Optimal achromatic wave retarders using two birefringent wave plates

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Two plates of different birefringence material can be combined to obtain an achromatic wave retarder. In this work, we achieve a correction for the overall retardation of the system that extends the relation to any azimuth. Current techniques for the design of achromatic wave retarders do not present a parameter that characterizes its achromatism on a range of wavelengths. Thus, an achromatic degree has been introduced, in order to determine the optimal achromatic design composed with retarder plates for a spectrum of incident light. In particular, we have optimized a quarter retarder using two wave plates for the visible spectrum. Our technique has been compared to previous results, showing significant improvement. © 2013 Optical Society of America

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1. Introduction

Circularly polarized light can be easily obtained with a linear polarizer and a quarter wave plate [1]. However, there exist applications where a high degree of circularity and stability is required, such as polarimetry, ellipsometry, optical activity, light polarization spectroscopy, optical communications cryptography, and photoelasticity [1,2]. In many applications, light is polychromatic. The use of achromatic circular polarizers on a region of the light spectrum has been studied for many years. One of the pioneers, Pancharatnam, proposed a configuration of three retarder plates, which produce an achromatic overall delay [3]. Hariharan achieved achromatic half-wave systems using retarders of quartz and mica [4]. Also, Gil and Bernabeu showed achromatic systems composed of two plates centered in a specific wavelength [5]. Hariharan and Malacara analyzed achromatic systems formed by two quarter-wave plates and one half-wave plate [6]. Boulbry *et al.* studied the errors associated with zero-order quarter wave plates [7]. Recently, Saha *et al.* analyzed different methods to optimize achromatic systems composed of two retarder plates in several configurations $\lambda/2 + \lambda/4$ and $\lambda/4 + \lambda/4$ [8].

In this paper, we consider a system composed of two wave plates, and a novel expression for the overall retardation is calculated. The obtained expression is a generalization of the currently used equation to any azimuth [8–11]. In former works, achromatism is achieved only for two wavelengths and does not have a parameter that characterizes the performance of these achromatic designs. We present a new strategy for achromatism, optimizing a merit function that considers all the spectrum. By minimizing this merit function, which measures the distance between the retardation of the system in terms of the wavelength to target overall retardation, we can determine the thickness of the wave plates. We see that different materials for the two wave plates are required in order to obtain good achromatic systems. A numerical analysis is developed for different wavelength ranges. Our results are compared to those obtained by other authors, showing a significant improvement. In addition, this technique is extrapolable to other systems with more than two wave plates.

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2. Theoretical Frame

Let us consider a system formed by two wave plates, represented by their respective Jones matrices, L_1 and L_2, and characterized by the retardations δ_1 and δ_2, and the relative inclination of their fast axis with the horizontal, ϕ_1 and ϕ_2, respectively. This system, shown in Fig. 1, is illuminated with a polychromatic light beam with spectrum $g(\lambda)$. The notation of subindices used for all articles will be $i = 1, 2$, for the first and second plates. In this section we obtain the overall retardation of the system. Using Jones formalism, an ideal retarder plate, L_i, is defined by the following 2×2 matrix:

$$L_i(\delta, \phi) = \begin{pmatrix} \cos(\delta/2) + i \sin(\delta/2) \cos 2\phi & i \sin(\delta/2) \sin 2\phi \\ i \sin(\delta/2) \sin 2\phi & \cos(\delta/2) - i \sin(\delta/2) \cos 2\phi \end{pmatrix}, \tag{1}$$

where the retardation, $\delta = 2\pi \Delta n(\lambda) d_i/\lambda$, depends on its thickness, the birefringence of the material $\Delta n(\lambda) = n_e(\lambda) - n_0(\lambda)$, and the wavelength. Refractive indices can be expressed using Sellmeier's equations [12]. $L_i(\delta, \phi)$ can be described in the basis of Pauli matrices, Σ, represented by four linearly independent matrices 2×2, $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2}$$

