

# Matrices

## Table of Contents

Matrices.....	2
Basic Properties (Theorem 2.2.6).....	2
More Basic Properties (Theorem 2.2.11).....	2
Matrix Multiplication Isn't Commutative (Remark 2.2.10.3).....	2
Zero Matrix Product (Remark 2.2.10.4).....	3
Square Matrices Multiplication.....	3
Powers of Square Matrices (Definition 2.2.12).....	3
Transpose Matrices.....	3
Basic Properties (Theorem 2.2.22).....	3
Inverse Matrices.....	4
Definition of Inverse Matrices (Definition 2.3.2).....	4
Basic Properties.....	4
Cancellation Laws (Remark 2.3.4).....	4
Checking for Singularity (Remark 2.4.10).....	5
Determinants.....	5
Geometrical Interpretation.....	5
Scalar Multiplication (Theorem 2.5.22.1).....	5
Matrix Multiplication (Theorem 2.5.22.2).....	5
Determinant of $2 \times 2$ Matrix (Example 2.5.4.1).....	5
Determinant of $3 \times 3$ Matrix (Remark 2.5.5).....	5
Determinants of Transpose of Square Matrices.....	5
Determinants of Triangular Matrices (Theorem 2.5.8).....	5
Identical Row or Columns (Theorem 2.5.12).....	6
Determinants & Elementary Row Operations (Theorem 2.5.15).....	6
Invertible Matrices (Theorem 2.5.19).....	6
Adjoins (Theorem 2.5.25).....	6

# Matrices

## Basic Properties (Theorem 2.2.6)

Let  $A, B, C$  be matrices of the same size and  $c, d$  are scalars.

1.  $A+B=B+A$  (commutative law for matrix addition)
2.  $A+(B+C)=(A+B)+C$  (associative law for matrix addition)
3.  $c(A+B)=cA+cB$
4.  $(c+d)A=cA+dA$
5.  $(cd)A=c(dA)=d(cA)$
6.  $A+0=0+A=A$
7.  $A-A=0$
8.  $0A=0$

## More Basic Properties (Theorem 2.2.11)

1. If  $A, B, C$  are  $m \times p, p \times q, q \times n$  matrices respectively, then  $A(BC)=(AB)C$  (associative law for matrix multiplication).
2. If  $A, B_1, B_2$  are  $m \times p, p \times n, p \times n$  matrices respectively, then  $A(B_1+B_2)=AB_1+AB_2$  (distribution laws for matrix addition & multiplication).
3. If  $A, C_1, C_2$  are  $p \times n, m \times p, m \times p$  matrices respectively, then  $(C_1+C_2)A=C_1A+C_2A$  (distribution laws for matrix addition & multiplication).
4. If  $A, B$  are  $m \times p, p \times n$  matrices respectively and  $c$  is a scalar, then  $c(AB)=(cA)B=A(cB)$ .

## Matrix Multiplication Isn't Commutative (Remark 2.2.10.3)

In general,  $AB$  and  $BA$  are 2 different matrices even if the products exists.

For example, let  $A = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ .

Then  $AB = \begin{pmatrix} -1 & -2 \\ 11 & 4 \end{pmatrix}$  and  $BA = \begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}$ . Hence,  $AB \neq BA$ .

Let  $A$  be a square matrix of order  $n$ .

Let  $r$  be the row of a matrix and let  $c$  be the column of a matrix.

$$A(r_1 \ r_2 \ \dots \ r_n) \neq (Ar_1 \ Ar_2 \ \dots \ Ar_n) \text{ but } (r_1 \ r_2 \ \dots \ r_n)A = (r_1A \ r_2A \ \dots \ r_nA).$$

$$A(c_1 \ c_2 \ \dots \ c_n) = (Ac_1 \ Ac_2 \ \dots \ Ac_n) \text{ but } (c_1 \ c_2 \ \dots \ c_n) \neq (c_1A \ c_2A \ \dots \ c_nA).$$

### Zero Matrix Product (Remark 2.2.10.4)

When  $AB=0$  , it is not necessary that  $A=0$  or  $B=0$  .

For example, let  $A=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$  and  $B=\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$  .

We have  $A \neq 0$  and  $B \neq 0$  but  $AB=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=0$  .

## Square Matrices Multiplication

Let  $A, B, C$  be square matrices of size  $n$  .  $AB=BA, AC=CA$  does not imply  $BC=CB$  .

