Linear Systems & Augmented Matrices

Solutions of Linear Equations

Given n real numbers $s_1, s_2, \ldots, s_n$, we say that $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ is a solution to a linear equation $a_1 x_1 = s_1, a_2 x_2 = s_2, \ldots, a_n x_n = b$ if the equation is satisfied when we substitute the values into the equation accordingly.

In other words, if $a_1 s_1 + a_2 s_2 + \ldots + a_n s_n = b$, $s_1, s_2, \ldots, s_n$ is a solution to $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$.

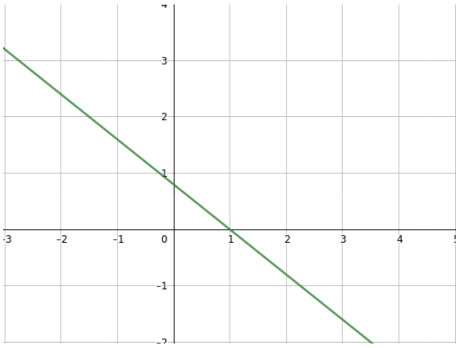
In even simpler words, for example given a linear equation, such as $4x + 2y + 5z = 2$ what are the numbers x, y, z that I can plug into the equation such that it is true?

Or if given $y = -0.8x + 0.8$, what are the numbers a, b that I can plug into the equation such that it is true?

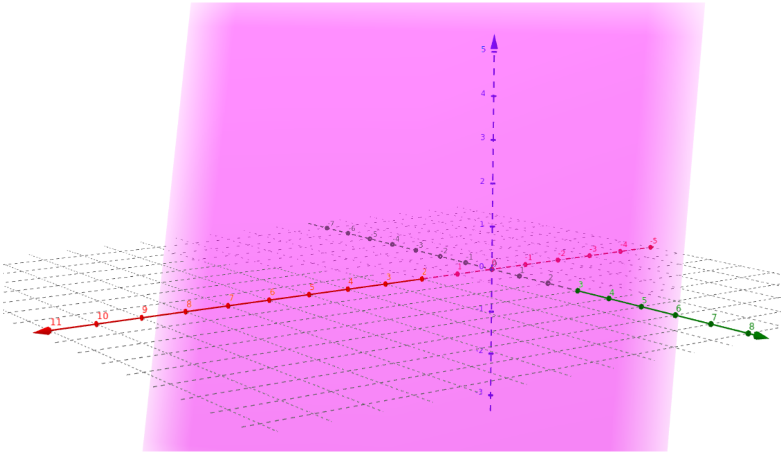
Given the equation $4x + 2y + 5z = 2$, $x = -1.75, y = 2, z = 1$ is one of the infinitely many possible solutions to the equation, because $4(-1.75) + 2(2) + 5(1) = 2$. Similarly, $x = -1.75, y = 2, z = 1$ is another possible solution since $4(0) + 2(-4) + 5(2) = 2$.

Geometrical Representation of Solutions of Linear Equations

The solutions of an equation can be visualised geometrically. For example, given the formula $y=-0.8x+0.8$, we can represent all the solutions as a line.



Similarly, for the general equation $3x+2y+z=6$, we can represent it's solutions as a plane.



You might recognise $3x+2y+z=6$ as a plane equation and that is no coincidence. Any point in that plane will satisfy the equation. Let's call this plane P. This means that the normal of our plane is $(3,2,1)$ and $(3,2,1)\cdot(x,y,z)=6 \mid (x,y,z)\in P$.

General Solutions of Linear Equations

Consider the linear equation $4x-2y=1$ which has infinitely many solutions. The general solution is, $x=t,\ y=2t-0.5$ where t is an arbitrary parameter. In other words, any point $(t,2t-0.5) \mid t\in\mathbb{R}$ lies on the line $4x-2y=1$.

Alternatively, we can also write the general solution as $\begin{matrix} x=0.5s+0.25 \\ y=s \end{matrix}$, where s is an arbitrary parameter. In other words, any point $(0.5s+0.25,s)\ \forall s\in\mathbb{R}$ lies on the line $4x-2y=1$.

Linear Systems

A finite set of linear equations in the variables $x_1, x_2, \ldots, x_n$ is called a **linear system** (also known as a **system of linear equations**).

A linear system looks like:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

where $a_{11}, a_{12}, \ldots, a_{mn}$ and $b_1, b_2, \ldots, b_m$ are real constants.

Some examples of linear systems are:

$$\begin{aligned} x+2y+3z &= 11 \\ 4x+5y+6z &= 16 \\ 7x+8y+9z &= 63 \end{aligned}$$

where x, y, z are the solutions to the linear system.

$$\begin{aligned} x+2y &= 4 \\ 3x+2y &= 11 \\ 6x+6y &= -43 \end{aligned}$$

where x, y are the solutions to the linear system.

$$\begin{aligned} w &= 0.432 \\ x &= 5 \\ y &= -1.7 \\ z &= -9 \end{aligned}$$

where w, x, y, z are the solutions to the linear system.

A linear system can only have either 0, 1 or infinitely many solutions. Therefore it is impossible for a linear system to only have 2, 3, 7, 46, 500 or 3213 unique solutions.

Homogeneous Linear Systems

A system of linear equations is homogeneous if all the constant terms are zero.

$$\begin{aligned} 1x+2y+3z &= 0 \\ 4x+5y+6z &= 0 \\ 7x+8y+9z &= 0 \end{aligned} \quad \text{or} \quad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right)$$

A homogeneous linear system always has the origin as the solution. The origin is called the trivial solution. Any other solution is called a non-trivial solution. Since any linear system may only have 0, 1 or inifinte number of solutions, and a homogenous linear system always has the origin as its solution, we can deduce that a homogenous linear system will either have 1 or inifinite number of solutions.

Augmented Matrices

A system of linear equations can be represented by an augmented matrix.

$$\begin{aligned} 1x+2y+3z&=11 \\ 4x+5y+6z&=16 \\ 7x+8y+9z&=63 \end{aligned} \text{ can be represented as } \left(\begin{array}{ccc|c} 1 & 2 & 3 & 11 \\ 4 & 5 & 6 & 16 \\ 7 & 8 & 9 & 63 \end{array}\right).$$

$$\begin{aligned} 1x+2y+3z&=0 \\ 4x+5y+6z&=0 \\ 7x+8y+9z&=0 \end{aligned} \text{ can be represented as } \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right).$$

Augemented matrices allow us to solve for the solutions of a linear system using **Gaussian elimination** and **Gauss-Jordan elimination**. You are expected to know **Gaussian elimination** and **Gauss-Jordan elimination**. They will not be covered in this document but is crucial to its understanding.