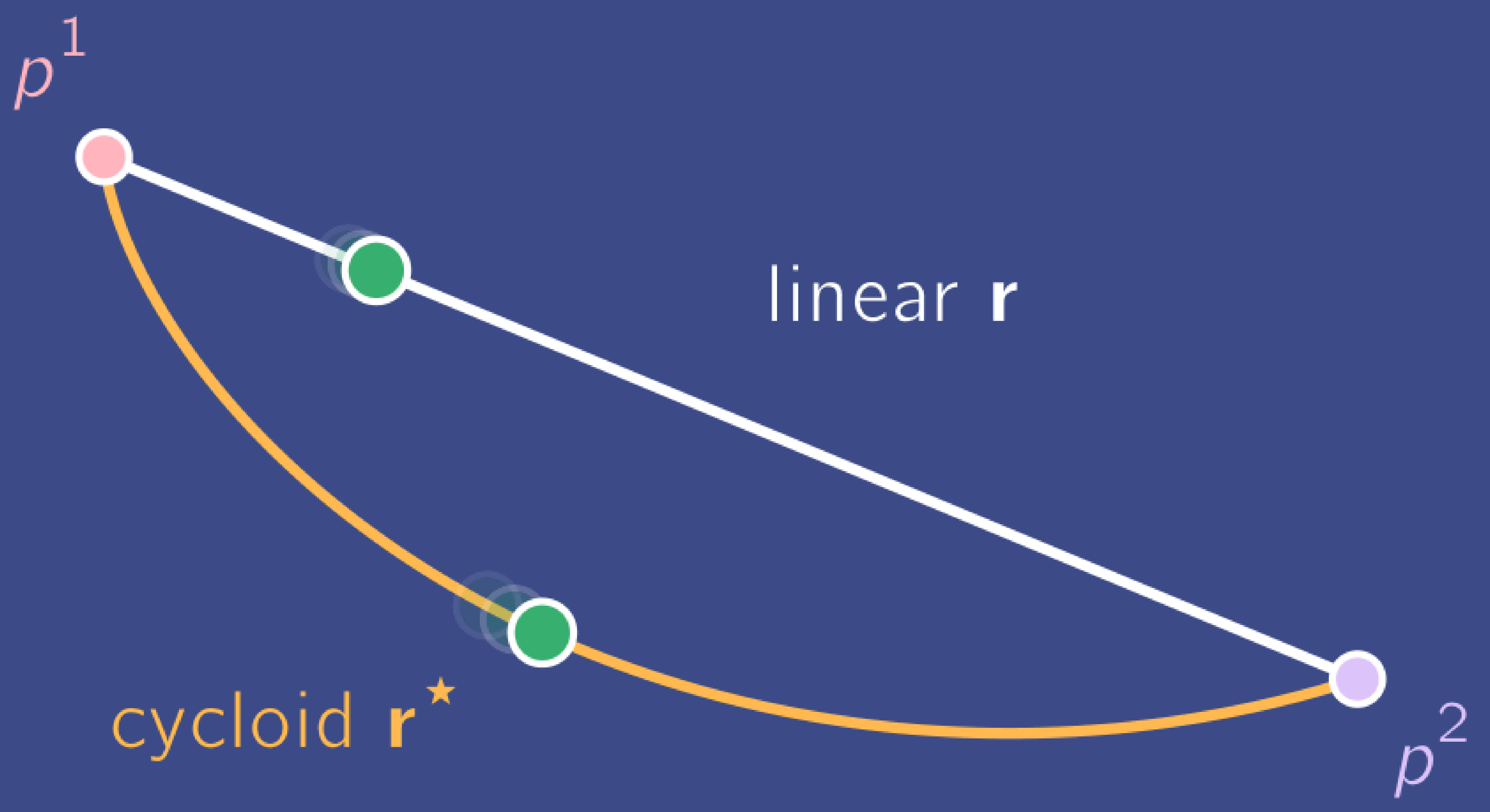
Brachistochrone Problem

Find the frictionless path of least time between two points under uniform gravity



Howard Heaton

Typal Academy

Overview

“Point-wise thinking ends here. Your variable just grew a time axis."

