Unified Theory of Polarization and Coherence in Composite Wave-Plate Systems

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Abstract

Optical systems composed of multiple waveplates are widely used in the design of zero-order and achromatic retarders, as well as in photonic quantum key distribution systems and quantum computing gates. However, despite the well-characterized transformations of fully polarized and coherent light through these systems, there is no general method for determining the transformations on partially polarized and incoherent light, considering depolarization effects. Thus, a theoretical methodology is proposed here to determine the transformations of partially polarized light and depolarization effects of a system of composite waveplates (CW) using coherence theory. Experimental support and a Python module that automates the process are presented.

Keywords: Coherence, Polarization, Waveplate, Depolarization

1 Introduction

Currently, to address the problem of describing partially polarized states associated with incoherent light, and its propagation in anisotropic media, the use of the Stokes-Mueller formalism along with coherence theory is the most utilized [1, 14]. In particular, this formalism has traditionally been used to describe depolarization due to birefringent media [8, 11, 12, 18, 29]. However, coherence theory can also be used in conjunction with other formalisms such as Jones calculus, polarization matrix, quaternions, or Pauli vectors, in the treatment of partially polarized light and its transformations [5, 17, 20, 24, 25, 31, 32]. The importance of representation lies in its convenience in the formulation and solution of some problems compared to Stokes-Mueller formalism. However, in the literature reviewed, despite having a theoretical model for the description and transformation of partially polarized states, these do not include depolarization comprehensively.

Light depolarization is a phenomenon that depends on the coherence properties of light and also on the medium structure through which the light propagates or reflects. If the medium is turbulent, porous, colloidal, or in general, posses random structure, depolarization occurs mainly because of the scattering of light [**?**]. On the other hand, depolarization also occurs when partially coherent light propagates trough birrefringent media. In general, complex birrefringent media as biological tissues, minerals, or thermoformed plastics can be considered as to be composed of layers that can be modeled as Composite Waveplate (CW) systems [?]. These models are very useful for studying complex systems, for example in ophthalmic [3, 4, 13], and communication channels [26, 30], where the properties of these media can be studied from the transformations that the polarization state undergoes and especially from the statistics presented in the depolarization of light when interacting with the media. Therefore, a methodology is needed to determine the polarization transformations for partially polarized and incoherent light for any system composed of multiple birefringent media or CW systems. Some works use a formalism based on coherence theory to describe the transformations of partially polarized light and depolarization in combination with the coherence matrices and the jones formalism [19, 22], as well as using a formalism of probability distributions on the Poincaré sphere and the probability density function for polarization[9, 16, 28]. However, these methodologies do not allow transitioning from one formalism to another within the framework of temporal coherence theory, nor has it been adapted to the general solution of the transformations of partially polarized light through CW systems that account for depolarization.

Thus, this work proposes a treatment of coherence and polarization theory, which can be applied in any of the possible algebraic representations of polarization, just as stokes vectors, Pauli vectors, Jones vectors, etc. to describe polarization transformations and depolarization in birrefringent media for partially polarized light. Polarization matrices and Jones matrices will primarily be used to model the transformations of the polarization state of incoherent light passing through CW. With the proposed methodology, the description of depolarization through a biplate, an optical system composed of two waveplates, will be addressed. Based on the description of CW, particularly a biplate for the coherent light regime [23], the solution for a single retarder and two retarders in the case of incoherent sources where depolarization is present is proposed.

2 Temporal coherence and partial polarization

introduccion

Every natural light source so as laser beams posses random fluctuations of its associated electric field $\mathbf{E}(t)$. The polarization state is then a random function of time, also called partial polarization, such that for each time t there is an instantaneous polarization state described by the Jones vector representation or another different algebraic formalism. However, since classically the ellipse of polarization is the usual geometrical interpretation of this property of light we are tempted to assume that the polarization state can only be completely defined when the electric field has evolved an enough quantity of time T such that $\mathbf{E}(t)$ describes a regular geometric locus. Moreover, in the quantum formalism each individual photon has defined polarization state and the partial polarization consists of an statistical ensemble of large numbers of photons. Thus, the time-evolution of a polarization state of a random electric field have to be considered as discrete in time, and the idealization of an instantaneous polarization state may seem nonphysical. Whether the correct physical interpretation of light polarization state may be discrete or continuous in time is a topic to be discussed in a deep research concerning the quantum nature of light.

