Poisson Distribution

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1 Introduction

The Poisson distribution is used to describe the number of events occurring within a fixed interval of time or space. The distribution is defined by the parameter λ, which represents the average number of events in the interval. The probability mass function (PMF) of the Poisson distribution for a random variable X representing the number of events is given by:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!},$$

where k is the number of occurrences of an event, and e is the base of the natural logarithm.

2 Proof of Expectation(Simple Ver.)

The expectation (mean) of the Poisson distribution can be found by summing over all possible values of k multiplied by their probabilities:

$$E(X) = \sum_{k=0}^{\infty} k \cdot P(X = k)$$

Substituting the PMF of the Poisson distribution into the expectation formula, we get:

$$E(X) = \sum_{k=0}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

Since the term for $k = 0$ contributes nothing to the sum, we start the sum at $k = 1$ and factor out $e^{-\lambda}$ and λ as they do not depend on k:

$$E(X) = e^{-\lambda}\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}$$

Notice that the sum is the Taylor series expansion of e^{λ}, thus:

$$E(X) = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

Therefore, the expectation of the Poisson distribution is λ, which completes the proof. $\square$

3 Proof by using Moment Generating Function

3.1 Establishing MGF

The moment generating function (MGF) for a Poisson-distributed random variable X is given by the expectation of e^{tX}, where t is a real number. Thus, the MGF of X is defined as:

$$\phi(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk} \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

Simplifying, we get:

$$\phi(t) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}$$

Recognizing the Taylor series expansion of $e^{\lambda e^t}$, the MGF simplifies to:

$$\phi(t) = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

3.2 Expectation (Mean)

The first derivative of the MGF with respect to t gives the first moment (mean) of the distribution:

$$E(X) = \phi'(0) = \frac{d}{dt} e^{\lambda(e^t-1)}\bigg|_{t=0} = \lambda e^t e^{\lambda(e^t-1)}\bigg|_{t=0} = \lambda$$

3.3 Variance

The second derivative of the MGF at $t = 0$ gives the second moment about the origin, which is used to calculate the variance:

$$E(X^2) = \phi''(0) = \frac{d^2}{dt^2} e^{\lambda(e^t-1)}\bigg|_{t=0} = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)}\bigg|_{t=0} = \lambda + \lambda^2$$

Thus, the variance of X, denoted as $Var(X)$, is the second moment about the mean, which is:

$$Var(X) = E(X^2) - [E(X)]^2 = (\lambda + \lambda^2) - \lambda^2 = \lambda \quad \square$$