Eigenvalues, Eigenvectors & Diagonalization

Eigenvalues & Eigenvectors

Let A be a square matrix of order n.

A non-zero column vector $u \in \mathbb{R}^n$ is called an eigenvector of A if $Au = \lambda u$ for some scalar λ.

The scalar λ is called an eigenvalue of A and u is said to be an eigenvector of A associated with the eigenvalue λ.

Basic Properties

Let A be a square matrix of order n.

1. If λ is an eigenvalue of A, λ^n is an eigenvalue of A^n.
2. If λ is an eigenvalue of A, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
3. A and A^T have the same eigenvalues.
4. A is diagonalizable if and only if A has n linearly independent eigenvectors.
5. If A is symmetric, its eigenvalues are guaranteed to be real numbers and not complex numbers.
6. If A is symmetric, eigenvectors from different eigenspaces of A are always orthogonal to each other.
7. If A is orthogonal, it's eigenvalues are 1 or -1.
8. If A is invertible, is not an eigenvalue of A.

Some Proofs

Prove that if λ is an eigenvalue of A, λ^n is an eigenvalue of A^n.

$Au = \lambda u$

Assume $A^k u = \lambda^k u$ is true. Then $A^{k+1} u = A A^k u = A \lambda^k u = \lambda^k A u = \lambda^k \lambda u = \lambda^{k+1} u$.

By mathematical induction, we proved that if λ is an eigenvalue of A, λ^n is an eigenvalue of A^n.

Prove that if λ is an eigenvalue of A , $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} .

$$
\begin{aligned}
Au&=\lambda u\\
u&=\lambda A^{-1}u\\
\frac{1}{\lambda}u&=A^{-1}u\\
A^{-1}u&=\frac{1}{\lambda}u
\end{aligned}
$$

Thus, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} .

Prove that A and A^T have the same eigenvalues.

$$
\begin{aligned}
det(\lambda I-A)&=0\\
det((\lambda I-A)^T)&=0\\
det(\lambda I-A^T)&=0
\end{aligned}
$$

Thus, if λ is an eigenvalue of A , λ is also an eigenvalue of A^T .

Prove that if A is orthogonal, it’s eigenvalues are 1 or -1.

$$
\begin{aligned}
\lambda u&=Au\\
\lambda A^T u&=A^T Au\\
\lambda A^T u&=u\\
A^T u&=\frac{1}{\lambda}u
\end{aligned}
$$

Since A and A^T have the same eigenvalues and $\lambda\neq 0$ (A is invertible), $\lambda=\frac{1}{\lambda}$.

$$
\begin{aligned}
\lambda&=\frac{1}{\lambda}\\
\lambda^2&=1\\
\lambda&=\pm 1
\end{aligned}
$$

Thus, if A is orthogonal, it’s eigenvalues are 1 or -1.

Diagonalization

All vectors in this explanation are column vectors.

Let A be a diagonalizable square matrix of order n .

Let $P=(p_1, p_2, \ldots, p_n)$ where $p_1, p_2, \ldots, p_n$ are basis vectors for the eigenspace of A .

Let $D=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues associated with $p_1, p_2, \ldots, p_n$ respectively.

The goal of diagonalization is to get a diagonal matrices due to the useful properties of diagonal matrices. We know that the product of A and its eigenvectors is a scalar of the eigenvectors.

$$AP=P\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}=P^{-1}AP$$

$$D=P^{-1}AP$$

Orthogonal Diagonalization

All vectors in this explanation are column vectors.

A square matrix A is called orthogonally diagonalizable if there exists an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

A square matrix A is orthogonally diagonalizable if and only if it is symmetric.