Bases & Coordinate Systems

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Let V be a vector space and $S=\{u_1, u_2, \ldots, u_k\}\subset V$.

S is called a basis (plural bases) for V if

1. S is linearly independent and
2. S spans V .

A basis for V can be used to build a coordinate system for V . We can think of bases as coordinate systems for a vector space. For $S=\{u_1, u_2, \ldots, u_k\}\subset V$, $u_1, u_2, \ldots, u_k$ are known as our basis vectors, or geometrically, the axes of our coordinate system.

Standard Bases

In a vector space R^n , the basis $\{\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}, \ldots, \begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\}$ is called the standard basis.

Example:

1. $\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$ is the standard basis for R^2 .
2. $\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \}$ is the standard basis for R^3 .

Let $[P]$ represent a column vector in the standard basis.

Let $[P]_S$ represent a column vector in basis $S=\{u_1, u_2, \ldots, u_k\}$.

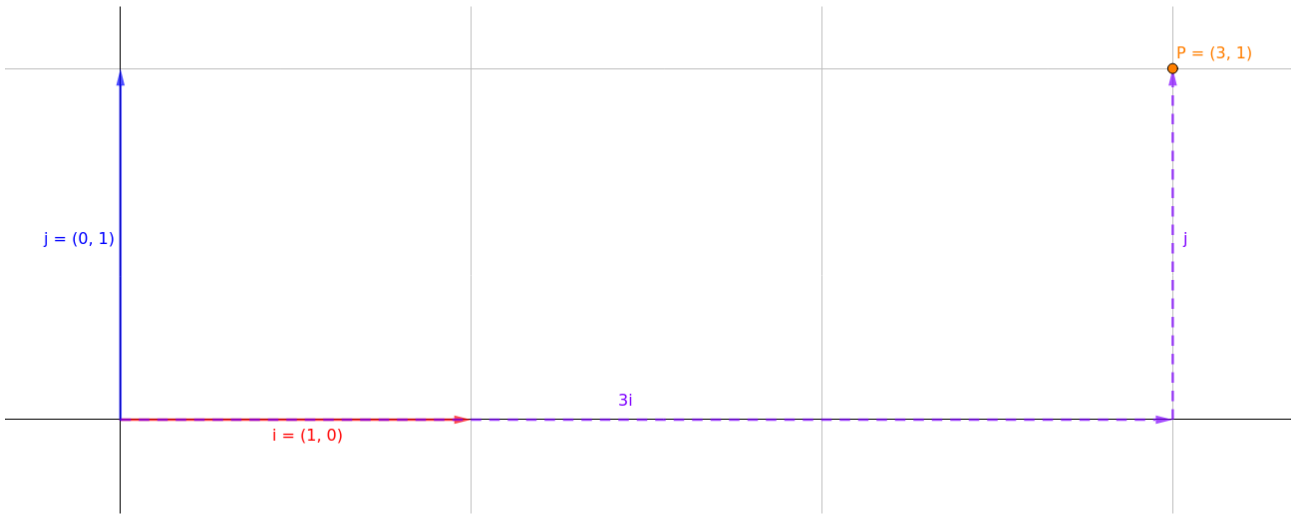

Figure 1

Shown in the figure above, we can arrive at $[P]=\begin{pmatrix}3\\1\end{pmatrix}$ by scaling our basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ by 3, and $\begin{pmatrix}0\\1\end{pmatrix}$ by 1.

Given a coordinate, we can think of it as a scalar of the axes of the coordinate system. For example, consider $[P]=\begin{pmatrix}3\\1\end{pmatrix}$ of the standard basis in R^2 . Starting from the origin, we can arrive at $[P]=\begin{pmatrix}3\\1\end{pmatrix}$ by scaling the horizontal axis $\begin{pmatrix}1\\0\end{pmatrix}$ by 3, and the vertical axis $\begin{pmatrix}0\\1\end{pmatrix}$ by 1. Writing it as an equation, we get $3\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$.

Here we see that $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are very important vectors, because our entire coordinate system is governed by assumptions around these vectors. We may think of $\begin{pmatrix}1\\0\end{pmatrix}$ as travelling 1 unit towards the right, and $\begin{pmatrix}0\\1\end{pmatrix}$ as travelling 1 unit upwards.

Non-Standard Bases

Let's say would like a different set of vectors as our basis vectors, is it possible? Good news, we can! For example, we may decide to use $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$. This will mean that we have a new coordinate system, one where coordinates are represented as scalars of $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$.