The set Σ is complete and orthogonal, and Eq. (1) results in

$$L_i(\delta, \phi) = \cos \frac{\delta}{2} \cdot \sigma_0 + i \sin \frac{\delta}{2} \cos 2\phi \cdot \sigma_1 + i \sin \frac{\delta}{2} \sin 2\phi \cdot \sigma_2, \tag{3}$$

where the σ_3 term does not appear. In order to simplify the expression, lets us consider, without loss of generality, $\phi_1 = 0$ and $\phi_2 = \phi$, ϕ being the angle between the fast axes of the second plate and the first plate. Therefore the characteristic matrix of the system is the product

$$M = L_2 L_1 = \begin{pmatrix} \cos \frac{\delta}{2} + i \sin \frac{\delta}{2} \cos 2\phi & i \sin \frac{\delta}{2} \sin 2\phi \\ i \sin \frac{\delta}{2} \sin 2\phi & \cos \frac{\delta}{2} - i \sin \frac{\delta}{2} \cos 2\phi \end{pmatrix} \times \begin{pmatrix} e^{i\delta_1/2} & 0 \\ 0 & e^{-i\delta_1/2} \end{pmatrix} = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \tag{4}$$

where $A = \exp(i\delta_1/2)[\cos(\delta_2/2) + i \sin(\delta_2/2) \cos 2\phi]$ and $B = i \exp(-i\delta_1/2) \sin(\delta_2/2) \sin 2\phi$.

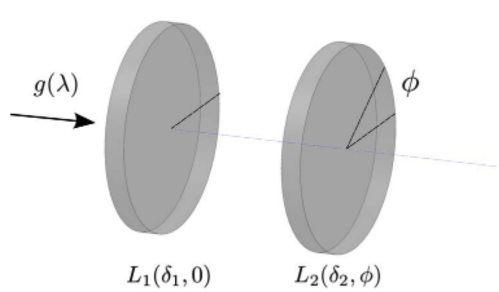

Fig. 1. (Color online) System to study. Two retarder plates L_1 and L_2 with retardations δ_1 and δ_2 and azimuths 0 and ϕ, respectively, are illuminated with a polychromatic light beam with spectrum $g(\lambda)$.

Let us analyze the conditions under which the system, M, consisting of two plates, behaves as a single wave plate of azimuth Ψ and gap Δ. First, we decompose Eq. (4) in Pauli matrices

$$M = l_0\sigma_0 + l_1\sigma_1 + l_2\sigma_2 + l_3\sigma_3, \tag{5}$$

where

$$l_0 = \cos \frac{\delta_1}{2} \cos \frac{\delta_2}{2} - \sin \frac{\delta_1}{2} \sin \frac{\delta_2}{2} \cos 2\phi,$$
$$l_1 = i\left(\cos \frac{\delta_1}{2} \sin \frac{\delta_2}{2} \cos 2\phi + \sin \frac{\delta_1}{2} \cos \frac{\delta_2}{2}\right),$$
$$l_2 = i \cos \frac{\delta_1}{2} \sin \frac{\delta_2}{2} \sin 2\phi,$$
$$l_3 = -i \sin \frac{\delta_1}{2} \sin \frac{\delta_2}{2} \sin 2\phi.$$

The equality $M(\delta_1, \delta_2, \phi) = L_i(\Delta, \Psi)$ implies that the component associated to σ_3 and l_3 must be zero, which only holds for $\phi = 0$ or $\phi = \pi/2$. Under this condition, equaling terms, we have that

$$\tan^2 \frac{\Delta}{2} = \frac{|\mathrm{Im}(A)|^2 + |\mathrm{Im}(B)|^2}{|\mathrm{Re}(A)|^2 + |\mathrm{Re}(B)|^2}, \qquad \Psi = 0. \tag{6}$$

These expressions have been incorrectly used under the condition $\phi \neq 0$ [8,11]. However, it is possible to calculate the overall delay, Δ, introduced for any orientation of the plates, $\phi \neq 0$, via diagonalization of M. Since matrix M is self-adjoint and unitary, the module of its eigenvalues, μ_i, is unit

$$\mu_1 = s_0 + \sqrt{s_0^2 - 1}, \qquad \mu_2 = s_0 - \sqrt{s_0^2 - 1}, \tag{7}$$

where

$$s_0 = \cos\frac{\delta_1}{2}\cos\frac{\delta_2}{2} - \sin\frac{\delta_1}{2}\sin\frac{\delta_2}{2}\cos 2\phi. \quad (8)$$