2.1 Instantaneous States of Polarization

The instantaneous polarization state of an electric wave is defined as a normalized Jones vector

$$\hat{\mathbf{e}}(t) = \begin{pmatrix} \varepsilon_x(t)\mathrm{e}^{i\phi_x(t)} \\ \varepsilon_y(t)\mathrm{e}^{i\phi_y(t)} \end{pmatrix}, \tag{1}$$

which can also be expressed in terms of the instantaneous ellipticity angles $\chi(t)$ and orientation $\alpha(t)$ associated with the polarization state at time t. Written on a basis of Horizontal and vertical states, the polarization state will be [2]:

$$\begin{aligned} \begin{pmatrix} E_x(t) \\ E_y(t) \end{pmatrix} &\equiv |\alpha, \chi\rangle \\ &= \begin{pmatrix} \cos\alpha(t)\cos\chi(t) - i\sin\alpha(t)\sin\chi(t) \\ \sin\alpha(t)\cos\chi(t) + i\cos\alpha(t)\sin\chi(t) \end{pmatrix}. \end{aligned} \tag{2}$$

moreover, a change of basis can be performed to write $|\alpha, \chi\rangle$ in a base of any pair of orthogonal elliptically polarized states. Then, we write

$$\begin{pmatrix} E_1(t) \\ E_2(t) \end{pmatrix} = \hat{T}^{-1} \begin{pmatrix} E_x(t) \\ E_y(t) \end{pmatrix}, \tag{3}$$

where the transformation matrix $\hat{T}$ is given by:

$$\hat{T} = \begin{bmatrix} \cos\alpha\cos\chi - i\sin\alpha\sin\chi & \cos\alpha\sin\chi + i\sin\alpha\cos\chi \\ \sin\alpha\cos\chi + i\cos\alpha\sin\chi & \sin\alpha\sin\chi - i\cos\alpha\cos\chi \end{bmatrix} \tag{4}$$

This is the starting point from which any of the algebraic representations of polarization can be accessed, such as Pauli vectors and Stokes parameters, in any polarization basis.

Definition 1 (Instantaneous Polarization Matrix)**.** *The instantaneous polarization matrix $J(t)$ is an equivalent representation of the dynamic polarization state $\hat{\mathbf{e}}(t)$ given in terms of intensities esto no es una intensidad!. It is constructed from the Jones vector as follows:*

$$\begin{aligned} \mathbf{J}(t) &= \begin{pmatrix} E_1(t) \\ E_2(t) \end{pmatrix} \begin{pmatrix} \overline{E_1(t)} & \overline{E_2(t)} \end{pmatrix} \\ &= \begin{pmatrix} E_1(t)\overline{E_1(t)} & E_1(t)\overline{E_2(t)} \\ E_2(t)\overline{E_1(t)} & E_2(t)\overline{E_2(t)} \end{pmatrix}, \end{aligned} \tag{5}$$

where $\overline{E_i}$ indicates the complex conjugated of E_i.

A convenient basis for $\mathbf{J}(t)$ are the Pauli matrices $\hat{\sigma}_i$, given by

$$\hat{\sigma}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\sigma}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{6}$$

Thus, we have

$$\mathbf{J}(t) = \sum_{i=0}^{3} \frac{1}{2} s_i(t)\hat{\sigma}_i, \tag{7}$$

where the expansion coefficients $s_i(t)$ are defined as instantaneous Stokes parameters. The Stokes parameters $s_i(t)$ can also be written in terms of the angles (α, χ) depending on the chosen polarization basis. As a result, the Stokes parameters $s_i(\alpha, \chi; t)$ are expressed as the parametric equations of a sphere; thus, the polarization state has a representation given by a four-vector $\mathbf{S}(t) = [s_0, \mathbf{s}(t)]^T$, whose geometric locus lies on the surface of a sphere of radius s_0; the Poincaré sphere. The dynamic polarization states can then be represented as a vector $\mathbf{s}(t)$ that evolves over time on the Poincaré sphere.

2.2 Effect of a waveplate

Definition 2 (Waveplate's Eigenstates)**.** *The waveplate's eigenstates are the eigenstates associated with the waveplate operator $\hat{R}_\delta$ which, in general, are a pair of orthogonal elliptically polarized states. quizas sea importante mencionar que se desprecia la absorcion.*

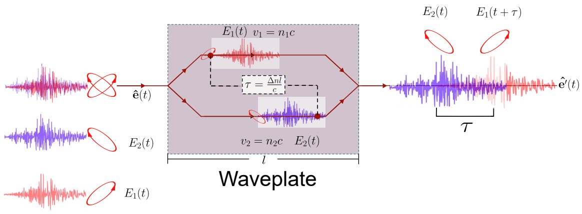

Fig. 1: Phenomenological scheme associated with the effect of a waveplate with elliptical eigenstates on a dynamical state of polarization.

