

Couples Problem

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1 Introduction

Suppose $n \geq 2$ couples are seated at random around a table with men and women alternating. Let N be the number of husbands seated next to their wife. Calculate $E[N]$ and the $Var(N)$.

2 Explanation

let A_i = couple i are together

$$N = \sum_{i=1}^n I[A_i]$$

$$E[I[A_i]] = 0 \cdot P(I[A_i] = 0) + 1 \cdot P(I[A_i] = 1) = \frac{2 \cdot (n-1)!(n-1)!}{n!(n-1)!} = \frac{2}{n}$$

$$E(N) = E\left(\sum_{i=1}^n I[A_i]\right) = \sum_{i=1}^n E(I[A_i]) = n \cdot \frac{2}{n} = 2$$

$$Var(N) = E(N^2) - (E(N))^2 = E\left(\left(\sum_{i=1}^n I[A_i]\right)^2\right) - 4$$

$$= E\left(\sum_{i=1}^n (I[A_i])^2 + \sum_{i \neq j} I[A_i]I[A_j]\right) - 4$$

$$= \sum_{i=1}^n E(I[A_i]) + \sum_{i \neq j} E(I[A_i]I[A_j]) - 4 \quad (\because I[A_i]^2 = I[A_i])$$

$$= \frac{2}{n} \cdot n + \frac{2}{n} \frac{2n-3}{(n-1)^2} n(n-1) - 4$$

$$= 2 - \frac{2}{n-1} \quad \square$$