Matrices

Basic Properties

Let A, B and C be matrices of the same size.

Let Z be a zero matrix of the same size as A, B and C.

Let f and g be scalars.

1. Commutative law for matrix addition.

   $A + b = B + A$

2. Associative law for matrix addition.

   $A + (B + C) = (A + B) + C$

3. $f(A + B) = fA + fB$
4. $(f + g)A = fA + gA$
5. $(fg)A = f(gA) = g(fA)$
6. $A + Z = Z + A = A$
7. $A – A = Z$
8. $0A = Z$

More Basic Properties

1. Associative law for matrix multiplication.

   $A(BC) = (AB)C$

2. Distribution law for matrix addition and multiplication.

   $A(B + C) = AB + AC$

   $(A + B)C = AC + BC$

3. Let A and B be matrices. Let c be a scalar.

   $c(AB) = (cA)B = A(cB)$

Matrix Multiplication Is Not Commutative

In general, AB and BA are 2 different matrices even if the products exists.

For example, let A = $\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$.

Then AB = $\begin{pmatrix} -1 & -2 \\ 11 & 4 \end{pmatrix}$ and BA = $\begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}$.

Hence, AB ≠ BA.

Zero Matrix Product

When AB = 0, it is not necessary that A = 0 or B = 0.

For example, let A = $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

We have A ≠ 0 and B ≠ 0 but AB = $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0$.

Powers of Square Matrices

1. Let A be a square matrix and n a non-negative integer.

   We define A^n as AA … A.

   For example, let A = $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$.

   Then A^3 = $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}=\begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$.

2. Let A be a square matrix and n, m a non-negative integer.

   $A^mA^n = A^{m+n}$

3. Since matrix multiplication is not commutative, in general, for 2 square matrix A and B of the same size, $(AB)^2$ and A^2B^2 may be different.

   For example, let A = $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   Then $(AB)^2$ = ABAB = $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and A^2B^2 = AABB = $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

Transpose Matrices

Let A be a m × n matrix.

1. $(A^T)^T = A$
2. If B is a m × n matrix, then $(A + B)^T = A^T + B^T$.
3. If c is a scalar, then $(cA)^T = cA^T$.
4. If B is a n × p matrix, then $(AB)^T = B^T A^T$.

Inverse Matrices

Let A be a square matrix of order n. Let I be an identity matrix.

1. A is said to be invertible if there exists a square matrix B of order n such that AB = I and BA = I.
2. The matrix B here is called the inverse of A.
3. Inverse matrices are unique. That means that an invertible matrix only has 1 inverse matrix.
4. A square matrix is called singular if it has no inverse.

Cancellation Laws for Matrix Multiplication

1. Let A be an invertible m × m matrix.
   a) If B_1 and B_2 are m × n matrices such that $AB_1 = AB_2$, then $B_1 = B_2$.
   b) If C_1 and C_2 are n × m matrices such that $C_1A = C_2A$, then $C_1 = C_2$.
2. If A is singular, the cancellation laws may not hold.

   For example, let A = $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, B_1 = $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and B_2 = $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

   Then $AB_1 = AB_2$, but $B_1 \neq B_2$.

Basic Properties of Inverse Matrices

Let A and B be 2 invertible matrices and c a non-zero scalar.

1. cA is invertible and $(cA)^{-1} = \frac{1}{c} A^{-1}$.
2. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
3. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
4. AB is invertible and $(AB)^{-1} = B^{-1} A^{-1}$.