Consider the effect of the operator $\hat{R}_\delta$ on a polarization matrix $\mathbf{J}(t)$ associated with a dynamic polarization state. In an arbitrary basis of elliptically polarized states, the Jones matrix associated with a retarder, with elliptical polarized eigenstates that coincide with this basis, has the following form:

$$\hat{R}_\delta = \begin{bmatrix} e^{i\delta} & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}$$

On the other hand, the transformation of the polarization state $\mathbf{J}$, due to the retarder $\hat{R}_\delta$, is given by [20]

$$\begin{aligned} \mathbf{J}'(t) &= \hat{R}_\delta \mathbf{J}(t) \hat{R}_\delta^\dagger \\ &= \begin{pmatrix} E_1(t)\overline{E_1(t)} & E_1(t)\overline{E_2(t)}e^{i\delta} \\ E_2(t)\overline{E_1(t)}e^{-i\delta} & E_2(t)\overline{E_2(t)} \end{pmatrix}. \end{aligned} \tag{9}$$

Now let's consider the result of the product of the factor $e^{i\delta}$ with the electric field $E_i(t)$. Writing the product in terms of the Fourier transform of $E_i(t)$, we find:

$$\mathrm{e}^{i\delta} E_i(t) = \mathrm{e}^{i\delta} \int_{\mathbb{R}} E_i(\nu) \mathrm{e}^{i2\pi\nu t} d\nu$$

$$= e^{i\frac{2\pi\tilde{\nu}}{c}\Delta nl}\int_{\mathbb{R}} E_i(\nu)e^{i2\pi\nu t}d\nu$$
$$= \int_{\mathbb{R}} E_i(\nu)e^{i2\pi\nu\left(t+\frac{\Delta nl}{c}\right)}d\nu, \quad (10)$$

where the factor $e^{i\frac{2\pi\nu}{c}\Delta l}$ has been inserted inside the integral under the condition that the polychromatic wave has a narrow spectrum [15]. Now, the integral 10 is an inverse Fourier transform that is temporally shifted, that is,

$$\int_{\mathbb{R}} E_i(\nu)e^{i2\pi\nu\left(t+\frac{\Delta nl}{c}\right)}d\nu = E_i\left(t+\frac{\Delta nl}{c}\right), \quad (11)$$

and consequently, the factor $e^{i2\pi\nu\tau}$ acts as a shift operator:

$$e^{i2\pi\nu\tau}E_i(t) = E_i\left(t+\tau\right), \quad (12)$$

where $\tau = \frac{\Delta nl}{c}$, with $\Delta n = n_1 - n_2$ being the material's birefringence. This is expected according to the phenomenology of the process, as observed in Fig.1, one of the components of the electric field is delayed with respect to the other as the wave propagates inside the material. creo conveniente explicar mejor la figura

Applying the result of Eq. 12 to Eq. 10, we obtain the following expression

$$\begin{aligned} \mathbf{J}'(t) &= \hat{R}_\delta \mathbf{J}(t)\hat{R}_\delta^\dagger \quad (13)\\ &= \begin{pmatrix} E_1(t)\overline{E_1(t)} & E_1(t)\overline{E_2(t+\tau)} \\ E_2(t)\overline{E_1(t-\tau)} & E_2(t)\overline{E_2(t)} \end{pmatrix}. \end{aligned}$$

This process can then be generalized for systems composed of multiple waveplates through the following operation

$$\mathbf{J}'(t) = \hat{R}_{\delta_N}...\hat{R}_{\delta_2}\hat{R}_{\delta_1}\mathbf{J}(t)\hat{R}_{\delta_1}^\dagger\hat{R}_{\delta_2}^\dagger...\hat{R}_{\delta_N}^\dagger. \quad (14)$$

Up to this point, this formalism solves the problem of transformations of dynamic polarization states through systems of composite waveplates (CW). However, this is only valid when the polarization dynamics are deterministic, that is, when the temporal functions $E_1(t)$ and $E_2(t)$ are known. For cases where these functions evolve randomly over time, it becomes necessary to consider the effects of coherence.

2.3 Coherence Matrix and Depolarization

The expected value, at a given time t, of an ensemble Ω of identical light sources whose polarization dynamics are given by the matrix $\mathbf{J}(t)$ is expressed as follows

$$\mathrm{E}\{\mathbf{J}\}(t) = \int_\Omega \mathcal{P}(\mathbf{J};t)\mathbf{J}(\alpha,\chi;t)d\alpha d\chi \quad (15)$$

where $\mathcal{P}(\mathbf{J};t)$ is the probability density function associated with the polarization state $\mathbf{J}(t)$ from the ensemble Ω. Generally, $\mathcal{P}(\mathbf{J};t)$ is a function explicitly dependent on time, and consequently so are its statistical moments. However, for light sources in thermodynamic equilibrium, the statistical formulation of polarization can be made assuming a hypothesis of stationarity in a broad sense, where at least the first two moments of $\mathcal{P}(\mathbf{J};t)$ are independent of time.

