Optimal achromatic wave retarders using two birefringent wave plates: comment

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In a recent publication, J. L. Vilas *et al.* [Appl. Opt. **52**, 1892 (2013)] described a combined system comprising two wave plates and proposed an expression for the overall retardation. However, to the best of my knowledge, the proposed theoretical expression for the total retardation is incorrect. The authors improperly derived that expression by simply diagonalizating the Jones matrix of the cascaded system. The mistakes are pointed out and discussed in these comments.

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The achromatic wave plate can be made in various ways; for example, from two or more different birefringent materials or from two or more wave plates of the same material but whose axes are oriented at appropriate angles with respect to each other.

In a recent publication, J. L. Vilas *et al.* [1] addressed the optimization of a combined system composed of two wave plates from different materials (quartz and MgF_2). They defined a merit function they called achromatism degree (AcD) to perform the optimization. The defined merit function is a measure of the spectrum-weighted distance between the overall retardation of the system and the target retardation. However, the authors improperly derived the expression for the overall retardation by simply diagonalizating the Jones matrix of the combined system. Therefore, they failed to put forward the mathematic solution for the overall azimuthal angle of the combined system under any orientation angles of the two plates.

In fact, the topic about the combined optical systems, which are composed of pure retarders, linear polarizers, and rotators, has been thoroughly discussed by R. C. Jones in his papers [2,3], which proved the theorem that an optical system containing any number of retarders and rotators is optically equivalent to a system containing only a retarder and a rotator [3].

According to Jones' theorem, the overall matrix representing an optical system composed of any number of wave plates with arbitrary individual azimuthal angles is unitary and can be expressed by [3]

$$\mathbf{U} = \begin{bmatrix} A & -B^* \\ B & A^* \end{bmatrix} = \mathbf{R}(\mu + \nu)\mathbf{R}(-\nu)\mathbf{J}(\Delta)\mathbf{R}(\nu), \quad (1)$$

where A and B are complex matrix elements, μ and ν are real angles, $\mathbf{R}(\cdot)$ is the rotation matrix

$$\mathbf{R}(\nu) = \begin{bmatrix} \cos \nu & -\sin \nu \\ \sin \nu & \cos \nu \end{bmatrix}, \quad (2)$$

and $\mathbf{J}(\Delta)$ is a diagonal matrix that represents the Jones matrix of a retarder under principal axes, and Δ is its phase retardation:

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$$\mathbf{J}(\Delta) = \begin{bmatrix} e^{i\Delta/2} & 0 \\ 0 & e^{-i\Delta/2} \end{bmatrix}. \tag{3}$$

Therefore the combined system is commonly equivalent to a cascaded system of a retarder followed by a rotator. The equivalent phase retardation of the resulting retarder will be

$$\tan^2 \frac{\Delta}{2} = \frac{\mathrm{Im}^2(A) + \mathrm{Im}^2(B)}{\mathrm{Re}^2(A) + \mathrm{Re}^2(B)}, \tag{4}$$

where $\mathrm{Re}(\cdot)$, $\mathrm{Im}(\cdot)$ mean the real part and imaginary part of a complex number, respectively. It is evident that this expression holds under the condition that the combined system is composed of any number of wave plates with arbitrary individual azimuthal angles. However, paper [1] made a wrong claim upon this point that Eq. (4) can only be used for the cases $\phi = 0$ and $\phi = \pi/2$. Actually, papers [4,5] have made the correct use of Eq. (4). In addition, the azimuthal angle of the resulting retarder is $-\nu$, and the rotation angle of the resulting rotator is $\mu + \nu$, which can be determined by

$$\tan(\mu + \nu) = \frac{\mathrm{Re}(B)}{\mathrm{Re}(A)}, \tag{5}$$

$$\tan(\mu - \nu) = \frac{\mathrm{Im}(B)}{\mathrm{Im}(A)}. \tag{6}$$

Generally, the rotation angle $\mu + \nu$ is not null, so the combined system does not behave as a single pure retarder. As a consequence, the overall matrix cannot be simply diagonalized to calculate the total phase retardation. This is the second mistake made in paper [1]. Although the overall matrix of the combined system is unitarily diagonalizable, such unitary similarity transformation does not possess actual physical significance in the field of polarization optics, since the basis of eigenvectors are complex orthonormal.

