangle$	$\|0\ \mathbf{10}\rangle$	$\|1\ \mathbf{10}\rangle$
$\|3\rangle$	$\|0\ 11\rangle$	$\|0\ 11\rangle$
$\|4\rangle$	$\|1\ 00\rangle$	$\|1\ 00\rangle$
$\|5\rangle$	$\|1\ 01\rangle$	$\|1\ 01\rangle$
$\|6\rangle$	$\|1\ \mathbf{10}\rangle$	$\|0\ \mathbf{10}\rangle$
$\|7\rangle$	$\|1\ 11\rangle$	$\|1\ 11\rangle$

Phase Oracle

$U_f|x\rangle = (-1)^{f(x)}|x\rangle$ $\quad \boldsymbol{a = 010}$

$\|i\rangle$	$\|x\rangle$	$U_f\|x\rangle$
$\|0\rangle$	$\|0\ 00\rangle$	$\|0\ 00\rangle$
$\|1\rangle$	$\|0\ 01\rangle$	$\|0\ 01\rangle$
$\|2\rangle$	$\|\mathbf{0\ 10}\rangle$	$-\|\mathbf{010}\rangle$
$\|3\rangle$	$\|0\ 11\rangle$	$\|0\ 11\rangle$
$\|4\rangle$	$\|1\ 00\rangle$	$\|1\ 00\rangle$
$\|5\rangle$	$\|1\ 01\rangle$	$\|1\ 01\rangle$
$\|6\rangle$	$\|1\ 10\rangle$	$\|1\ 10\rangle$
$\|7\rangle$	$\|1\ 11\rangle$	$\|1\ 11\rangle$

Construct the matrix for XOR Oracle $a = 10$

$$\boldsymbol{base} = \underbrace{\{|0\rangle, |0\rangle, \dots, |n-1\rangle\}}_{\boldsymbol{base\ vector\ numbers} \Leftrightarrow \boldsymbol{decimal\ numbers}}$$

$$\langle 0||0\rangle \Leftrightarrow \langle 0|0\rangle = 1$$

$$|j\rangle - row,\ \langle i| - column$$

$$\delta_{ji} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$$

$$U_f = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \langle j|U_f|i\rangle |j\rangle\langle i|$$

$$\langle 2|\, U_f|6\rangle = \langle 2| \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \langle j|U_f|i\rangle|j\rangle\langle i|6\rangle = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \langle j|U_f|i\rangle\langle 2|j\rangle\langle i|6\rangle =$$

$$= \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \langle j|U_f|i\rangle \delta_{2j}\delta_{i6} = \sum_{i=0}^{N-1} \langle j|U_f|6\rangle \delta_{2j} = \langle 2|\boxed{U_f|6\rangle} = \langle 010|010\rangle = 1$$

$$U_{f\ j,i} = \langle j|U_f|i\rangle$$

$$U_{f20} = \langle 2|\, U_f|0\rangle = \langle 010|000\rangle = 0$$

$$U_{f21} = \langle 2|\, U_f|1\rangle = \langle 010|001\rangle = 0$$

$$U_{f22} = \langle 2|\, U_f|2\rangle = \langle 010|110\rangle = 0$$

$$U_{f26} = \langle 2|\, U_f|6\rangle = \langle 010|010\rangle = 1$$

$$U_{f62} = \langle 6|\, U_f|2\rangle = \langle 110|110\rangle = 1$$

$U_f\ (|y\rangle \otimes |x\rangle)$
See table

Create the Matrix of U_f operation

XOR Oracle

$$\begin{array}{ccc} & & j\ column \\ & & \begin{matrix}0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\end{matrix} \\ i\ row & \begin{matrix}0\\1\\2\\3\\4\\5\\6\\7\end{matrix} & \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\Downarrow$$

quantum gate

Phase Oracle

$$\begin{array}{ccc} & & j\ column \\ & & \begin{matrix}0 & 1 & 2 & 3 & 4 & 5 & 6 & 7\end{matrix} \\ i\ row & \begin{matrix}0\\1\\2\\3\\4\\5\\6\\7\end{matrix} & \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}\end{array}$$

$$\Downarrow$$

quantum gate

Hint Task 3-5

$$|0\rangle_{n=3} = |000\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle \qquad {}_{n=3}\langle 0| = \langle 000| = \langle 0| \otimes \langle 0| \otimes \langle 0|$$

$$|0\rangle_{n=3} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Construct the matrix for a 2 bits NOT-gate (N=2)

- $|j\rangle - row,\ \langle i| - column$ (22.6) , V $|x\rangle_n = (-1)^{f(x)}|x\rangle_n$ (23.9)

$$U_{NOT} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\langle j|\,U_{NOT}|i\rangle|j\rangle\langle i| =$$

$$= \langle 0|U_{NOT}|0\rangle|0\rangle\langle 0| + \langle 1|U_{NOT}|0\rangle|1\rangle\langle 0| + \langle 0|U_{NOT}|1\rangle|0\rangle\langle 1| + \langle 1|U_{NOT}|1\rangle|1\rangle\langle 1| =$$

$$= \langle 0||1\rangle|0\rangle\langle 0| + \langle 1||1\rangle|1\rangle\langle 0| + \langle 0||0\rangle|0\rangle\langle 1| + \langle 1||0\rangle|1\rangle\langle 1| =$$

$$= 0 * |0\rangle\langle 0| + 1 * |1\rangle\langle 0| + 1 * |0\rangle\langle 1| + 0 * |1\rangle\langle 1| = |1\rangle\langle 0| + |0\rangle\langle 1| =$$

$$= \begin{pmatrix}0\\1\end{pmatrix} * (1\ 0) + \begin{pmatrix}1\\0\end{pmatrix} * (0\ 1) = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} + \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

[źródło: H. Y. Wong, *Introduction to Quantum Computing*, Springer 2022]

The End