Definition 3 (Coherence Matrix)**.** *aqui creo necesario citar a los pioneros The coherence matrix $\mathbf{\Gamma}(t_1,t_2)$ is the average polarization state, or the first statistical moment, determined by the expected value of the Polarization Matrix $\mathbf{J}$ (the instantaneous polarization state):*

$$\begin{aligned} \mathrm{E}\{\mathbf{J}\} &= \mathbf{\Gamma}(t_1,t_2) \quad (16)\\ &= \begin{pmatrix} \mathrm{E}\{E_1(t_1)\overline{E_1(t_1)}\} & \mathrm{E}\{E_1(t_1)\overline{E_2(t_2)}\} \\ \mathrm{E}\{E_2(t_2)\overline{E_1(t_1)}\} & \mathrm{E}\{E_2(t_2)\overline{E_2(t_2)}\} \end{pmatrix}, \end{aligned}$$

where the expected values $\mathrm{E}\{E_i(t_i)\overline{E_j(t_j)}\}$ of each element from matrix $\mathbf{J}(t_1,t_2)$ are the correlation functions between the random signals $E_i(\alpha,\chi;t_i)$ and $E_j(\alpha,\chi;t_j)$, that is, the first-order coherence function of the electric field. These correlation functions, by virtue of statistical stationarity, depend only on the difference $t_i - t_j$.

The elements $\Gamma_{ij}(t_i - t_j)$ of coherence matrix can be written in terms of the mean value $\mathrm{E}\{E_i(t_i)\} = \mathrm{E}\{E_i\}$ of each signal as follows

$$\begin{aligned} \Gamma_{ij}(t_i - t_j) &= \mathrm{E}\{E_i\}\mathrm{E}\{\overline{E_j}\}\,\gamma(t_i - t_j) \quad (17)\\ &= \Gamma_{ij}(0)\gamma(t_i - t_j) \quad (18) \end{aligned}$$

where $\gamma(t_i - t_j)$ is the complex degree of coherence, with $0 < |\gamma(t_i - t_j)| \leq 1$. On the other hand, $\Gamma_{ij}(0) = \mathrm{E}\{E_i(t_i)\}\mathrm{E}\{\overline{E_j(t_j)}\}$ is the normalization factor of the coherence function. This factor consists of the product of the average values of each signal, thus, it is equivalent to the elements of a coherence matrix associated with a fully polarized state; that is, the case where there is no decorrelation due to phase differences $t_i - t_j$ between the components of the electric field. This only occurs when the difference $t_i - t_j = 0$; hence the notation in which $\Gamma_{ij}(0)$ is evaluated at zero.

Thus, the first statistical moment of the polarization state is the first-order coherence matrix $\mathbf{\Gamma}$, which is equivalent to the well-known Wiener-Wolf polarization matrix. From it, the expected values of the associated Stokes parameters can be extracted [33, 34]:

$$\mathrm{E}\{s_i(t)\} = \mathrm{Tr}\{\hat{\sigma}_i \mathbf{\Gamma}\}, \tag{19}$$

and we can determine the degree of polarization P

$$P = \sqrt{1 - \frac{4\det \mathbf{J}}{\mathrm{Tr}\{\mathbf{J}\}^2}}. \tag{20}$$

Then, by using this theoretical methodology, we can determine the first order observables of the polarization state, such as the average Stokes parameters $\mathrm{E}\{s_i(t)\}$ and the degree of polarization P and its transformations, associated with a partially polarized incoherent light source.

3 Depolarization through composite waveplates

Following the proposed methodology, the transformation of a retarder $\hat{R}_\delta$, whose eigenstates are elliptically polarized states, on an incoherent light beam with an initial polarization state given by $\mathbf{J}(t_1, t_2)$ can be studied. The polarization state is transformed to $\mathbf{J}'(t_1, t_2)$ according to Eq. 14. The expected value of $\mathbf{J}'$ is then given by:

$$\begin{aligned}\mathrm{E}\{\mathbf{J}'\} &= \mathbf{\Gamma}' \\ &= \begin{pmatrix} \mathrm{E}\{E_1(t_1)\overline{E_1(t_1)}\} & \mathrm{E}\{E_1(t_1)\overline{E_2(t_2+\tau)}\} \\ \mathrm{E}\{E_2(t_2)\overline{E_1(t_1-\tau)}\} & \mathrm{E}\{E_2(t_2)\overline{E_2(t_2)}\} \end{pmatrix},\end{aligned} \tag{21}$$

whose elements are given by

$$\begin{aligned}\Gamma'_{ij} &= \mathrm{E}\{E_i(t_i)\overline{E_j(t_j + \varepsilon_{ij}\tau)}\} & (22)\\ &= \mathrm{E}\{E_i(t_i)\}\mathrm{E}\{E_j(t_j)\}\,\gamma(t_i - t_j + \varepsilon_{ij}\tau) & (23)\end{aligned}$$

definir ε_{ij} However, by virtue of the result given by Eq.12 and Eq.18, it can be written as follows:

$$\begin{aligned}\Gamma'_{ij} &= \mathrm{E}\{E_i(t_i)\}\mathrm{E}\{E_j(t_j)\}\,\gamma(t_i - t_j)e^{i2\pi\nu\varepsilon_{ij}\tau} & (24)\\ &= \Gamma_{ij}(t_i - t_j)e^{i2\pi\nu\varepsilon_{ij}\tau}\,. & (25)\end{aligned}$$