Consequently, the trace of the matrix product results in

$$\mathrm{Tr}(M) = 2s_0. \quad (9)$$

Finally, the overall retardation, Δ, is obtained by equaling a diagonalized matrix with a diagonal matrix with eigenvalues $\mu_1 = e^{i\Delta/2}$ and $\mu_2 = e^{-i\Delta/2}$; that is,

$$\begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} = \begin{pmatrix} e^{i\Delta/2} & 0 \\ 0 & e^{-i\Delta/2} \end{pmatrix}, \quad (10)$$

and, as a consequence,

$$\cos\frac{\Delta}{2} = \frac{\mathrm{Tr}\,M}{2} = s_0. \quad (11)$$

Equation (11) can be used for an arbitrary orientation ϕ between the plates. Nevertheless, when $\phi = 0$ or $\phi = \pi/2$, there is an equivalence with Eq. (6).

3. Achromatism

Achromatic systems are commonly formed by two wave plates with different birefringence, and they present two configurations: in the first configuration, the fast axis of one of the wave plates is parallel to the slow axis of the other wave plate. In the second configuration, the fast axis of both wave plates is parallelly aligned [4,7]. Also, there are some works that use other azimuths ϕ [8]. In these works, achromatism is achieved with a contour condition applied to only two wavelengths without considering the spectrum of incident light. In addition, they do not have a parameter that characterizes the performance of these achromatic designs. The condition about the spectrum of incident light can be critical in order to optimize the behavior of achromatic retarders. For that reason, in the present work a merit function, achromatism degree (AcD), is defined in order to optimize the behavior in all the wavelength range

$$\mathrm{AcD} = \frac{\left(\int |\Delta(\phi, d_1, d_2, \lambda) - \Delta_0|^2 g(\lambda)\mathrm{d}\lambda\right)^{1/2}}{\int g(\lambda)\mathrm{d}\lambda}, \quad (12)$$

where Δ_0 is the desired retardation and $g(\lambda)$ represents the spectrum of incident light. This merit function is a measure of the distance between the retardation of the system in terms of wavelength and a target overall retardation Δ_0. This distance is weighted by the spectrum, in order to consider only those wavelengths with significant intensity. Therefore, the optimization of AcD will be adapted to the spectrum. Moreover, the AcD parameter represents a criterion in order to determine the achromaticity of optic systems. When the retardation obtained for all wavelengths is equal to the desired retardation Δ_0, AcD is null. As this case is not fulfilled in the general case, when AcD presents a minimum, we will consider that we obtain the best achromatic result for the given spectrum.

In general, Eq. (12) cannot be solved analytically since, as shown in Eqs. (1) and (4), there is a complex relationship between Δ and the wavelength. Nevertheless, we can minimize AcD with respect to ϕ, $\partial(\mathrm{AcD})/\partial\phi = 0$. For this minimization, there are two possible solutions for the azimuth, $\phi = 0$ or $\phi = \pi/2$, which are in agreement with the common experimental setups. Only under this result are Eqs. (11) and (6) equivalent. For the particular cases $\phi = 0$ and $\phi = \pi/2$, Eq. (11) simplifies to

$$\Delta = \delta_1 + \delta_2 \Leftrightarrow \phi = 0, \qquad \Delta = \delta_1 - \delta_2 \Leftrightarrow \phi = \pi/2. \quad (13)$$

Since thickness cannot be negative, the minus sign in the second equation is related to an azimuth $\phi = \pi/2$. As a consequence, both solutions are equivalent.

4. Results

After minimizing AcD with respect to ϕ, we need to minimize with respect to the thickness of the wave plates (d_1, d_2). The standard technique to obtain achromatic devices is making $|\Delta(\phi, d_1, d_2, \lambda) - \Delta_0| = 0$ of Eq. (12) for just two wavelengths (λ_1, λ_2), which are normally placed at the edges of the spectrum [4]. Then a system of two equations with two unknowns is obtained. For example, from Eq. (13), for a quarter-wave retarder, this results in

$$\begin{aligned} \Delta n_1(\lambda_1)d_1 \pm \Delta n_2(\lambda_1)d_2 &= \lambda_1/4, \\ \Delta n_1(\lambda_2)d_1 \pm \Delta n_2(\lambda_2)d_2 &= \lambda_2/4. \end{aligned} \quad (14)$$

Note that this system can be solved only when the birefringence of both plates is different. Therefore, the retarder plates must be manufactured with different materials.

