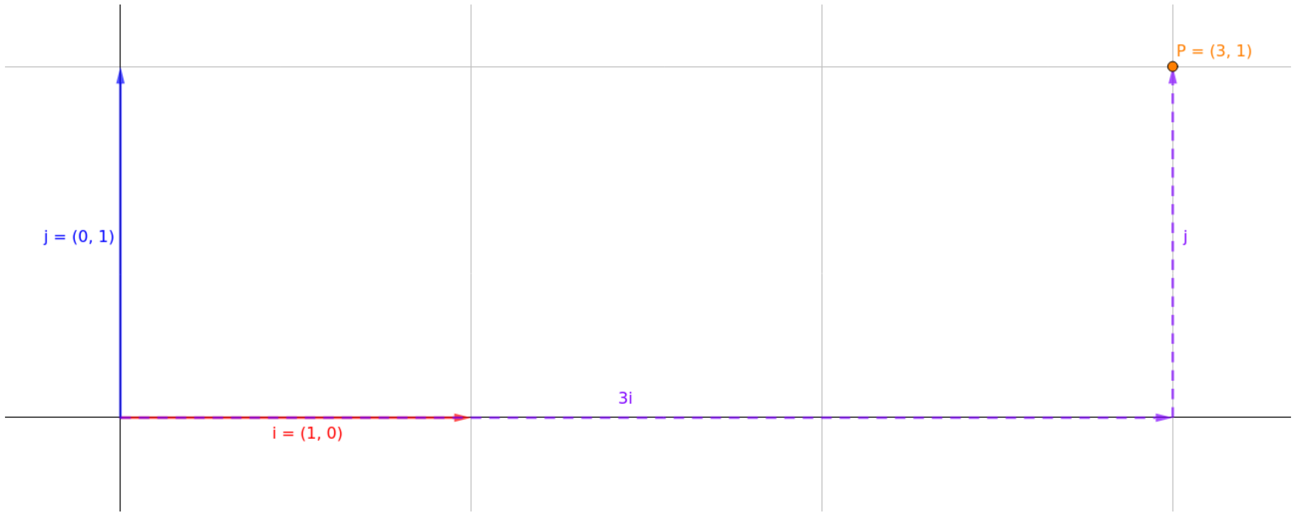
Basis Vectors

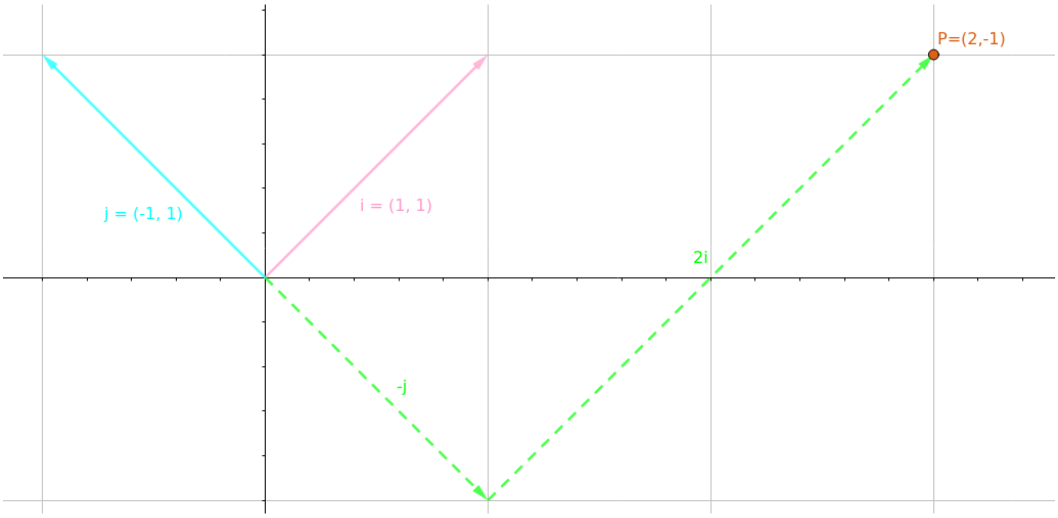
Given a coordinate, we can think of it as a scalar of the axes. For example, consider $P(3,1)$. Starting from the origin, we can arrive at P by scaling the horizontal axis $\begin{pmatrix}1\\0\end{pmatrix}$ by 3, and the vertical axis $\begin{pmatrix}0\\1\end{pmatrix}$ by 1. Writing it as an equation, we get $3\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$.

Here we see that $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are very important vectors, because our entire coordinate system is governed by assumptions around these vectors. We may think of $\begin{pmatrix}1\\0\end{pmatrix}$ as travelling 1 unit towards the right, and $\begin{pmatrix}0\\1\end{pmatrix}$ as travelling 1 unit upwards. These 2 vectors are known as the "basis vectors" of our coordinate system.