–Variational Evangelist

Scope Introductory derivation of solution to brachistochrone problem

Outline

- Problem Statement
- Problem Reformulation
- First-Order Optimality / Euler-Lagrange
- Beltrami Identity
- Differential Equation Solution

This lecture introduces concepts in the calculus of variations.

Problem Statement

Given two points p^1 and p^2, find the curve $\mathbf{r}^\star$ a frictionless bead would slide on from p^1 to p^2 in the least time T, *i.e.* using dots for time derivatives, solve

$$\mathbf{r}^\star = \operatorname*{argmin}_{\mathbf{r},T} T \quad \text{such that} \quad \mathbf{r}(0) = p^1, \quad \mathbf{r}(T) = p^2, \quad \dot{\mathbf{r}}(0) = 0,$$

$$\mathcal{E} = \tfrac{1}{2} m \|\dot{\mathbf{r}}(t)\|^2 + mg\mathbf{r}_y(t) \quad \text{for all } t.$$

Note

- We assume p^2 is lower than and not directly below p^1.
- For each time t, the point $\mathbf{r}(t) = (\mathbf{r}_x(t), \mathbf{r}_y(t))$ is in $\mathbb{R}^2$.
- Constraint $\dot{\mathbf{r}}(0) = 0$ means bead starts from rest (*i.e.* velocity is zero).
- Total energy $\mathcal{E}$ is conserved as bead travels from p^1 to p^2.

Derivative for Horizontal Motion

Assume the curve is given by $\mathbf{r}(t) = (x(t), y(x(t)))$ for some function $y(x)$.[†]

This gives the speed formula

$$\|\dot{\mathbf{r}}\| = \left\|\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right\| = \left\|\left(1, y'\right)\right\| \frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{1 + (y')^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}.$$

By the conservation of energy $\mathcal{E}$,

$$\mathcal{E} = \frac{1}{2}m\|\dot{\mathbf{r}}\|^2 + mgy \qquad \iff \qquad \|\dot{\mathbf{r}}\| = \sqrt{2(\mathcal{E}/m - gy)}.$$

Together, these two relations imply

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sqrt{\frac{2(\mathcal{E}/m - gy)}{1 + (y')^2}}.$$

[†] Although terms may depend on x or t, we henceforth leave dependencies implicit.

Problem Reformulation

Set $p^1 = (x_1, y_1)$ and $p^2 = (x_2, y_2)$, taking $x_2 > x_1$. Then

$$T = \int_0^T \mathrm{d}t = \int_{x_1}^{x_2} \frac{\mathrm{d}t}{\mathrm{d}x}\,\mathrm{d}x = \int_{x_1}^{x_2} \sqrt{\frac{1 + (y')^2}{2(\mathcal{E}/m - gy)}}\,\mathrm{d}x = \int_{x_1}^{x_2} \mathcal{L}(y, y')\,\mathrm{d}x,$$

where $\mathcal{L}$ is the Lagrangian for this problem. It is common to define the functional

$$J(y) = \int_{x_1}^{x_2} \mathcal{L}(y, y')\,\mathrm{d}x$$

and let $\mathcal{A}$ denote a set of "nice" functions y with $y(x_1) = y_1$ and $y(x_2) = y_2$.