A better solution can be obtained when the whole spectrum $g(\lambda)$ is considered for the minimization. Then, by numerical computations we can determine the thickness. As an example, let us consider an achromatic quarter-wave retarder, $\Delta_0 = \pi/2$, using two wave plates made of quartz and MgF_2. These materials have been considered due to the abundant literature about them and are commonly used as retarder systems [7,13–18]. The ordinary, n_o, and extraordinary, n_e, refractive indices are defined by the Sellmeier's relations, which for MgF_2 are

$$\begin{aligned} n_o^2 - 1 &= \frac{0.48755108\lambda^2}{\lambda^2 - 0.04338408^2} + \frac{0.39875031\lambda^2}{\lambda^2 - 0.09461442^2} + \frac{2.3120353\lambda^2}{\lambda^2 - 23.793604^2}, \\ n_e^2 - 1 &= \frac{0.41344023\lambda^2}{\lambda^2 - 0.03684262^2} + \frac{0.50497499\lambda^2}{\lambda^2 - 0.09076162^2} + \frac{2.4904862\lambda^2}{\lambda^2 - 23.771995^2}, \end{aligned} \quad (15)$$

and for quartz,

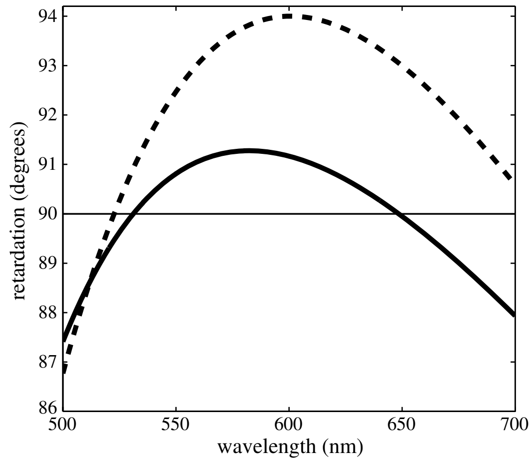

Fig. 2. Optimal overall retardation of the system for a Gaussian and plane spectrum (continuous curve) centered at $\lambda_0 = 600$ nm and a width $\delta\lambda = 100$ nm, with wavelength [500,700] nm. Both curves are coincident. A previous result of overall retardation is shown with a dashed curve [8].

$$n_o^2 - 1 = \frac{0.663044\lambda^2}{\lambda^2 - (0.0600)^2} + \frac{0.517852\lambda^2}{\lambda^2 - 0.1060^2} + \frac{0.175912\lambda^2}{\lambda^2 - 0.1190^2} + \frac{0.565380\lambda^2}{\lambda^2 - 8.844^2} + \frac{1.675299\lambda^2}{\lambda^2 - 20.742^2},$$
$$n_e^2 - 1 = \frac{0.665721\lambda^2}{\lambda^2 - 0.0600^2} + \frac{0.503511\lambda^2}{\lambda^2 - 0.1060^2} + \frac{0.214792\lambda^2}{\lambda^2 - 0.1190^2} + \frac{0.539173\lambda^2}{\lambda^2 - 8.792^2} + \frac{1.8076613\lambda^2}{\lambda^2 - 19.70^2}. \quad (16)$$

With the definition of refractive indices given by the Sellmeier's relations, Eqs. (15) and (16), the AcD can be optimized in configurations of two retarder plates. For that reason, we applied Eq. (13) in (12), finding the thickness d_1 and d_2 such that Eq. (12) is minimal for a spectrum $g(\lambda)$. Since the integral Eq. (12) cannot be solved in the general case, a numerical calculus has been performed. This computation consists of determining AcD for each pair (d_1, d_2) for an interval from 0 to 1000 μm in steps of 1 μm. The best pair will be that whose parameter AcD is minimum. Sampling in steps of 1 μm gives variation in the overall retardation less than or equal to 0.6°, which in consequence is a good approximation. As an example, the optimization will be performed in the wavelength interval [500,700] nm to achieve a quarter-wave plate.