Therefore, the transformation due to a retarder on a partially polarized state is rewritten as

$$\begin{aligned}\Gamma'_{ij}(t_i - t_j + \varepsilon_{ij}\tau) &= \Gamma_{ij}(0)\gamma(t_i - t_j)e^{i2\pi\nu\varepsilon_{ij}\tau} & (26)\\ &= \Gamma_{ij}(0)\gamma(t_i - t_j + \varepsilon_{ij}\tau) & (27)\end{aligned}$$

From this result, the transformations on a partially polarized state by a system of CW, are formulated as follows

$$\mathbf{\Gamma}' = \prod_i \hat{R}_{\delta_i}\mathbf{\Gamma}(t_1, t_2)\prod_i \hat{R}^\dagger_{\delta_i} \tag{28}$$

where $\hat{R}_{\delta_i}$ are the operators associated with each waveplate that conform the CW, with a phase-shift δ_i, and different eigenstates. To find the form of the operator $\hat{R}_{\delta_i}$ given the mentioned characteristics, the representations of the polarization states in terms of the angles α and χ, and the change of basis mentioned in the first section of this article can be used.

3.1 Depolarization in a waveplate with elliptical eigenstates

Let's examine the case of the depolarization experienced by a light beam, initially elliptically polarized, when propagating through a waveplate whose eigenstates of polarization are generally elliptical. We take the result obtained in Eq. 27 with $t_i - t_j = 0$ since the initial coherence matrix $\mathbf{\Gamma}$ must correspond to that of a fully polarized state, and therefore, the degree of correlation between the orthogonal components of the electric field is $|\gamma_{12}| = 1$. Thus, the elements of the resulting coherence matrix $\mathbf{\Gamma}'$ are:

$$\begin{aligned}\Gamma'_{ij} &= \Gamma_{ij}(0)e^{i\varepsilon_{ij}\delta} = \Gamma_{ij}(\varepsilon_{ij}\delta) \\ &= \langle E_i\rangle\overline{\langle E_j\rangle}\,\gamma(\varepsilon_{ij}\tau),\end{aligned} \tag{29}$$

where $\tau = \Delta nl/c$ is the retardance introduced by the waveplate and $\langle E_i \rangle$ and $\langle E_j \rangle$ are the components of the Jones vector in the selected elliptical eigenstate basis.

Thus, from the transformation given by Eq.29, the Stokes parameters and the degree of polarization associated with the emerging polarization state can be determined and written as a function of the input polarization state and the characteristics of the waveplate used. Using Eq.20, the expression for the degree of polarization associated with $\mathbf{\Gamma}'$ is obtained:

$$P = \sqrt{1 - 4\langle E_1 \rangle^2 \langle E_2 \rangle^2 (1 - |\gamma_{12}(\tau)|)}, \qquad (30)$$

where the degree of polarization P depends on the squared amplitudes of each electric field component $\langle E_i \rangle^2$ and the modulus of the complex degree of coherence $|\gamma_{12}(\tau)|$, evaluated at the retardance τ introduced by the retarder. The dependence of P on the amplitudes $\langle E_i \rangle^2$ implies that the degree of polarization of the emerging beam depends on the incident polarization state. In the case when the polarization state of the incident beam coincides with the waveplate eigenstates, then either $\langle E_1 \rangle = 0$ or $\langle E_2 \rangle = 0$, so $P = 1$, and there is no depolarization. This fact is consistent with the very definition of eigenstates, as these are states that remain invariant and therefore do not undergo depolarization. However, despite an individual retarder having eigenstates that do not undergo the depolarization process, this is generally not the case for general CW systems.

3.2 Depolarization in a biplate

We apply the method to describe the depolarization of a incident partially coherent light beam with elliptical polarization through a biplate consisting of two waveplates with linear eigenstates. The angle ϕ between the principal axes of each waveplate is $\phi = \theta_1 - \theta_2$ as shown in Fig. 2. The biplate can rotate with an angle θ, so the two linear retarders rotate simultaneously while maintaining the angle ϕ between their principal axes. In this example, we want to study the variation of the degree of polarization $P(\alpha, \chi)$ and the Stokes parameters of the emerging beam as a function of the rotation angle θ and the incident polarization state $|\alpha, \chi\rangle$.

