

Vectors

Linear Combinations

Definition

Let u_1, u_2, \dots, u_k be vectors in \mathbb{R}^n .

For any real numbers c_1, c_2, \dots, c_k , the vector $c_1 u_1, c_2 u_2, \dots, c_k u_k$ is called a linear combination of u_1, u_2, \dots, u_k .

Coordinates as Linear Combinations

In the standard basis, the vector $\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$ can be thought of as the linear combination of

$$5 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $u_1 = (2, 1, 3), u_2 = (1, -1, 2), u_3 = (3, 0, 5)$.

Is $v = (3, 3, 4)$ a linear combination of u_1, u_2, u_3 ?

$$(3, 3, 4) = a(2, 1, 3) + b(1, -1, 2) + c(3, 0, 5) = (2a + b + 3c, a - b, 3a + 2b + 5c)$$

Therefore, we can represent this as
$$\begin{aligned} 2a + b + 3c &= 3 \\ a - b + 0c &= 3 \\ 3a + 2b + 5c &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 1 & -1 & 0 & 3 \\ 3 & 2 & 5 & 4 \end{array} \right] \rightarrow \text{Gaussian Elimination} \rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 3 & 3 \\ 0 & -1.5 & -1.5 & 1.5 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore, $(a, b, c) = (2 - t, -1 - t, t)$.

For example, we have particular solutions $(2, -1, 0)$, $(1, -2, 1)$, etc.

So we can write v as linear combinations $v = 2u_1 - u_2 + 0u_3$, $v = u_1 - 2u_2 + u_3$, etc.

Linear Spans

Let $S = \{u_1, u_2, \dots, u_k\}$ be a set of vectors in \mathbb{R}^n . The set of all linear combinations of u_1, u_2, \dots, u_k $\{c_1 u_1 + c_2 u_2 + \dots + c_k u_k \mid c_1, c_2, \dots, c_k \in \mathbb{R}\}$ is called a linear span of S and is denoted by $\text{span}(S)$.

Another way to put it is that $\text{span}(S)$ is all the coordinates you can travel to using a linear combination of u_1, u_2, \dots, u_k .

Subsets

Let $B = \{(u_1, u_2, u_3, u_4) \mid u_1 = 0 \text{ and } u_2 = u_4\}$. It means that B is a subset of \mathbb{R}^4 such that $\{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4\} \Leftrightarrow u_1 = 0 \text{ and } u_2 = u_4$. Explicitly, we can write $B = \{(0, a, b, a) \mid a, b \in \mathbb{R}\}$.

Thus, $(0, 0, 0, 0), (0, 0, 10, 0), (0, 1, 3, 1), (0, 0.5, 0.5, 0.5) \in B$ but

$(1, 2, 3, 4), (0, 10, 0, 0), (0, 1, 3, 2), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin B$.

If a system of linear equation has n variables, then its solution set is a subset (may be empty) of \mathbb{R}^n .

For example, the general solution of the linear system $\begin{matrix} x+y+z=0 \\ x-y+2z=1 \end{matrix}$ can be expressed in vector form $(x, y, z) = (0.5 - 1.5t, -0.5 + 0.5t, t) \mid t \in \mathbb{R}$.

The **solution set** can be written as:

$\{(x, y, z) \mid x+y+z=0 \text{ and } x-y+2z=1\}$ (implicit form)

or $\{(0.5 - 1.5t, -0.5 + 0.5t, t) \mid t \in \mathbb{R}\}$ (explicit form).

Finite Sets

Let S be a finite set.

We use $|S|$ to denote the number of elements contained in S .

For example, let $S_1 = \{1, 2, 3, 4\}, S_2 = \{\{1, 2, 3, 4\}\}, S_3 = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. Then $|S_1| = 4, |S_2| = 1, |S_3| = 2$.

Subspace

Let V be a subset of \mathbb{R}^n . V is called a subspace of \mathbb{R}^n if $V = \text{span}(S)$ where $S = \{u_1, u_2, \dots, u_k\}$ for some vectors $u_1, u_2, \dots, u_k \in \mathbb{R}^n$.

More precisely, we say that V is subspace spanned by S ,

or V is a subspace spanned by u_1, u_2, \dots, u_k .

We also say that S spans V ,

or u_1, u_2, \dots, u_k spans V .

To determine if a subset V is a subspace, it must satisfy the following 3 conditions.

1. $\forall u_1, u_2 \in V \ ((u_1 + u_2) \in V)$ (V is closed under addition)
2. $\forall u \in V, \forall t \in \mathbb{R} \ (tu \in V)$ (V is closed under scalar multiplication)
3. It must contain the zero vector. (Otherwise the previous 2 conditions will fail.)

Trivial Subspace

Let 0 be the zero vector of \mathbb{R}^n .

The set $\{0\} = \text{span}\{0\}$ is a subspace of \mathbb{R}^n and is known as the zero space.

Let $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ..., $e_n = (0, \dots, 0, 1)$ be vectors in \mathbb{R}^n .

Any vector $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ can be written as $u = u_1 e_1 + u_2 e_2 + \dots + u_n e_n$.

Thus $\mathbb{R}^n = \text{span}\{e_1, e_2, \dots, e_n\}$ is a subspace of \mathbb{R}^n .

Solution Spaces

The solution set of a homogenous linear system in n variables is a subspace of \mathbb{R}^n .