Matrices

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Basic Properties (Theorem 2.2.6)

Let A, B, C be matrices of the same size and c, d are scalars.

1. $A+B=B+A$ (commutative law for matrix addition)
2. $A+(B+C)=(A+B)+C$ (associative law for matrix addition)
3. $c(A+B)=cA+cB$
4. $(c+d)A=cA+dA$
5. $(cd)A=c(dA)=d(cA)$
6. $A+0=0+A=A$
7. $A-A=0$
8. $0A=0$

More Basic Properties (Theorem 2.2.11)

1. If A, B, C are $m\times p, p\times q, q\times n$ matrices respectively, then $A(BC)=(AB)C$ (associative law for matrix multiplication).
2. If A, B_1, B_2 are $m\times p, p\times n, p\times n$ matrices respectively, then $A(B_1+B_2)=AB_1+AB_2$ (distribution laws for matrix addition & multiplication).
3. If A, C_1, C_2 are $p\times n, m\times p, m\times p$ matrices respectively, then $(C_1+C_2)A=C_1A+C_2A$ (distribution laws for matrix addition & multiplication).
4. If A, B are $m\times p, p\times n$ matrices respectively and c is a scalar, then $c(AB)=(cA)B=A(cB)$.

Matrix Multiplication Isn't Commutative (Remark 2.2.10.3)

In general, AB and BA are 2 different matrices even if the products exists.

For example, let $A=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ 3 & 0\end{pmatrix}$.

Then $AB=\begin{pmatrix}-1 & -2\\ 11 & 4\end{pmatrix}$ and $BA=\begin{pmatrix}3 & 6\\ -3 & 0\end{pmatrix}$. Hence, $AB\neq BA$.

Let A be a square matrix of order n .

Let r be the row of a matrix and let c be the column of a matrix.

$A(r_1\ \ r_2\ \ \ldots\ \ r_n)\neq(Ar_1\ \ Ar_2\ \ \ldots\ \ Ar_n)$ but $(r_1\ \ r_2\ \ \ldots\ \ r_n)A=(r_1A\ \ r_2A\ \ \ldots\ \ r_nA)$.

$A(c_1\ \ c_2\ \ \ldots\ \ c_n)=(Ac_1\ \ Ac_2\ \ \ldots\ \ Ac_n)$ but $(c_1\ \ c_2\ \ \ldots\ \ c_n)\neq(c_1A\ \ c_2A\ \ \ldots\ \ c_nA)$.

Zero Matrix Product (Remark 2.2.10.4)

When $AB=0$, it is not necessary that $A=0$ or $B=0$.

For example, let $A=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$ and $B=\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$.

We have $A\neq 0$ and $B\neq 0$ but $AB=\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=0$.

Square Matrices Multiplication

Let A, B, C be square matrices of size n . $AB=BA, AC=CA$ does not imply $BC=CB$.

One example is the scenario where A is an identity or zero-matrix, while B and C may be any 2 non-commutative matrices.

Powers of Square Matrices (Definition 2.2.12)

1. Let A be a square matrix and n a non-negative integer.

   We define A^n as $A\times A\times ...\times A$.

   For example, let $A=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$, then $A^3=\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}=\begin{pmatrix}11 & 30\\ 15 & 41\end{pmatrix}$.

2. Let A be a square matrix and m, n a non-negative integer.

   Then, $A^m A^n=A^{(m+n)}$.

3. Since matrix multiplication is not commutative, in general, for 2 square matrix A, B of the same size, $(AB)^2$ and $A^2 B^2$ may be different.

   For example, let $A=\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ and $B=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

   Then $(AB)^2=ABAB=\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$, and $A^2 B^2=AABB=\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}$.

Transpose Matrices

Basic Properties (Theorem 2.2.22)

Let A be a $m\times n$ matrix.

1. $(A^T)^T=A$
2. If B is a $m\times n$ matrix, then $(A+B)^T=A^T+B^T$.
3. If c is a scalar, then $(cA)^T=cA^T$.
4. If B is a $n\times p$ matrix, then $(AB)^T=B^T A^T$.

Inverse Matrices

Definition of Inverse Matrices (Definition 2.3.2)

Let A be a square matrix of order n. Let I be an identity matrix.

1. A is said to be invertible if there exists a square matrix B of order n such that $AB=I$ and $BA=I$.
2. The matrix B here is called the inverse of A.
3. Inverse matrices are unique. That means that an invertible matrix only has 1 inverse matrix.
4. A square matrix is called singular if it has no inverse.

For any square matrix A, the following statements are equivalent. This means that if one of the statements is true, all of them are true. If one of the statements is false, all of them are false.