To perform the theoretical calculation of the transformation, and according to the experimental setup, we consider a particular case in which the two linear retarders that make up the biplate introduce the same phase-shift δ. The first retarder is oriented horizontally, and therefore has an associated operator $\hat{R}_\delta$. The second retarder is rotated with respect to the first by an angle ϕ, so its associated operator will be $\hat{R}^{-1}(\phi)\,\hat{R}_\delta\,\hat{R}(\phi)$, where the operator $\hat{R}(\phi)$ represents a rotation matrix of the coordinate system by an angle ϕ. Therefore, the operator representing the optical system, i.e., the biplate that has the freedom to rotate by an angle θ, is given by:

$$\hat{B} = \hat{R}^{-1}(\theta)\left[\hat{R}_\delta\left(\hat{R}^{-1}(\phi)\,\hat{R}_\delta\,\hat{R}(\phi)\right)\right]\hat{R}(\theta) \quad (31)$$

Subsequently, the coherence matrix $\mathbf{\Gamma}$ associated with the incident light beam elliptically polarized must be transformed using the operator $\hat{B}$ as follows:

$$\mathbf{\Gamma}' = \hat{B}\begin{pmatrix} \Gamma_{xx}(0) & \Gamma_{xy}(0) \\ \Gamma_{yx}(0) & \Gamma_{yy}(0) \end{pmatrix}\hat{B}^\dagger. \qquad (32)$$

The development of this matrix product and the shift of the coherence functions due to the factors $e^{i\delta}$ was carried out using a Python module of our own authorship, which allows performing these transformations for any type of CW [6]. According to the method proposed here, when performing the proper operation, phase terms $e^{i\delta}$ will be found multiplying each element $\Gamma_{ij}(0)$ of the initial matrix; the respective shift of the coherence function must be made in each case.

To perform calculations with this theoretical model and subsequently compare them with experimental data, it is necessary to know the complex degree of coherence $\gamma(\tau)$. Since the source used in the experiment is an LED source with a central wavelength $\lambda_0 = 0.633[\mu m]$, γ_τ this fits very well with Gaussian-Lorentzian behavior coherence function [21], given by:

$$\gamma(\tau) = \exp\left\{-i\omega_0\tau - \frac{\pi}{2}\left(\frac{\tau}{\tau_G}\right)^2 - \frac{|\tau|}{\tau_L}\right\}, \quad (33)$$

where $\omega_0 = 0.633[\mu m]c/2\pi$ is the central frequency of the LED spectrum, τ is a retardance introduced, in this case by the biplate, between the components of the electric field, $\tau_G = 9.4[\mu m]/c$ and $\tau_L = 25[\mu m]/c$ are the parameters associated with the LED coherence time. These parameters were obtained by directly measuring the absolute value of the complex degree of coherence through a depolarization process in a variable retarder, following a method proposed in one of our previous works [16]. The data treatment and the resources used to determine this coherence function and the results from this experiment are available in the following repository [7].

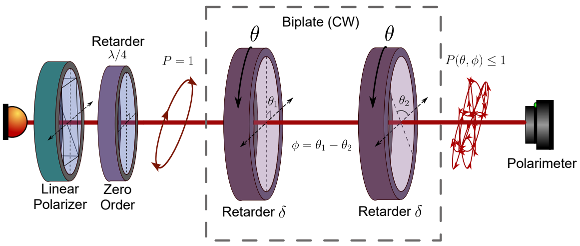

Fig. 2: Experimental setup for measuring the depolarization of an incoherent light beam incident on a biplate composed of two identical waveplates (with retardance $\tau = (n+1/4)\lambda_0$) with linear eigenstates. The biplate rotates at an angle θ.

The optical setup of the biplate was reproduced experimentally. It consists of a LED source *M625L4-C1 - 625 nm*, two $\lambda/4$ waveplates *WPMQ05M-633*, which form the biplate, and the polarimeter *PAX5710VIS-T* which measures the average Stokes parameters with a resolution of 0.05; all elements manufactured by *Thorlabs inc.*. Initially, an angle $\phi \approx -39°$ was chosen between the two linear waveplates. The light beam from the LED source was collimated and horizontally polarized and then incident on the biplate, then we measure the polarization state of the emerging beam for different values of θ, particularly for the range $\theta = [0°, 180°]$ with steps of $4° \pm 2°$. Subsequently, the emerging polarization state was measured for the same range of θ angles when the incident polarization state is right circular and elliptical.

