Vectors

Linear Combinations

Definition

Let $u_1, u_2, \ldots, u_k$ be vectors in $\mathbb{R}^n$.

For any real numbers $c_1, c_2, \ldots, c_k$, the vector $c_1 u_1, c_2 u_2, \ldots, c_k u_k$ is called a linear combination of $u_1, u_2, \ldots, u_k$.

Coordinates as Linear Combinations

In the standard basis, the vector $\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ can be thought of as the linear combination of $5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Let $u_1 = (2,1,3), u_2 = (1,-1,2), u_3 = (3,0,5)$.

Is $v = (3,3,4)$ a linear combination of u_1, u_2, u_3 ?

$(3,3,4) = a(2,1,3) + b(1,-1,2) + c(3,0,5) = (2a+b+3c, a-b, 3a+2b+5c)$

Therefore, we can represent this as $\begin{array}{r} 2a+b+3c=3 \\ a-b+0c=3 \\ 3a+2b+5c=4 \end{array}$.

$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array}\right]$ → Gaussian Elimination → $\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -1.5 & -1.5 & 1.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

Therefore, $(a,b,c) = (2-t, -1-t, t)$.

For example, we have particular solutions $(2,-1,0)$, $(1,-2,1)$, etc.

So we can write v as linear combinations $v = 2u_1 - u_2 + 0u_3$, $v = u_1 - 2u_2 + u_3$, etc.

Linear Spans

Let $S=\{u_1,u_2,\dots,u_k\}$ be a set of vectors in $\mathbb{R}^n$. The set of all linear combinations of $u_1,u_2,\dots,u_k\{c_1u_1+c_2u_2+\dots+c_ku_k \mid c_1,c_2,\dots,c_k\subset\mathbb{R}\}$ is called a linear span of S and is denoted by $span(S)$.

Another way to put it is that $span(S)$ is all the coordinates you can travel to using a linear combination of $u_1,u_2,\dots,u_k$.

Subsets

Let $B=\{(u_1,u_2,u_3,u_4) \mid u_1=0 \text{ and } u_2=u_4\}$. It means that B is a subset of $\mathbb{R}^4$ such that $\{(u_1,u_2,u_3,u_4)\in\mathbb{R}^4\}\Leftrightarrow u_1=0 \text{ and } u_2=u_4$. Explicitly, we can write $B=\{(0,a,b,a) \mid a,b\in\mathbb{R}\}$.

Thus, $(0{,}0{,}0{,}0)$, $(0{,}0{,}10{,}0)$, $(0{,}1{,}3{,}1)$, $(0,0.5,0.5,0.5)\in B$ but $(1{,}2{,}3{,}4)$, $(0{,}10{,}0{,}0)$, $(0{,}1{,}3{,}2)$, $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})\notin B$.

If a system of linear equation has n variables, then its solution set is a subset (may be empty) of $\mathbb{R}^n$.

For example, the general solution of the linear system $\begin{matrix} x+y+z=0 \\ x-y+2z=1 \end{matrix}$ can be expressed in vector form $(x,y,z)=(0.5-1.5t,-0.5+0.5t,t) \mid t\in\mathbb{R}$.

The **solution set** can be written as:

$\{(x,y,z) \mid x+y+z=0 \text{ and } x-y+2z=1\}$ (implicit form)

or $\{(0.5-1.5t,-0.5+0.5t,t) \mid t\in\mathbb{R}\}$ (explicit form).

Finite Sets

Let S be a finite set.

We use $|S|$ to denote the number of elements contained in S.

For example, let $S_1=\{1,2,3,4\},S_2=\{(1,2,3,4)\},S_3=\{(1,2,3),(2,3,4)\}$. Then $|S_1|=4$, $|S_2|=1$, $|S_3|=2$.

Subspace

Let V be a subset of $\mathbb{R}^n$. V is called a subspace of $\mathbb{R}^n$ if $V = span(S)$ where $S = \{u_1, u_2, \ldots, u_k\}$ for some vectors $u_1, u_2, \ldots, u_k \in \mathbb{R}^n$.

More precisely, we say that V is subspace spanned by S,

or V is a subspace spanned by $u_1, u_2, \ldots, u_k$.

We also say that S spans V,

or $u_1, u_2, \ldots, u_k$ spans V.

To determine if a subset V is a subspace, it must satisfy the following 3 conditions.

1. $\forall u_1, u_2 \in V \ \left((u_1 + u_2) \in V\right)$ (V is closed under addition)
2. $\forall u \in V, \forall t \in \mathbb{R} \ (tu \in V)$ (V is closed under scalar multiplication)
3. It must contain the zero vector. (Otherwise the previous 2 conditions will fail.)

Trivial Subspace

Let 0 be the zero vector of $\mathbb{R}^n$.

The set $\{0\} = span\{0\}$ is a subspace of $\mathbb{R}^n$ and is known as the zero space.

Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots, e_n = (0, \ldots, 0, 1)$ be vectors in $\mathbb{R}^n$.

Any vector $u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n$ can be written as $u = u_1 e_1 + u_2 e_2 + \ldots + u_n e_n$.

Thus $\mathbb{R}^n = span\{e_1, e_2, \ldots, e_n\}$ is a subspace of $\mathbb{R}^n$.

Solution Spaces

The solution set of a homogenous linear system in n variables is a subspace of $\mathbb{R}^n$.

Linear Independence

A set of vectors are linearly independent if there are **no redundant vectors**. This means that none of the vectors can be expressed as a linear combination of the other vectors. Geometrically, this means that every vector in the set gives a new dimension to the span of the set.

Let $S=\{u_1,u_2,\ldots,u_k\}\subset\mathbb{R}^n$.

Consider the homogeneous vector equation $c_1u_1+c_2u_2+\ldots+c_ku_k=0$ where $c_1,c_2,\ldots,c_k$ are variables.

The homogeneous vector equation is linearly independent if and only if it has only the trivial solution $c_1=0,c_2=0,\ldots,c_k=0$.

Determine whether the vectors $(1,-2,3)$, $(5,6,-1)$, $(3,2,1)$ are linearly independent.

$$c_1(1,-2,3)+c_2(5,6,-1)+c_3(3,2,1)=(0,0,0)\Leftrightarrow\left[\begin{array}{ccc|c}1&5&3&0\\-2&6&2&0\\3&-1&1&0\end{array}\right]$$

By Gaussian Elimination, we find that there are infinitely many solutions. So the vectors are linearly dependent.

Determine whether the vectors $(1,0,0,1)$, $(0,2,1,0)$, $(1,-1,1,1)$ are linearly independent.

$$c_1(1,0,0,1)+c_2(0,2,1,0)+c_3(1,-1,1,1)=(0,0,0,0)\Leftrightarrow\left[\begin{array}{ccc|c}1&0&1&0\\0&2&-1&0\\0&1&1&0\\1&0&1&0\end{array}\right]$$

By Gaussian Elimination, we find that there is only the trivial solution. So the vectors are linearly independent.