1. A is invertible.
2. The linear system $Ax=0$ has only the trivial solution.
3. The reduced row-echelon form of A is an identity matrix.
4. A can be expressed as a product of elementary matrices.

Basic Properties

Let A and B be 2 invertible matrices and c a non-zero scalar.

1. cA is invertible and $(cA)^{-1}=\frac{1}{c}A^{-1}$.
2. A^T is invertible and $(A^T)^{-1}=(A^{-1})^T$.
3. A^{-1} is invertible and $(A^{-1})^{-1}=A$.
4. AB is invertible and $(AB)^{-1}=B^{-1}A^{-1}$.

Cancellation Laws (Remark 2.3.4)

1. Let A be an invertible $m\times m$ matrix.
   a) If B_1 and B_2 are $m\times n$ matrices such that $AB_1=AB_2$, then $B_1=B_2$.
   b) If C_1 and C_2 are $n\times m$ matrices such that $C_1A=C_2A$, then $B_1=B_2$.
2. If A is singular, the cancellation laws may not hold.

   For example, let $A=\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$, $B_1=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ and $B_2=\begin{pmatrix}2 & 0\\ 1 & 3\end{pmatrix}$.

   Then $AB_1=AB_2$, but $B_1\neq B_2$.

Checking for Singularity (Remark 2.4.10)

If a square matrix A is invertible, a row-echelon form of A has at least 1 zero row.

Determinants

Geometrical Interpretation

The determinant represents how much an area or volume is scaled. In 2D, this is equivalent to the area, and in 3D, this is equivalent to the volume.

Scalar Multiplication (Theorem 2.5.22.1)

Let A be an $n \times n$ matrix, and c be a scalar. Then, $det(cA) = c^n \times det(A) = det(A) \times c^n$.

Matrix Multiplication (Theorem 2.5.22.2)

If A and B are square matrices of the same size then,
$det(AB) = det(A) \times det(B) = det(B) \times det(A) = det(BA)$.

Determinant of 2×2 Matrix (Example 2.5.4.1)

Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we can calculate the determinant using this formula:

$$det(A) = ad - bc$$

Determinant of 3×3 Matrix (Remark 2.5.5)

Given matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, we can calculate the determinant using this formula:

$$det(A) = aei + bfg + cdh - ceg - afh - bdi$$

Determinants of Transpose of Square Matrices (Theorem 2.5.10)

The determinant of a square matrix is equal to the determinant of its transpose.

$$det(A^T) = det(A)$$

Determinants of Inverse Matrices (Theorem 2.5.22.3)

If A is an invertible matrix, then $det(A^{-1}) = \frac{1}{det(A)}$.

Determinants of Triangular Matrices (Theorem 2.5.8)

If A is an $n \times n$ triangular matrix, then $det(A) = a_{11} \times a_{22} \times ... \times a_{nn}$.

Example: $A=\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ or $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

This also means that the reduced row-echelon form of an invertible matrix is the identity matrix.

This also means that a row-echelon form of a singular matrix must have at least 1 zero row.

Identical Row or Columns (Theorem 2.5.12)

The determinant of a square matrix with 2 identical rows or columns is 0.

The following matrices have zero determinant:

- $\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$
- $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 4 \\ 1 & 2 & 4 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -3 & -3 & 9 \\ 2 & 4 & 4 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix}$

Determinants & Elementary Row Operations (Theorem 2.5.15)

Let A be an $n\times n$ matrix, E be an $n\times n$ elementary matrix and $B=EA$.

Elementary Matrix (E)	Elementary Row Operation	Effect on Determinant
$B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix}A$	Scale a row by k .	$det(B)=k\times det(A)$
$B=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}A$	Swap 2 rows.	$det(B)=-det(A)$
$B=\begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}A$	Add the multiple of one row to another.	$det(B)=det(A)$

Invertible Matrices (Theorem 2.5.19)

A square matrix A is invertible if and only if $det(A)\neq 0$.

Adjoints (Theorem 2.5.25)

If A is an invertible matrix, then $A \times adj(A) = det(A) \times I$ and $A \times [\frac{1}{det(A)} adj(A)] = I$.

Cofactor Expansions (Theorem 2.5.6)

Let $A = (a_{ij})$ be an $n \times n$ matrix.

Let M_{ij} be an $(n-1) \times (n-1)$ matrix obtained from A by deleting the i^{th} and the j^{th} column and $A_{ij} = (-1)^{i+j} \times det(M_{ij})$.

Then for any $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$,
$det(A) = a_{i1} A_{i1} + a_{i2} A_{i2} + \dots + a_{¿} A_{¿} = a_{1j} A_{1j} + a_{2j} A_{2j} + \dots + a_{nj} A_{nj}$.