Now let us consider under what conditions the combined system will be equivalent to just a pure retarder. It is evident that if the rotation angle $\mu + \nu = 0$, then there exists no resulting rotator. From Eq. (5), it is easy to see that $\mathrm{Re}(B)$ must be equal to zero, which implies that the matrix element B should be a pure imaginary number. Another additional condition that $\mathrm{Re}(A) > 0$ will ensure $\mu + \nu \neq \pm\pi$, but this is not a necessary condition since when the rotation angles are $\pm\pi$, the resulting rotator actually does not play a role in the system. Under these conditions, the overall Jones matrix $\mathbf{U}$ of the combined system can be diagonalized by an orthogonal matrix $\mathbf{Q}$ with real entries:

$$\mathbf{J}(\Delta) = \mathbf{Q}^{-1}\mathbf{U}\mathbf{Q} = \begin{bmatrix} \mathrm{Re}(A) + i\sqrt{\mathrm{Im}^2(A) + \mathrm{Im}^2(B)} & 0 \\ 0 & \mathrm{Re}(A) - i\sqrt{\mathrm{Im}^2(A) + \mathrm{Im}^2(B)} \end{bmatrix}. \tag{7}$$

From Eq. (7) and by recalling Eq. (3), we immediately get

$$\tan \frac{\Delta}{2} = \frac{\sqrt{\mathrm{Im}^2(A) + \mathrm{Im}^2(B)}}{\mathrm{Re}(A)}. \tag{8}$$

Only in this case, the calculated retardation in Eq. (8) from the diagnalized version of overall matrix $\mathbf{U}$ is equivalent to Eq. (4). From Eq. (1), it is easy to find that $\mathbf{Q} = \mathbf{R}(-\nu)$. In fact, for example, a particular combination of three wave plates can behave as a pure retarder [6], which meets the requirements of both $\mathrm{Re}(B) = 0$ and $\mathrm{Re}(A) > 0$.

At this point, it also should be pointed out that Eq. (7) in paper [4] is incorrect. It only holds for $\phi = 0$ and $\phi = \pi/2$ in [4]. The correct azimuthal angle of the combined system should come from Eqs. (5) and (6) in this paper as well.

Next, let us give a numerical example to verify the above theoretical discussion. Assuming the combined system contains only two wave plates. The first one is followed by the second. The phase retardations are δ_1 and δ_2, and the azimuthal angles are 0 and ϕ, respectively. If $\delta_1 = \pi/2$, $\delta_2 = 3\pi/4$, $\phi = \pi/3$, then the overall Jones matrix of this combined system is

$$\mathbf{U} = \begin{bmatrix} 0.5972 - 0.0560i & 0.5658 + 0.5658i \\ -0.5658 + 0.5658i & 0.5972 + 0.0560i \end{bmatrix}. \tag{9}$$

From Eqs. (4) to (6) we will obtain $\Delta = 1.2094$, $\mu = 0.4556$, and $\nu = -1.2139$ (in radians). Thus the combined system is similar to a pure retarder with phase retardation of $\Delta = 1.2094$ (in radians) and azimuthal angle of $\psi = 1.2139$(in radians), followed by a rotator with rotation angle of $\theta = -0.7583$ (in radians). It is easy to verify that such a retarder-rotator combination has the same Jones matrix as that shown in Eq. (9). Therefore the effectiveness of Eqs. (4) to (6) has been confirmed.

However, since ϕ is unequal to either 0 or $\pi/2$ in this example, the matrix element $B = -0.5658 + 0.5658i$ in Eq. (9) is not a pure imaginary number. If we use the Eq. (11) in Ref. [1], we will get a wrong result $\Delta = 1.8615$ (in radians), and the unitary transformation matrix will be

$$\begin{bmatrix} 0.4822 - 0.4822i & 0.7314 \\ 0.7314 & -0.4822 - 0.4822i \end{bmatrix}, \tag{10}$$

whose columns are the eigenvectors corresponding to Eq. (10) in Ref. [1]. It is impossible to extract the azimuthal angle of the resulting retarder from Eq. (10). In addition, if using Eqs. (6) and (7) in Ref. [4], we will get $\Delta = 1.2094$, $\psi = -0.7360$(in radians), and then the corresponding Jones matrix for such a retarder is

$$\begin{bmatrix} 0.8227 + 0.0560i & -0.5658i \\ -0.5658i & 0.8227 - 0.0560i \end{bmatrix}. \tag{11}$$

This is obviously different from Eq. (9). So such a retarder is not an equivalence to the original combined system.

Finally, it should be noted that the resulting rotator generally cannot be simply ignored if the combined wave-plate system is followed by other polarization-dependent optical systems, since the rotator will rotate the polarization ellipse and, as a result, change the polarization state of the optical beam in most cases.

In summary, some mistakes in a recent paper [1] are pointed out and discussed in detail. The authors of paper [1] considered the two-wave-plate system to be a single pure retarder and did not take the effect of the rotator into account. As a result, they incorrectly derived the expression of the equivalent phase retardation by unitarily diagonalizing the overall Jones matrix of the combined system. Also, they could not provide the expression of the azimuthal angle of the combined system due to their mistakes.

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