We have considered two different light spectra: a Gaussian spectrum $g(\lambda) = \exp[-(\lambda - \lambda_0)^2/(2\delta\lambda)]$ centered at $\lambda_0 = 600$ nm and a width $\delta\lambda = 100$ nm, and a plane spectrum, also centered at $\lambda_0 = 600$ nm and a width $\delta\lambda = 100$ nm. With the minimization of Eq. (12), we obtain the following thicknesses: $d_1 = 371$ μm and $d_2 = 299$ μm, for the both spectra. In Fig. 2 we can see the retardation produced by the system for the different wavelengths. The maximum difference, ϵ, between the target retardation and the obtained, defined as

$$\epsilon = \sup|\Delta - \Delta_0|, \quad (17)$$

results in $\epsilon = 2.6°$. The technique used by Saha to obtain an achromatic system produced a maximum difference $\epsilon = 4°$ for the same range of wavelengths [8].

Finally, let us see which is the maximum difference, ϵ, for different spectrum bandwidths, using the spectra quoted, plane and Gaussian. We have considered the previous example for $\delta\lambda = 25$, 50, 75, and 100 nm. For each bandwidth, an optimization of Eq. (12) has been performed. In Fig. 3(a) we show the retardation curves $\Delta(\lambda)$ for these intervals. As the bandwidth intervals are smaller, the retardation curve is more plane. The maximum difference for the Gaussian or plane spectrum with the target retardation for the Gaussian or plane spectrum results in $\epsilon(50\ \text{nm}) = 0.5°$, $\epsilon(75\ \text{nm}) = 1.2°$, and $\epsilon(100\ \text{nm}) = 2.6°$. For small intervals, the differences between

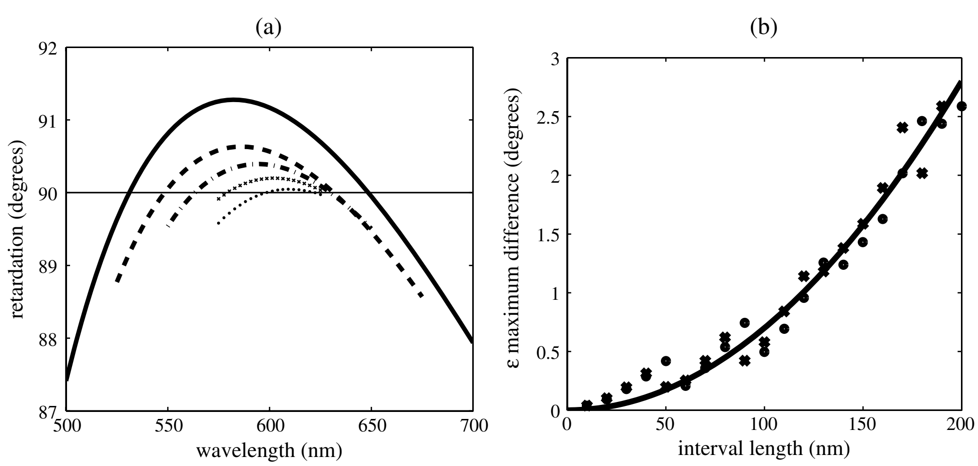

Fig. 3. (a) Overall retardation for plane or Gaussian spectra using several bandwidths: [500,700] nm (continuous), [525,675] nm (dashed), [550,650] nm (dash–dotted), and [575,625] nm (crosses for plane spectrum and dots for Gaussian spectrum). (b) Quadratic fit of maximum difference committed by the retarder as a function of the length interval for a Gaussian spectrum (dot points: continuous curve) and for a plane spectrum (cross points: dashed curve). Maximum difference, ϵ, in terms of the bandwidth $\Delta\lambda$ for a plane spectrum (crosses) and a Gaussian spectrum (dots), and quadratic fit (continuous curve).