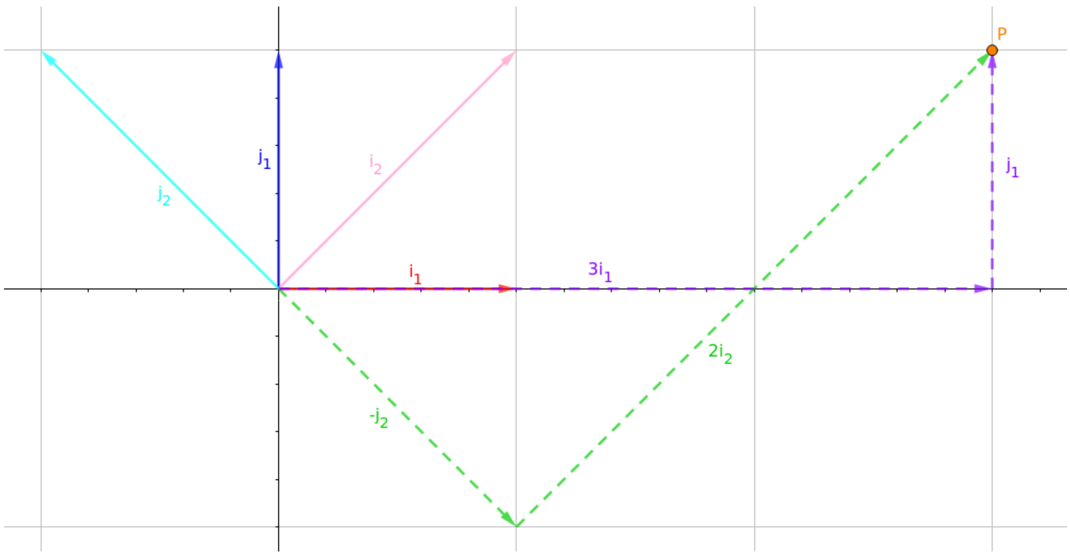
Shown in the figure above, we can arrive at $P(3,1)$ by scaling our basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ by 3, and $\begin{pmatrix}0\\1\end{pmatrix}$ by 1.

Let's say would like a different set of vectors as our basis vectors, is it possible? Good news, we can! For example, we may decide to use $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$. This will mean that we have a new coordinate system, one where coordinates are represented as scalars of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$.



With $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ as our basis vectors, P 's coordinates has changed. In our new system, P 's coordinates is $\begin{pmatrix}2\\-1\end{pmatrix}$. This means that from the origin, we can arrive at P 's location in space by scaling our basis vectors $\begin{pmatrix}1\\1\end{pmatrix}$ by 2, and $\begin{pmatrix}-1\\1\end{pmatrix}$ by -1.

It is important to understand P hasn't actually moved at all. It is still in the same location it always was. The only thing that changed is how it is being represented due to the change in coordinate systems. We can prove this if we overlap the 2 figures.



Looking back at the 2nd figure, you might be thinking this: ‘Woah, wait a minute… Scaling our “horizontal axis” by 2 and our “vertical axis” by -1 should give me $\begin{pmatrix}2\\-1\end{pmatrix}$, but if I write this down as an equation, I get $2\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$. And as far as I can tell, $\begin{pmatrix}2\\-1\end{pmatrix}$ and $\begin{pmatrix}3\\1\end{pmatrix}$ are pretty freaking different things!’

Well, yea… One point of confusion is how we represent the basis vectors versus how we represent P.

Let’s give the 2 different coordinates systems a name. The original coordinates system using $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ as the basis vectors will be called the “standard” coordinates system. We will name $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ as i_1 and j_1 respectively.

The new coordinates system using $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ will be called the “local” coordinates system. We will name $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ as i_2 and j_2 respectively.

This is where things get a little messy. P can be represented in “local” or “standard” coordinates, but i_2 and j_2 are represented in “standard” coordinates, even though they are the basis vectors of the “local” coordinates. This means that when we wrote $P(2,-1)$, P is represented in terms of i_2 and j_2. But i_2 and j_2 themselves are represented in terms of i_1 and j_1. This is because there is no other way of showing the difference between the “standard” and “local” coordinates. One coordinate system has to be relative to another, otherwise we would have no way of relating them to each other.

Matrix Transformation

Let's consider the following transformation which can have different meanings, depending on the context.

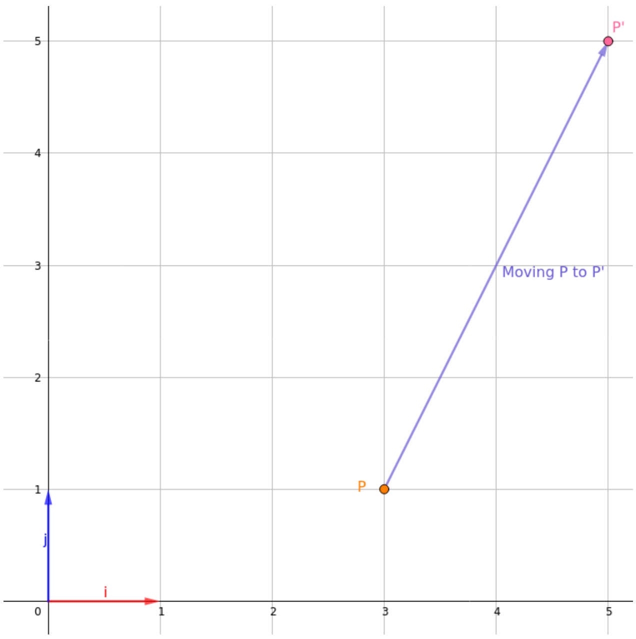
$$\begin{pmatrix} i_x & j_x \\ i_y & j_y \end{pmatrix} \begin{pmatrix} P_x \\ P_y \end{pmatrix} = \begin{pmatrix} P'_x \\ P'_y \end{pmatrix}$$

Let's take a look at some of the different meanings. As per above, we shall use the term "standard" to represent the coordinates where our horizontal axis is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and our vertical axis is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Coordinates represented by any other arbitrary axes shall be referred to as "local" coordinates.

