Row Operations & Preservation of Row Spaces

The following are geometric proofs that elementary row operations does not modify the row space of a matrix.

Consider the matrix $M=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$. The span of its row space is $span(\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\})$ as shown in Figure 1.

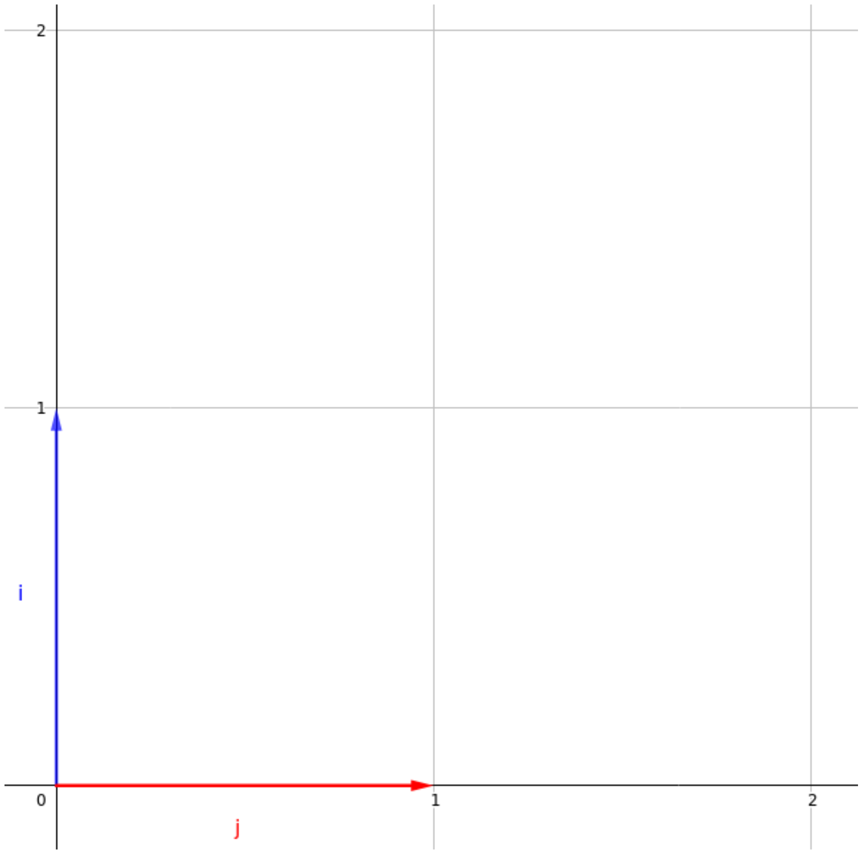

Figure 1

Multiplying A Row By A Non-Zero Constant

Scaling the first row by 2, we get the matrix $M'=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The span of its row space is $span(\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\})$ as shown in Figure 2.