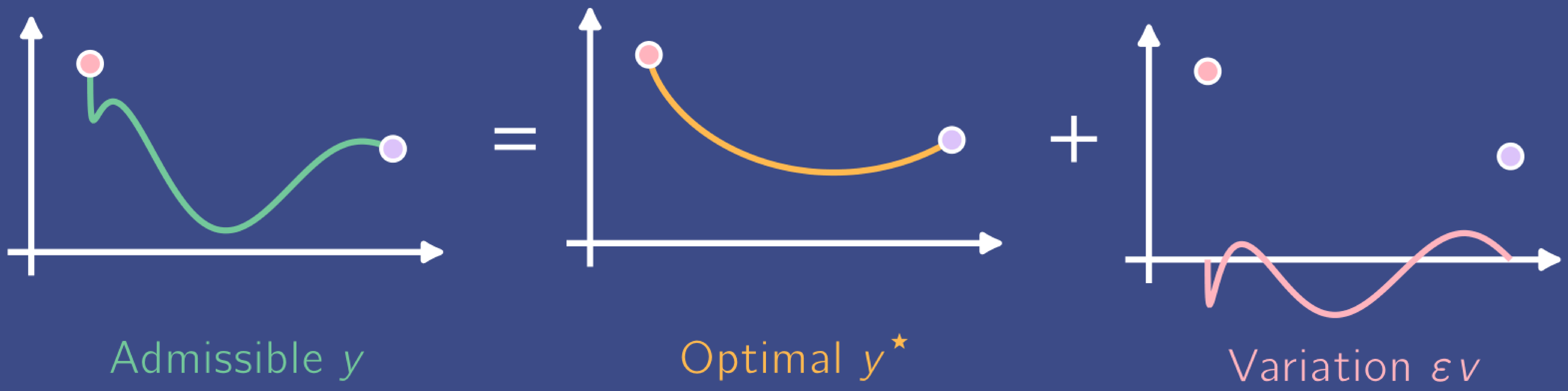
Our problem is[†]

$$\min_{y \in \mathcal{A}} J(y).$$

[†] The constraint $\dot{\mathbf{r}}(0) = 0$ is automatically enforced by taking $\mathcal{E} = mgy_1$.

Variation Function

Each admissible $y \in \mathcal{A}$ can be written as $y = y^{\star} + \varepsilon v$, using the optimal $y^{\star} \in \mathcal{A}$, $\varepsilon > 0$ and a variation function v satisfying $v(x_1) = v(x_2) = 0$.



Decomposition of Feasible Function

First Variation and Optimality

The first variation of J at y in the direction v is the directional derivative

$$\delta J(y, v) = \lim_{\varepsilon \to 0} \frac{J(y + \varepsilon v) - J(y)}{\varepsilon} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big[J(y + \varepsilon v)\Big]_{\varepsilon = 0}.$$

Fix a variation function v and define $f(\varepsilon) = J(y^\star + \varepsilon v)$. Since $y^\star$ is optimal,

$$f(0) = J(y^\star) \le J(y^\star + \varepsilon v) = f(\varepsilon) \quad \text{for all } \varepsilon \in \mathbb{R}.$$

This shows 0 minimizes f and, thus, $f'(0) = 0$. Due to this and the facts v is arbitrary and $f'(0) = \delta J(y^\star, v)$, it follows that

$$0 = \delta J(y^\star, v), \quad \text{for all admissible variations } v. \qquad \text{(optimality condition)}$$

First Variation Formula

Using subscripts for partial derivatives, the first variation δJ is given by

$$\begin{aligned}
\delta J(y, v) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big[J(y + \varepsilon v)\Big]_{\varepsilon=0} && \text{(substitute definition of } \delta J\text{)} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\int_{x_1}^{x_2} \mathcal{L}(y + \varepsilon v, y' + \varepsilon v')\ \mathrm{d}x\right]_{\varepsilon=0} && \text{(substitute definition of } J\text{)} \\
&= \int_{x_1}^{x_2} \frac{\mathrm{d}\mathcal{L}}{d\varepsilon}(y + \varepsilon v, y' + \varepsilon v')\Big|_{\varepsilon=0}\ \mathrm{d}x && \text{(pull derivative inside)} \\
&= \int_{x_1}^{x_2} \mathcal{L}_y v + \mathcal{L}_{y'} v'\ \mathrm{d}x && \text{(evaluate derivative)} \\
&= \int_{x_1}^{x_2} \left(\mathcal{L}_y - \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{y'}\right) v\ \mathrm{d}x + \big[v \cdot \mathcal{L}_{y'}\big]_{x_1}^{x_2} && \text{(integrate by parts)} \\
&= \int_{x_1}^{x_2} \left(\mathcal{L}_y - \frac{\mathrm{d}}{\mathrm{d}x}\mathcal{L}_{y'}\right) v\ \mathrm{d}x. && \text{(apply } v(x_1) = v(x_2) = 0\text{)}
\end{aligned}$$

