Matrices

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Basic Properties

Let A, B and C be matrices of the same size.

Let Z be a zero matrix of the same size as A, B and C.

Let f and g be scalars.

1. Commutative law for matrix addition.

   $A + b = B + A$

2. Associative law for matrix addition.

   $A + (B + C) = (A + B) + C$

3. $f(A + B) = fA + fB$
4. $(f + g)A = fA + gA$
5. $(fg)A = f(gA) = g(fA)$
6. $A + Z = Z + A = A$
7. $A – A = Z$
8. $0A = Z$

More Basic Properties

1. Associative law for matrix multiplication.

   $A(BC) = (AB)C$

2. Distribution law for matrix addition and multiplication.

   $A(B + C) = AB + AC$

   $(A + B)C = AC + BC$

3. Let A and B be matrices. Let c be a scalar.

   $c(AB) = (cA)B = A(cB)$

Even More Basic Properties

Let A be a square matrix of order n.

Let c_n be the column of a matrix.

$A(c_1\ c_2\ \ldots\ c_n) = (Ac_1\ Ac_2\ \ldots\ Ac_n)$.

But, $(c_1\ c_2\ \ldots\ c_n)A \neq (c_1A\ c_2A\ \ldots\ c_nA)$.

Matrix Multiplication Is Not Commutative

In general, AB and BA are 2 different matrices even if the products exists.

For example, let A = $\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$.

Then AB = $\begin{pmatrix} -1 & -2 \\ 11 & 4 \end{pmatrix}$ and BA = $\begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}$.

Hence, AB ≠ BA.

Zero Matrix Product

When AB = 0, it is not necessary that A = 0 or B = 0.

For example, let A = $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

We have A ≠ 0 and B ≠ 0 but AB = $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0$.

Powers of Square Matrices

1. Let A be a square matrix and n a non-negative integer.

   We define A^n as AA … A.

   For example, let A = $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$.

   Then A^3 = $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}=\begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$.

2. Let A be a square matrix and n, m a non-negative integer.

   $A^mA^n = A^{m+n}$

3. Since matrix multiplication is not commutative, in general, for 2 square matrix A and B of the same size, $(AB)^2$ and A^2B^2 may be different.

   For example, let A = $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Then $(AB)^2$ = ABAB = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and A^2B^2 = AABB = $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

More Square Matrices

Let A, B and C be square matrices of size n. AB = BA, AC = CA, does not imply BC = CB.

One example is the scenario where A is an identity or zero-matrix, while B and C may be any 2 non-commutative matrices.

Transpose Matrices

Let A be a $m \times n$ matrix.

1. $(A^T)^T = A$
2. If B is a $m \times n$ matrix, then $(A + B)^T = A^T + B^T$.
3. If c is a scalar, then $(cA)^T = cA^T$.
4. If B is a $n \times p$ matrix, then $(AB)^T = B^T A^T$.

Inverse Matrices

Let A be a square matrix of order n. Let I be an identity matrix.

1. A is said to be invertible if there exists a square matrix B of order n such that AB = I and BA = I.
2. The matrix B here is called the inverse of A.
3. Inverse matrices are unique. That means that an invertible matrix only has 1 inverse matrix.
4. A square matrix is called singular if it has no inverse.

For any square matrix A, the following statements are equivalent. This means that if one of the statements is true, all of them are true. If one of the statements is false, all of them are false.

1. A is invertible.
2. The linear system $Ax = 0$ has only the trivial solution.
3. The reduced row-echelon form of A is an identity matrix.
4. A can be expressed as a product of elementary matrices.

Cancellation Laws for Matrix Multiplication

1. Let A be an invertible $m \times m$ matrix.
   a) If B_1 and B_2 are $m \times n$ matrices such that $AB_1 = AB_2$, then $B_1 = B_2$.
   b) If C_1 and C_2 are $n \times m$ matrices such that $C_1A = C_2A$, then $C_1 = C_2$.
2. If A is singular, the cancellation laws may not hold.

   For example, let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

   Then $AB_1 = AB_2$, but $B_1 \neq B_2$.

Basic Properties of Inverse Matrices

Let A and B be 2 invertible matrices and c a non-zero scalar.

1. cA is invertible and $(cA)^{-1} = \frac{1}{c} A^{-1}$.
2. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
3. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
4. AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$.

Determinants

- A square matrix has the same determinant as its transpose.
- The determinant of a square matrix with 2 identical rows is 0.
- The determinant of a square matrix with 2 identical columns is 0.
- The following matrices have 0 determinant.
  - $\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$
  - $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 5 \\ 1 & 2 & 4 \end{pmatrix}$
  - $\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -3 & -3 & 9 \\ 2 & 4 & 4 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix}$
- If A and B are square matrices of the same size then,
  det(AB) = det(A) × det(B) = det(B) × det(A) = det(BA).
- Let A be an n × n matrix, and c be a scalar. det(cA) = c^n × det(A).
- A × adj(A) = det(A) × I.

Determinants & Elementary Row Operations

Let A be an n × n matrix.

Let E be an n × n elementary matrix.

Let B = EA.

Elementary Matrix (E)	Elementary Row Operation	Effect on Determinant
$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} A$	Scale a row by k.	det(B) = k × det(A) det(B) = det(E) × det(A) det(B) = det(EA)
$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A$	Swap 2 rows.	det(B) = -det(A) det(B) = det(E) × det(A) det(B) = det(EA)

$B=\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$	Add the multiple of one row to another.	det(B) = det(A) det(B) = det(E) × det(A) det(B) = det(EA)