The Fig.3 shows the graph on the Poincaré sphere of the emerging polarization state as a function of the rotation angle θ for the three types of incident polarization states. The experimental data represented as points and the red curves correspond to the theoretical model obtained from equation 32. For ease of visualization, the Stokes parameters plotted in each

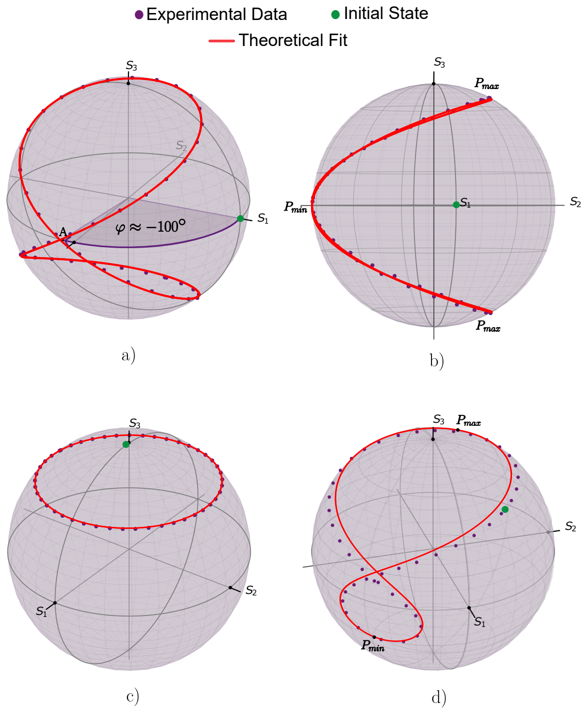

Fig. 3: Graph of the average direction vector $\hat{\mathbf{s}}' = \mathrm{E}\{\mathbf{s}'(t)\}/P$ of the beam emerging from the biplate with an angle $\phi \approx -39°$. The red curve corresponds to the theoretical model used, assuming Gauss-Lorentz coherence where the retardance of each waveplate is $\tau \approx 38.995\frac{\lambda_0}{4c}$. The dark points are the experimental data, and the green points are the incident polarization state in each case. a) Incident linear polarization state. b) Side view of graph a. c) The graph corresponds to an incident right circular polarization state $\chi_0 \approx 44.5°$. d) The graph corresponds to an incident elliptical polarization state $\chi_0 \approx 21.7°$ and $\alpha_0 \approx 27.7°$.

case correspond to the mean direction vector $\hat{\mathbf{s}} = (\mathrm{E}\{s_1\}/P, \mathrm{E}\{s_2\}/P, \mathrm{E}\{s_3\}/P)$, which consists of a mean Stokes parameters normalization with respect to the degree of polarization.

In Fig. 3a, we can notice that the trajectory of the vector $\hat{\mathbf{s}}$ is similar to the coherent case, i.e., when depolarization does not occur; the formed trajectory is that of a symmetrical figure-eight. In fact, for this case, with an incident linearly polarized state, regardless of whether depolarization occurs or not, it holds that the angle φ formed between the apex of the trajectory of $\hat{\mathbf{s}}$ and the incident state, and which is associated with the optical activity of the biplate, depends on the orientation angle ϕ between the principal axes of each waveplate composing the biplate, as already studied in [23]. However, in this case where the light beam undergoes depolarization, the trajectory of $\hat{\mathbf{s}}$ on the Poincaré sphere cannot be obtained through a geometric law like the

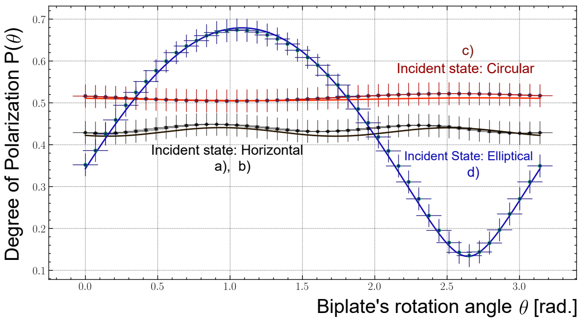

Fig. 4: Degree of polarization as a function of the biplate rotation angle θ. The points are the experimental data (1024 measurements for each θ, the degree of polarization measurement error is $\Delta P = \pm 0.02$ which is the polarimeter resolution), and the curves are the theoretical calculation.

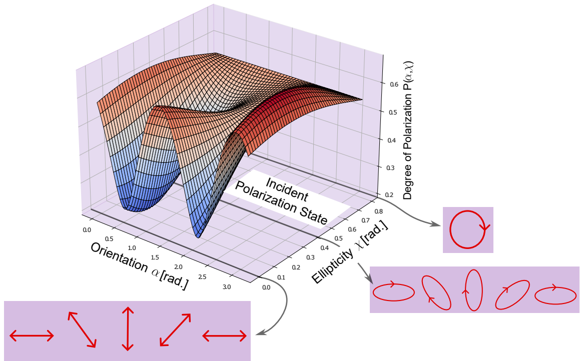

Fig. 5: Graph of the degree of polarization as a function of the incident polarization state characterized by the angles (α, χ) in the range $\alpha = [0, \pi]$ and $\chi = [0, \pi/4]$.

intersection of a cone, as previously studied for the coherent case in [23, 27].