Euler-Lagrange

Plugging the formula for δJ into the optimality condition yields

$$0 = \delta J(y^\star, v) = \int_{x_1}^{x_2} \left(\mathcal{L}_y - \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{L}_{y'} \right) v \, \mathrm{d}x, \quad \text{for all admissible variations } v.$$

The yellow term is zero as, if not, we could pick v to make the integral nonzero.[†]

Thus, we obtain the famous differential equation satisfied by optimal $y^\star$:

$$\boxed{0 = \mathcal{L}_y - \frac{\mathrm{d}\mathcal{L}_{y'}}{\mathrm{d}x} \quad \text{for all } x \in (x_1, x_2).} \qquad \text{(Euler-Lagrange)}$$

[†] This is a standard result, typically proven in a real analysis course.

Beltrami Identity

Multiplying the Euler-Lagrange equation by y' reveals

$$0 = y'\left(\mathcal{L}_y - \frac{\mathrm{d}\mathcal{L}_{y'}}{\mathrm{d}x}\right) = \mathcal{L}_y y' + \mathcal{L}_{y'} y'' - \frac{\mathrm{d}}{\mathrm{d}x}\left(y'\mathcal{L}_{y'}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathcal{L} - y'\mathcal{L}_{y'}\right].$$

This implies there is a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} C &= \mathcal{L} - y'\mathcal{L}_{y'} && \text{(Beltrami Identity)} \\ &= \sqrt{\frac{1 + (y')^2}{2(\mathcal{E}/m - gy)}} - y' \cdot \frac{y'}{\sqrt{2(\mathcal{E}/m - gy)(1 + (y')^2)}} \\ &= \frac{1}{\sqrt{2(\mathcal{E}/m - gy)(1 + (y')^2)}}. \end{aligned}$$

Differential Equation Parametrization

From the previous slide, optimal $y^\star$ solves the differential equation[†]

$$\frac{1}{2gC^2} = \left(\frac{\mathcal{E}}{mg} - y\right)\left(1 + (y')^2\right) = (y_1 - y)\left(1 + (y')^2\right).$$

Set $u = y_1 - y$ and $R = 1/4gC^2$ so that $u' = -y'$ and

$$u\left(1 + (u')^2\right) = 2R.$$

Parametrize u with respect to θ via $u = 2R\sin^2(\theta/2)$ so that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 = (u')^2 = \frac{2R}{u} - 1 = \frac{2R - u}{u}.$$

[†] From the initial conditions, $y_1 = \mathcal{E}/mg$.

Solution for Horizonal Component

Substituting in the parametrization of u,

$$(\mathrm{d}x)^2 = \frac{u}{2R-u} \cdot (\mathrm{d}u)^2 = \tan^2\left(\frac{\theta}{2}\right) \cdot \left(2R \cdot \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta\right)^2.$$

We may simplify this via†

$$(\mathrm{d}x)^2 = \left(\frac{1-\cos(\theta)}{\sin(\theta)} \cdot R\sin(\theta)\,\mathrm{d}\theta\right)^2 = \Big(R\big(1-\cos(\theta)\big)\,\mathrm{d}\theta\Big)^2.$$

With the conditions $x(0) = x_1$ and $x_2 > x_1$, we integrate to deduce

$$x = \int R\big(1-\cos(\theta)\big)\,\mathrm{d}\theta = R\big(\theta - \sin(\theta)\big) + x_1.$$

†We use the identities $\tan(\theta/2) = (1-\cos(\theta))/2$ and $\sin(\theta) = 2\sin(\theta/2)\cos(\theta/2)$.