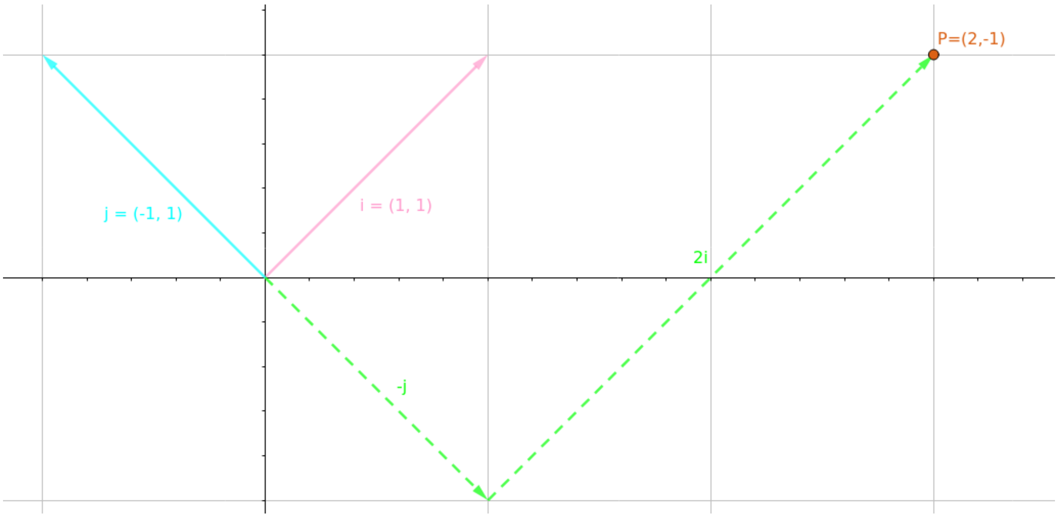

Figure 2

With $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ as our basis vectors, P 's <u>coordinates</u> has changed. In our new system, P 's <u>coordinates</u> is $\begin{pmatrix}2\\-1\end{pmatrix}$. This means that from the origin, we can arrive at P 's <u>location</u> in space by scaling our basis vectors $\begin{pmatrix}1\\1\end{pmatrix}$ by 2, and $\begin{pmatrix}-1\\1\end{pmatrix}$ by -1.

<u>It is important to understand P hasn't actually moved at all. It is still in the same location it always was. The only thing that changed is how it is being represented due to the change in coordinate systems. We can prove this if we overlap the 2 figures.</u>

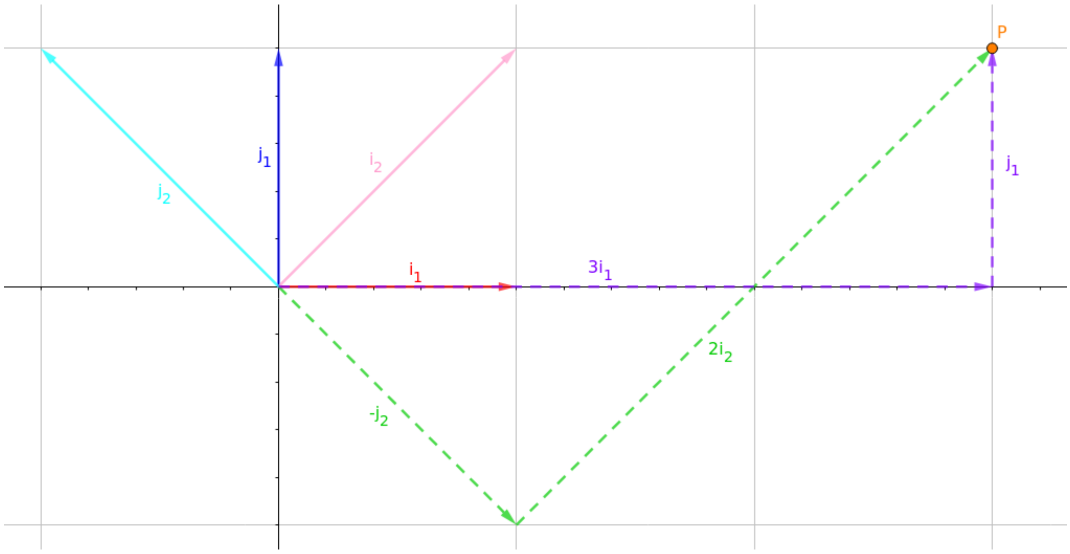

Figure 3

Common Cause of Confusion

Looking back at the Figure 2, you might be thinking this: ‘Woah, wait a minute… Scaling our “horizontal axis” by 2 and our “vertical axis” by -1 should give me $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, but if I write this down as an equation, I get $2\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. And as far as I can tell, $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ are pretty freaking different things!’

Well, yea… One common point of confusion is how we represented the basis vectors versus how we represented P .