On the other hand, the degree of polarization as a function of the rotation angle θ depends on the incident polarization state as seen in Fig. 4. Additionally, the maximum and minimum points of the degree of polarization curve $P(\theta)$ have a connection with the trajectory on the Poincaré sphere in each case. For example, for the horizontal incident state, as observed in Fig. 3b, the emerging polarization states that coincide with the extreme points of the figure-eight trajectory have the maximum degree of polarization, while the emerging state that coincides with the central point of the trajectory has the minimum value. For the elliptically incident state, the trajectory in Fig. 3d shows that the emerging state corresponding to the upper end of the red curve has the maximum degree of polarization, while the emerging state at the lower end of the curve has the minimum value. Finally, for an approximately circular incident state, the degree of polarization remains constant for any angle θ, and in Fig. 3c it is observed that the red curve coincides with one of the parallels of the Poincaré sphere.

Something interesting to note from Fig. 4 is that for none of the three incident states is there an angle θ for which the emerging state has a polarization state $P = 1$, meaning that it does not suffer depolarization. Using the theoretical model, we can graph the degree of polarization $P(\alpha, \chi)$ as a function of the incident polarization state, characterized by the angles (α, χ), for some constant angle ϕ of the biplate. Fig. 5 shows the graph of $P(\alpha, \chi)$ obtained theoretically. The main result is that no incident state does not depolarize after interacting with the biplate. In terms of the definitions we have used in this work, this result implies that, in general, CW systems do not have eigenstates of polarization, and therefore, these systems are not equivalent to another waveplate.

These results have a direct implication on methodologies where it is attempted to model continuous media presenting birefringence as a discrete composition of waveplates. An example is the work [10], where the birefringence with elliptical eigenstates presented by a single-mode optical fiber is modeled as a system of composite waveplates. The result obtained in the present work indicates that the CW model used for a continuous medium will not be valid when using a source sufficiently incoherent to generate depolarization. This is because CW systems do not have a polarization eigenstate as has already been shown.

4 Conclusions

The standard polarization algebra, such as Jones vectors and polarization matrices, was used to describe the dynamic 2D polarization state associated with a partially polarized source. It was found that the effect of the operator associated with a waveplate on this type of polarization state is to perform a temporal translation of the electric field components. This result can be extended to other algebraic representations of polarization states, such as Stokes parameters, Pauli vectors, quaternions, etc. Thus, it is possible to describe dynamic polarization states and their transformations through, in general, composite waveplates in any of the equivalent algebraic formalism.

For the case of partially polarized incoherent light, the polarization state is described by the ensemble average of its dynamic polarization state. In this work, the representation of the polarization matrix was taken, whose ensemble average results in the well-known Wiener-Wolf Coherence-Polarization matrix. However, if working with another algebraic representation, an equivalent ensemble average representation must be reached. The effect of the operator associated with a waveplate on the Coherence matrix results in the shift of the coherence functions. This quantifies not only the change in the mean polarization state but also the change in the degree of polarization, i.e., depolarization is taken into account through waveplate systems. This makes it possible to determine the polarization transformations due to any composite waveplate system in general for incoherent and partially polarized sources.

The application of this methodology allows for the study of the properties of composite waveplate systems on incident polarization states. In particular, for a waveplate with elliptical eigenstates, it is found that an incident polarization state that coincides with one of the waveplate eigenstates does not undergo depolarization. On the other hand, it was shown both experimentally and theoretically that discrete waveplate systems, such as a biplate, generally do not have eigenstates of polarization. Therefore, any fully polarized state incident on a CW system always undergoes a depolarization process. Thus, this indicates that caution should be exercised when using discrete waveplate systems as a model to analyze the birefringence of continuous media, since, if their transformations are analyzed for an incoherent source, the depolarization effects of the discrete system are not the same as those of the continuous system.

Composite waveplate systems are gaining significant importance today due to their use in quantum key distribution and communication systems where it is necessary to understand their depolarization effects, which can bring both advantages and disadvantages. Therefore, the theoretical model proposed in this work can be used to study these effects when multiple waveplates are involved. Additionally, new methods for quantum key distribution and communication can be found by leveraging the variation of depolarization as a function of the parameters of the retarder system.

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Appendix A Section title of first appendix

An appendix contains supplementary information that is not an essential part of the text itself but which may be helpful in providing a more comprehensive understanding of the research problem or it is information that is too cumbersome to be included in the body of the paper.

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