Time Scaling

We now solve for t in terms of θ. Observe

$$\begin{aligned}
\mathrm{d}t &= \sqrt{\frac{1 + (y')^2}{2g(y_1 - y)}}\,\mathrm{d}x \\
&= \sqrt{\frac{(\mathrm{d}x)^2 + (\mathrm{d}y)^2}{2gu}} \\
&= \sqrt{\frac{(R[1 - \cos(\theta)])^2 + (R\sin(\theta))^2}{2gR(1 - \cos(\theta))}}\mathrm{d}\theta \\
&= \sqrt{\frac{R}{g}}\,\mathrm{d}\theta.
\end{aligned}$$

Integrating gives $\theta = t\sqrt{g/R}$.

Summary – Solution Derivation

- Optimal $\mathbf{r}^\star$ corresponds to $y^\star$ satisfying $\delta J(y^\star, v) = 0$.
- This optimality condition is equivalent to $\mathcal{L}_y - \mathrm{d}\mathcal{L}_{y'}/\mathrm{d}x = 0$.
- This Euler-Lagrange equation is solved by the cycloid curve[†]

$$x = x_1 + R\big(\omega t - \sin(\omega t)\big),$$
$$y = y_1 - 2R\sin^2\left(\frac{\omega t}{2}\right) = y_1 - R\big(1 - \cos(\omega t)\big).$$

- Since each relation was established via equivalences/equalities, this is the unique solution (up to equivalent parametrizations).

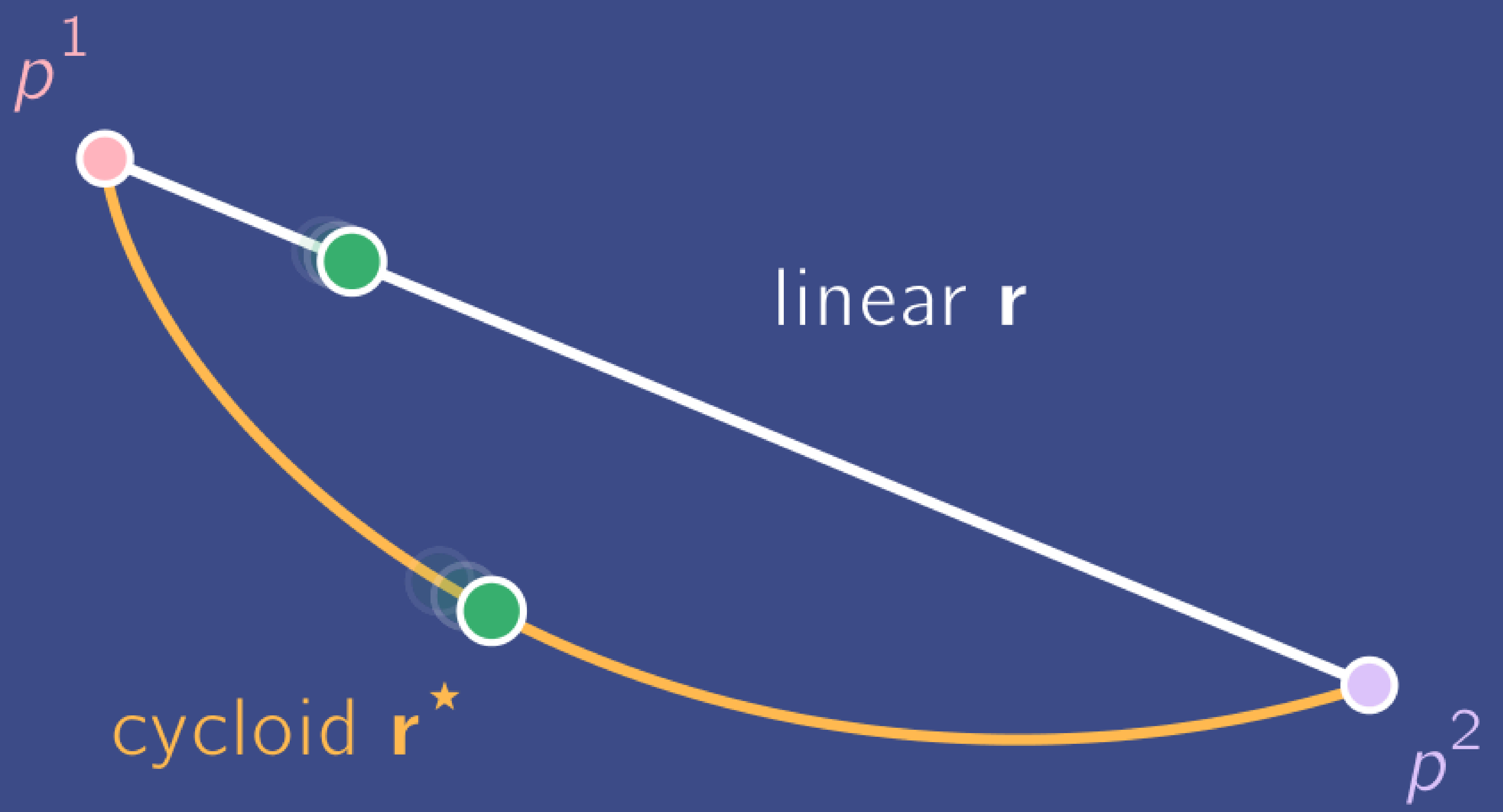
[†] We take $\omega = \sqrt{g/R}$ and use a trigonometric identity to simplify y.

