Variance of Sum

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1 Introduction

Two rods of unknown lengths a,b. A rule can measure the length but with error having 0 mean(unbiased) and variance σ^2. Errors are independent from measurement to measurement. To estimate a,b we could take separate measurement A,B of each rod.

$$E[A] = a, Var(A) = \sigma^2,\ E[B] = b, Var(B) = \sigma^2,$$

We can do better using two measurement, by measuring $a+b$ and $a-b$. WHY?

2 Explanation

let $X = A + B, Y = A - B$ then,

$$E(X) = a + b, Var(X) = \sigma^2, E(Y) = a - b, Var(Y) = \sigma^2$$

Furthermore,

$$E(X^2) = V(X) + (E(X))^2 = \sigma^2 + (a + b)^2$$
$$E(Y^2) = V(Y) + (E(Y))^2 = \sigma^2 + (a - b)^2,$$

$$E(XY) = E(A^2 - B^2) = E(A^2) - E(B^2) = Var(A) + (E(A))^2 - (Var(B) + (E(B))^2) = a^2 - b^2$$

$$Var(a) = Var(\frac{X+Y}{2}) = E(\left(\frac{X+Y}{2}\right)^2) - \{E\left(\frac{X+Y}{2}\right)\}^2$$
$$= \frac{1}{4}(E(X^2) + E(Y^2) + 2E(XY)) - \{\frac{1}{2}E(X) + E(Y)\}^2$$
$$= \frac{1}{4}(\sigma^2 + (a+b)^2 + \sigma^2 + (a-b)^2 + 2(a^2 - b^2)) - a^2$$
$$= \frac{\sigma^2}{2}$$

similarly, $Var(b) = Var(\frac{X-Y}{2}) = \frac{\sigma^2}{2}$

Therefore, using X,Y has smaller Variance while mean is same compared to using A,B. $\square$