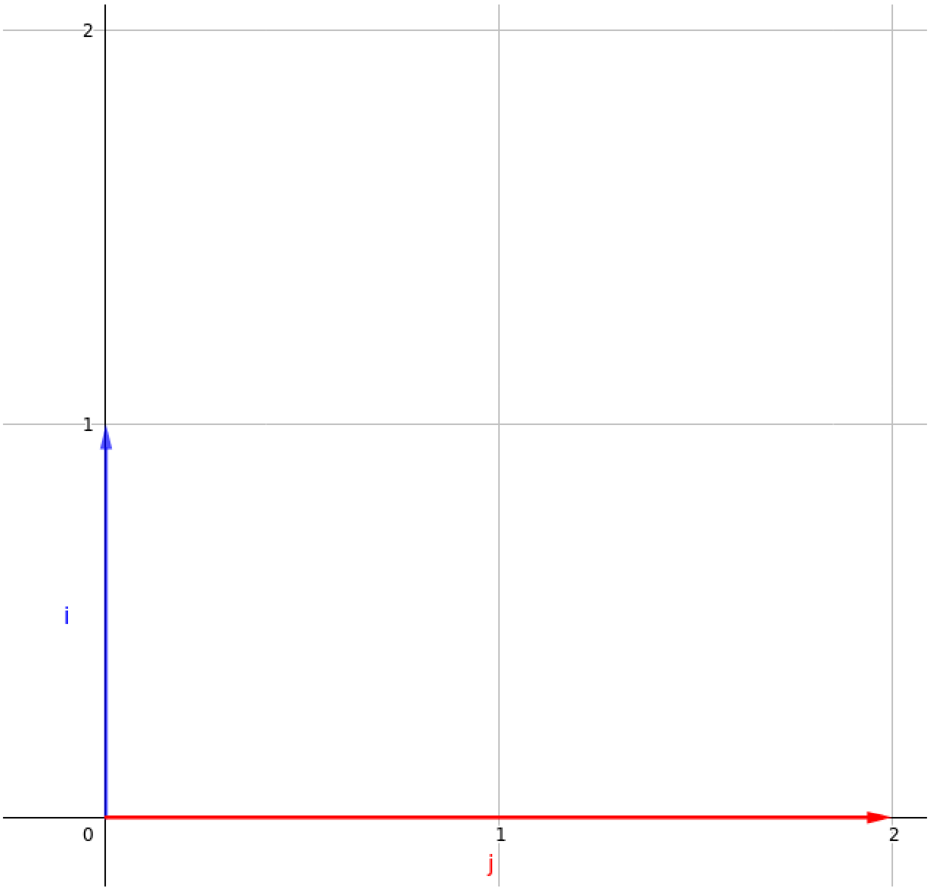

Figure 2

As can be seen, the span of M and M' are the same.

Interchanging 2 Rows

Interchanging the 2 rows, we get the matrix $M' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The span of its row space is $span(\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\})$ as shown in Figure 3.