Summary – Brachistochrone Solution

The optimal curve $\mathbf{r}^\star$ that solves the brachistochrone problem is

$$\mathbf{r}^\star(t) = \Big(x_1 + R\big(\omega t - \sin(\omega t)\big),\ y_1 - R\big(1 - \cos(\omega t)\big)\Big), \quad \text{for all } t \in [0, T],$$

with $\omega = \sqrt{g/R}$ and $R > 0$ and $T > 0$ determined† by the condition $\mathbf{r}^\star(T) = p^2$.



†See Appendix for an example.

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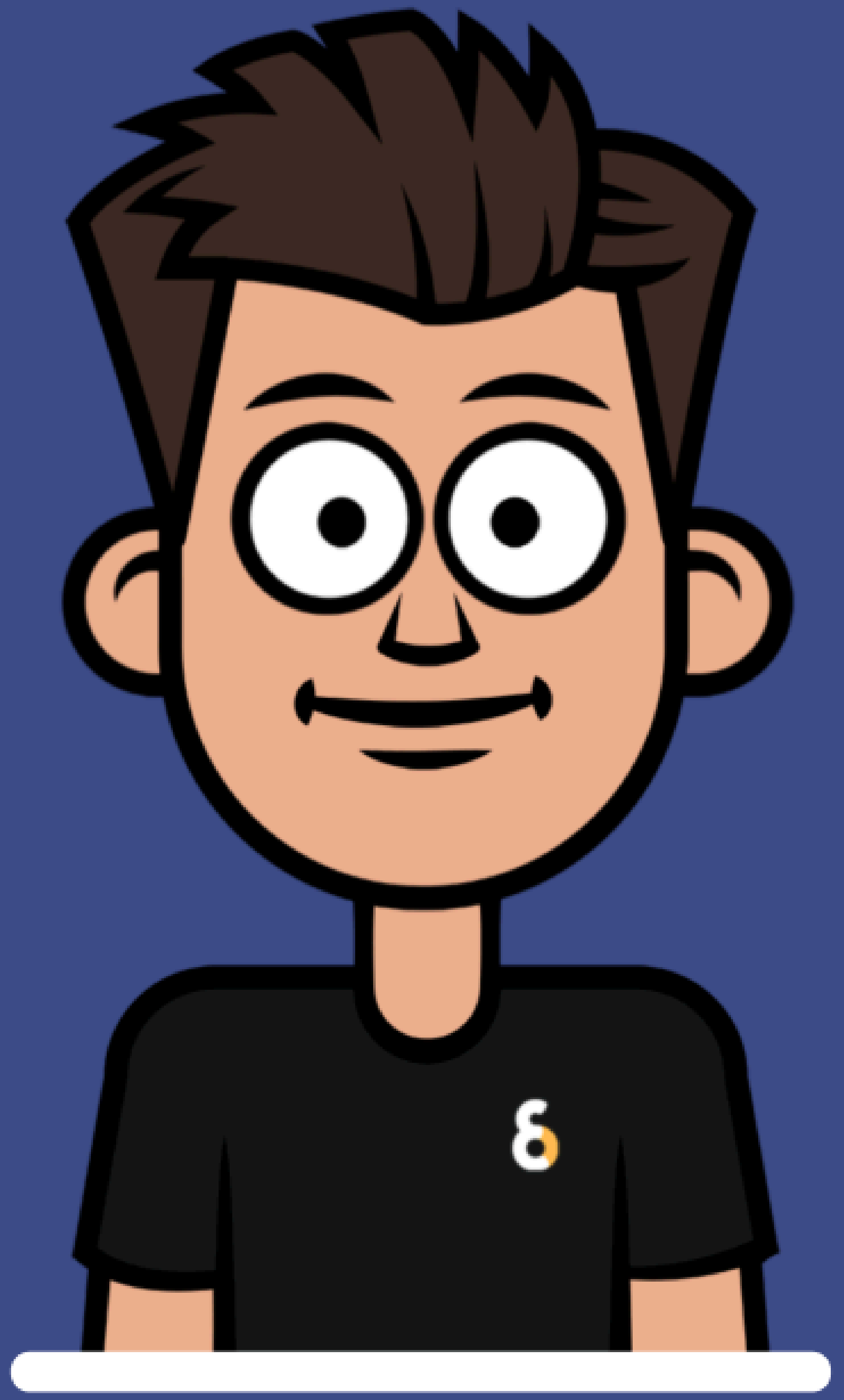