Let $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the axes of our standard basis be i_1 and j_1 respectively.

Let $S = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \}$ and let $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ be i_2 and j_2 respectively.

This is where things get a little messy. The vectors i_2 and j_2 are represented in standard coordinates even though they are the basis vectors of S . This means that when we write $[P]_S = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, P is represented in terms of i_2 and j_2 . But i_2 and j_2 themselves are represented in terms of i_1 and j_1 . This is because there is no other way of showing the difference between the standard coordinates and S . One coordinate system has to be relative to another, otherwise we would have no way of relating them to each other. Usually, the axes of non-standard bases are relative to the standard basis itself.

Matrix Transformation

Let’s consider the following transformation which can have different meanings, depending on the context.

$$\begin{pmatrix} i_x & j_x \\ i_y & j_y \end{pmatrix}\begin{pmatrix} P_x \\ P_y \end{pmatrix}=\begin{pmatrix} P'_x \\ P'_y \end{pmatrix}$$

Transforming A Standard Point

Case 1

Consider the point $[P]=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. In order to move this point, we can apply a transformation matrix onto it.

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

By applying the transformation matrix, we have moved P from $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in the standard basis to $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ in the standard basis.

Case 2

Consider another case, where we once again want to move the point $[P]=\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. This time, we apply $\begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}^{-1}$.

$$\begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix}=\begin{pmatrix} 5 \\ 5 \end{pmatrix}$$

By applying the transformation matrix, we have moved P from $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ in the standard basis to $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$ in the standard basis.

This is basically the same as the previous example, but I just want to point out that even when given an inverse matrix, we might still only be doing something as trivial as moving a point or vector.

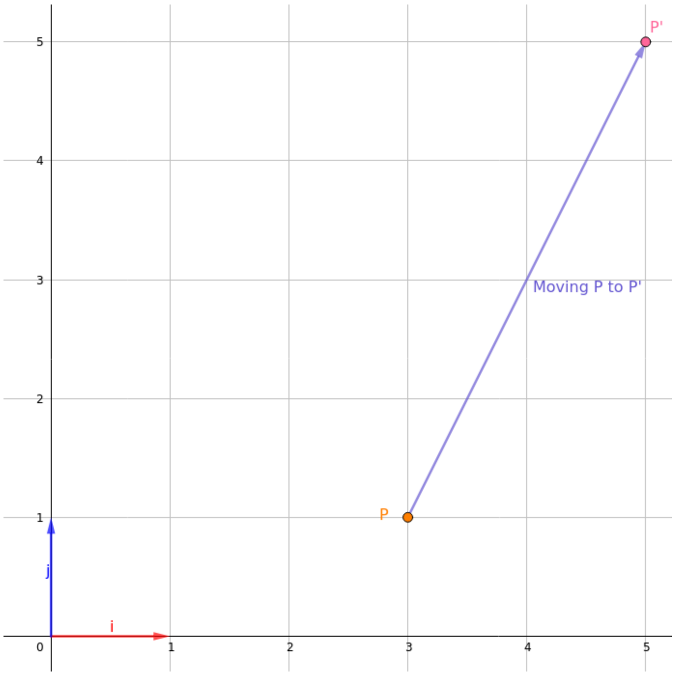

Figure 4

Change of Basis

Convert From Basis *S* To Standard Basis

Let $S=\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\}$ and $[P]_S=\begin{pmatrix}3\\1\end{pmatrix}$. What is $[P]$?

Since we know that P is the sum of $\begin{pmatrix}1\\2\end{pmatrix}$ scaled by 3 and $\begin{pmatrix}2\\-1\end{pmatrix}$ scaled by 1, we can derive that in standard space, it is $3\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}2\\-1\end{pmatrix}=\begin{pmatrix}5\\5\end{pmatrix}$. Alternatively, we can write this using matrices $\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix}=\begin{pmatrix}5\\5\end{pmatrix}$.

Formula for change of basis from $S=\{u_1,u_2,\ldots,u_k\}$ to the standard basis is $[P]=\begin{pmatrix}u_1 & u_2 & \cdots & u_k\end{pmatrix}[P]_S$.