Transforming A Standard Point

Consider the point $P(3,1)$ located in standard coordinates. In order to move this point, we can apply a transformation matrix onto it.

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$



By applying the transformation matrix, we have moved P from $(3,1)$ in standard coordinates to $(5,5)$ in standard coordinates.

Here is another case, where we once again want to move the point $P(3,1)$ in standard space. This time, we apply $\begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}^{-1}$.

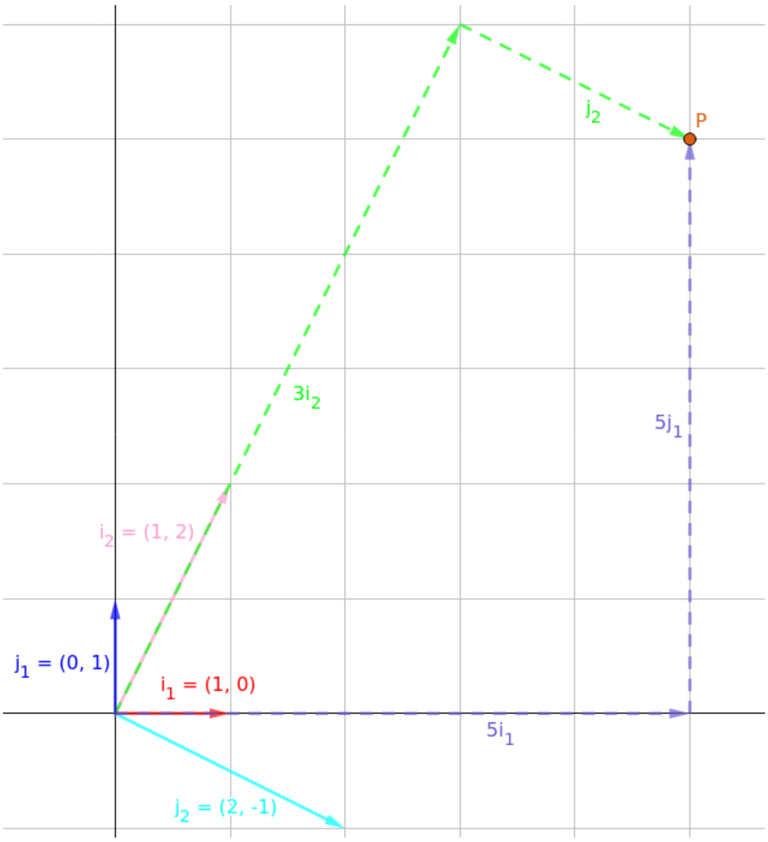
$$\begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

By applying the transformation matrix, we have moved P from $(3,1)$ in standard space to $(5,5)$ in standard space.

This is basically the same as the previous example, but I just want to point out that even when given an inverse matrix, we might still only be doing something as trivial as moving a point.

Change of Basis

Consider the point $P(3,1)$ located in the local coordinates of the axes $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. What if we want to know the coordinates of P in standard space? Since we know that P is the sum of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ scaled by 3 and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ scaled by 1, we can derive that in standard space, it is $3\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$. Alternatively, we can write this using matrices $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$.



What if we are given the standard coordinates $\begin{pmatrix}5\\5\end{pmatrix}$ and want to find the local coordinates in terms of the axes $\begin{pmatrix}1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-1\end{pmatrix}$? Since $\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}P_x\\P_y\end{pmatrix}=\begin{pmatrix}5\\5\end{pmatrix}$, and we know that the inverse of a square matrix undoes its transformation, we get the equation $\begin{pmatrix}P_x\\P_y\end{pmatrix}=\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}5\\5\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$.

Summary

In summary, when shown the transformation $\begin{pmatrix}i_x & j_x\\i_y & j_y\end{pmatrix}\begin{pmatrix}P_x\\P_y\end{pmatrix}=\begin{pmatrix}P'_x\\P'_y\end{pmatrix}$, there are 4 possible things that may be happening:

1. $[M]\times P_{global}=P'_{global}$ (point moves, no change of basis)
2. $[M]^{-1}\times P_{global}=P'_{global}$ (point moves, no change of basis)
3. $[M]\times P_{local}=P_{global}$ (point does not move, change of basis)
4. $[M]^{-1}\times P_{global}=P_{local}$ (point does not move, change of basis)

https://github.com/TypeDefinition