References

- D. Smith. *Variational Methods in Optimization*. Prentice-Hall. 1974.
- I. Gelfand and S. Fomin. *Calculus of Variations*. Dover. 2000.
- _. *Brachistochrone curve*. Wikipedia. 2025.

Appendix – Solving for R and T

Set $\Theta = T\sqrt{g/R}$,

$$\Delta x = x_2 - x_1 = R\big(\Theta - \sin(\Theta)\big) \quad \text{and} \quad \Delta y = y_2 - y_1 = R\big(\cos(\Theta) - 1\big)$$

so that $\Delta x > 0$, $\Delta y < 0$, and

$$0 = (R - R) \cdot \big(\Theta - \sin(\Theta)\big) = \Delta x - \Delta y \cdot \frac{\Theta - \sin(\Theta)}{\cos(\Theta) - 1}.$$

For $\theta \in (0, 2\pi)$, set

$$f(\theta) = \Delta x - \Delta y \cdot \frac{\theta - \sin(\theta)}{\cos(\theta) - 1}.$$

We seek Θ such that $f(\Theta) = 0$.

Appendix – Solving for R and T

Since f is continuous and

$$\lim_{t \to 0^+} f(\theta) = \Delta x > 0 \quad \text{and} \quad \lim_{t \to 2\pi^-} f(\theta) = -\infty < 0,$$

the intermediate value theorem states there is $\Theta \in (0, 2\pi)$ such that $f(\Theta) = 0$.

Because $f'(\theta) < 0$, the root Θ is unique.

Solve for Θ via bisection method, using f and interval $(0, 2\pi)$. Then

$$T = \Theta \sqrt{\frac{R}{g}} \quad \text{and} \quad R = \frac{\Delta x}{\Theta - \sin(\Theta)}.$$

Sample code: github.com/TypalAcademy/brachistochrone

Appendix – Numerical Integration

To get discrete steps along the trajectory $\mathbf{r}(t)$, we numerically integrate.

For a step size $\Delta t > 0$, the simplest approach is to use Forward Euler estimates

$$x(t + \Delta t) \approx x(t) + \Delta t \cdot \frac{\mathrm{d}x}{\mathrm{d}t}(t).$$

Note

Since the bead starts from rest, $\mathrm{d}x/\mathrm{d}t = 0$ at $t = 0$. However, to make things "nice" numerically, we can use small $\varepsilon > 0$ and $\mathcal{E} = \varepsilon$ to ensure $\mathrm{d}x/\mathrm{d}t > 0$ at $t = 0$ and get an estimate of the desired trajectory.