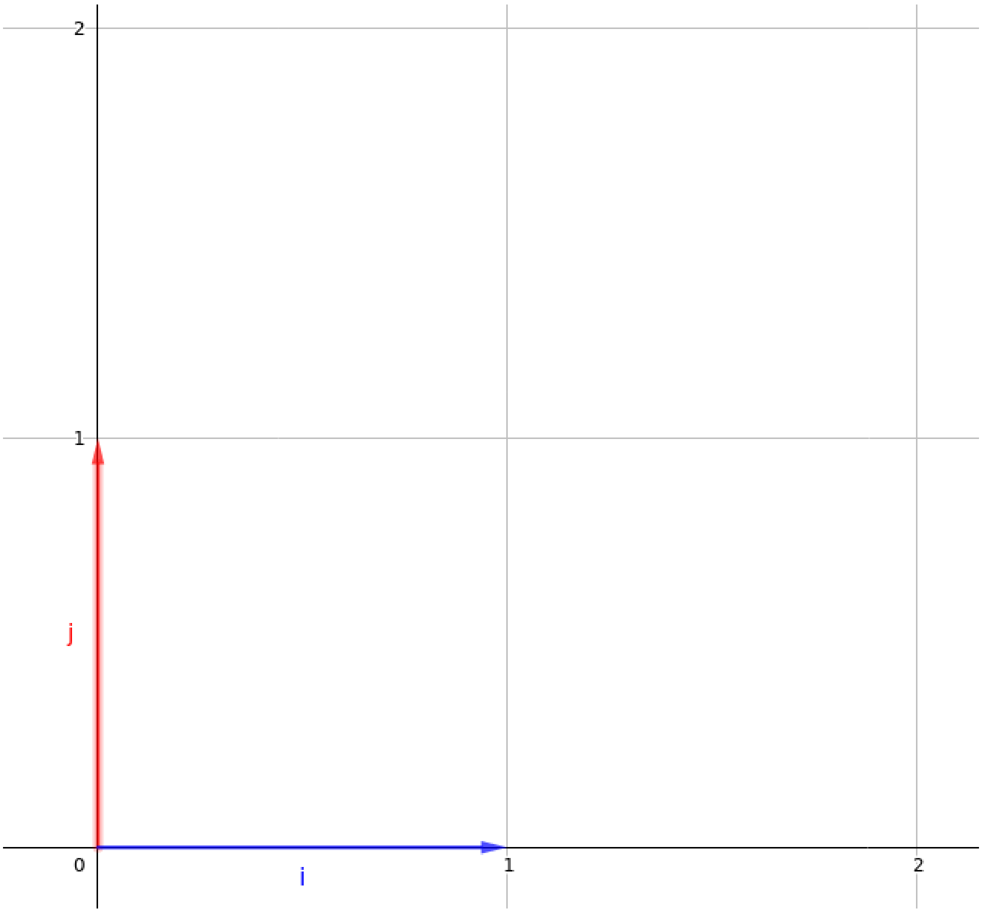

Figure 3

As can be seen, the span of M and M' are the same.

Adding A Multiple Of A Row To Another Row

Adding the first row to the second, we get the matrix $M' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ .The span of its row space is $span(\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\})$ as shown in Figure 4.

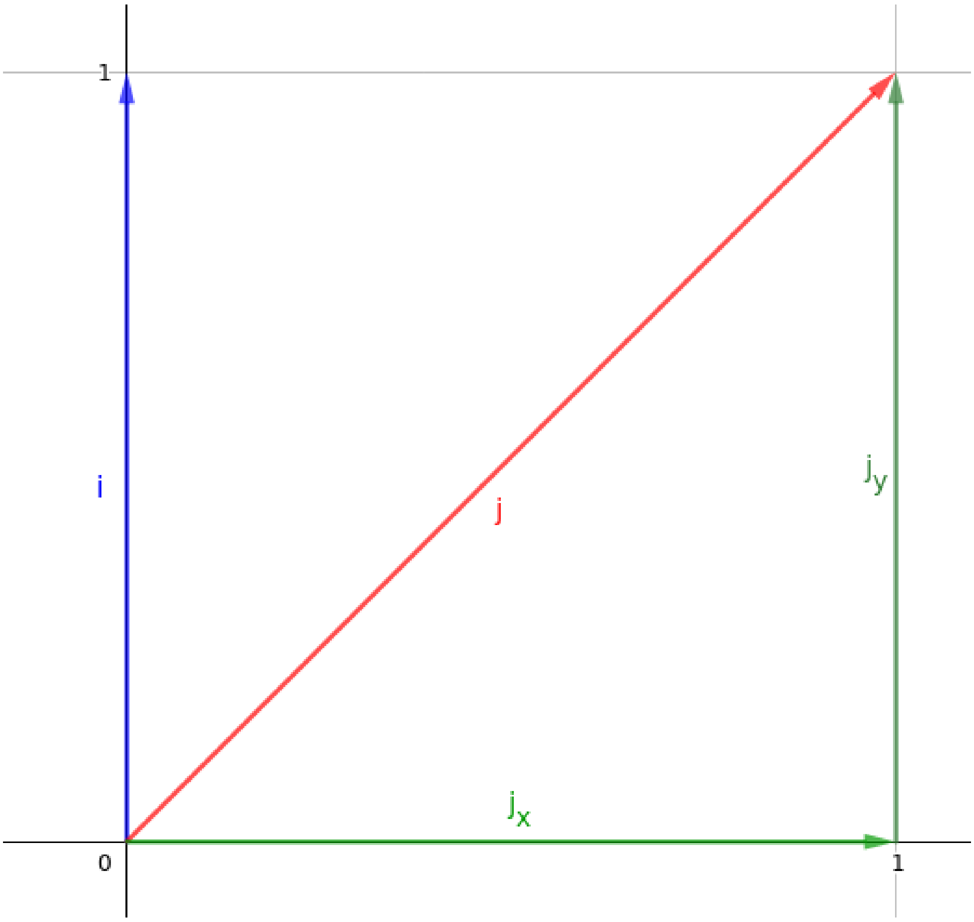

Figure 4

Let $v = k\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Since v is a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, any vector which can be expressed as $s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ can also be expressed as $s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (t\begin{pmatrix} 1 \\ 1 \end{pmatrix} - tk\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = (s - tk)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Thus $span(\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}) = span(\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\})$, and the span of M and M' are the same.