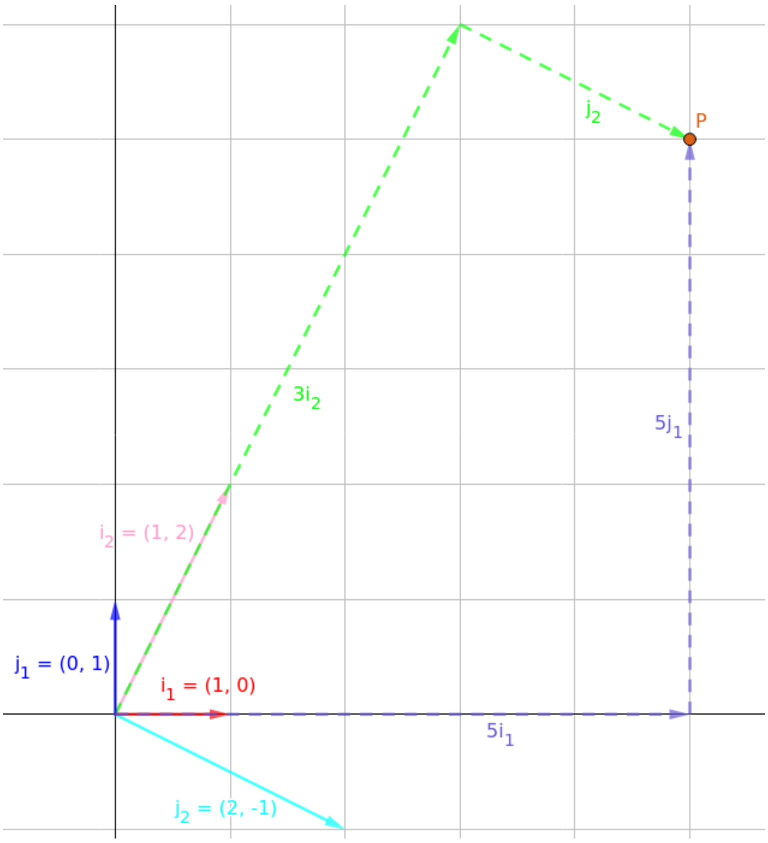

Figure 5

Convert From Standard Basis To Basis *S*

Let $S=\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\}$ and $[P]=\begin{pmatrix}5\\5\end{pmatrix}$. What is $[P]_S$?

Since $\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}[P]_S=\begin{pmatrix}5\\5\end{pmatrix}$, and we know that the inverse of a square matrix undoes its transformation, we derive the equation $[P]_S=\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}^{-1}[P]=\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}5\\5\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$.

Formula for change of basis from $S=\{u_1,u_2,\ldots,u_k\}$ to the standard basis is $[P]_S=\begin{pmatrix}u_1 & u_2 & \cdots & u_k\end{pmatrix}^{-1}[P]$.

Convert From Basis *S* To Basis *T*

Let $S=\{\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}2\\-1\end{pmatrix}\}$ and $[P]_S=\begin{pmatrix}3\\1\end{pmatrix}$. Let $T=\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}-1\\1\end{pmatrix}\}$. What is $[P]_T$?

To find $[P]_T$, we can split it into 2 steps.

1. Convert from $[P]_S$ to $[P]$.
2. Convert from $[P]$ to $[P]_T$.

$$[P]=\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}[P]_S$$
$$[P]_T=\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}[P]$$

Thus, we can derive a <u>transition matrix</u> $W=\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 2\\2 & -1\end{pmatrix}$ such that $[P]_T=W[P]_S$.

We can easily calculate W using Gauss-Jordan Elimination.

$\left(\begin{array}{cc|cc}1 & 1 & 1 & 2\\-1 & 1 & 2 & -1\end{array}\right)$ → Gauss-Jordan Elimination → $\left(\begin{array}{cc|cc}1 & 0 & -0.5 & 1.5\\0 & 1 & 1.5 & 0.5\end{array}\right)$

Thus, $W=\begin{pmatrix}-0.5 & 1.5\\1.5 & 0.5\end{pmatrix}$.

Formula for transition matrix W for change of basis from $S=\{u_1,u_2,\ldots,u_k\}$ to $T=\{v_1,v_2,\ldots,v_k\}$ is $\begin{pmatrix}v_1 & v_2 & \cdots & v_k\end{pmatrix}^{-1}\begin{pmatrix}u_1 & u_2 & \cdots & u_k\end{pmatrix}$.

Summary

In summary, when shown the transformation $\begin{pmatrix} i_x & j_x \\ i_y & j_y \end{pmatrix}\begin{pmatrix} P_x \\ P_y \end{pmatrix}=\begin{pmatrix} P'_x \\ P'_y \end{pmatrix}$, there are 4 possibilities:

1. $M \times P_{standard}=P'_{standard}$ (point moves, no change of basis)
2. $M^{-1} \times P_{standard}=P'_{standard}$ (point moves, no change of basis)
3. $M \times P_S=P_{standard}$ (point does not move, change of basis)
4. $M^{-1} \times P_{standard}=P_S$ (point does not move, change of basis)