One example is the scenario where  $A$  is an identity or zero-matrix, while  $B$  and  $C$  may be any 2 non-commutative matrices.

## Powers of Square Matrices (Definition 2.2.12)

1. Let  $A$  be a square matrix and  $n$  a non-negative integer.

We define  $A^n$  as  $A \times A \times \dots \times A$  .

For example, let  $A=\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$  , then  $A^3=\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}=\begin{pmatrix} 11 & 30 \\ 15 & 41 \end{pmatrix}$  .

2. Let  $A$  be a square matrix and  $m, n$  a non-negative integer.

Then,  $A^m A^n = A^{(m+n)}$  .

3. Since matrix multiplication is not commutative, in general, for 2 square matrix  $A, B$  of the same size,  $(AB)^2$  and  $A^2 B^2$  may be different.

For example, let  $A=\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$  and  $B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  .

Then  $(AB)^2=ABAB=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  , and  $A^2 B^2=AABB=\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$  .

## Transpose Matrices

### Basic Properties (Theorem 2.2.22)

Let  $A$  be a  $m \times n$  matrix.

1.  $(A^T)^T = A$
2. If  $B$  is a  $m \times n$  matrix, then  $(A+B)^T = A^T + B^T$  .
3. If  $c$  is a scalar, then  $(cA)^T = cA^T$  .
4. If  $B$  is a  $n \times p$  matrix, then  $(AB)^T = B^T A^T$  .

# Inverse Matrices

## Definition of Inverse Matrices (Definition 2.3.2)

Let  $A$  be a square matrix of order  $n$ . Let  $I$  be an identity matrix.

1.  $A$  is said to be invertible if there exists a square matrix  $B$  of order  $n$  such that  $AB=I$  and  $BA=I$ .
2. The matrix  $B$  here is called the inverse of  $A$ .
3. Inverse matrices are unique. That means that an invertible matrix only has 1 inverse matrix.
4. A square matrix is called **singular** if it has **no inverse**.

For any square matrix  $A$ , the following statements are equivalent. This means that if one of the statements is true, all of them are true. If one of the statements is false, all of them are false.

1.  $A$  is invertible.
2. The linear system  $Ax=0$  has only the trivial solution.
3. The reduced row-echelon form of  $A$  is an identity matrix.
4.  $A$  can be expressed as a product of elementary matrices.

## Basic Properties

Let  $A$  and  $B$  be 2 invertible matrices and  $c$  a non-zero scalar.

1.  $cA$  is invertible and  $(cA)^{-1} = \frac{1}{c} A^{-1}$ .
2.  $A^T$  is invertible and  $(A^T)^{-1} = (A^{-1})^T$ .
3.  $A^{-1}$  is invertible and  $(A^{-1})^{-1} = A$ .
4.  $AB$  is invertible and  $(AB)^{-1} = B^{-1} A^{-1}$ .

## Cancellation Laws (Remark 2.3.4)

1. Let  $A$  be an **invertible**  $m \times m$  matrix.
  - a) If  $B_1$  and  $B_2$  are  $m \times n$  matrices such that  $AB_1 = AB_2$ , then  $B_1 = B_2$ .
  - b) If  $C_1$  and  $C_2$  are  $n \times m$  matrices such that  $C_1 A = C_2 A$ , then  $B_1 = B_2$ .
2. If  $A$  is **singular**, the cancellation laws may not hold.

For example, let  $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ ,  $B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$  and  $B_2 = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ .

Then  $AB_1 = AB_2$ , but  $B_1 \neq B_2$ .

## Checking for Singularity (Remark 2.4.10)

If a square matrix  $A$  is invertible, a row-echelon form of  $A$  has at least 1 zero row.

## Determinants

### Geometrical Interpretation

The determinant represents how much an area or volume is scaled. In 2D, this is equivalent to the area, and in 3D, this is equivalent to the volume.

### Scalar Multiplication (Theorem 2.5.22.1)

Let  $A$  be an  $n \times n$  matrix, and  $c$  be a scalar. Then,  $\det(cA) = c^n \times \det(A) = \det(A) \times c^n$ .

### Matrix Multiplication (Theorem 2.5.22.2)

If  $A$  and  $B$  are square matrices of the same size then,

$$\det(AB) = \det(A) \times \det(B) = \det(B) \times \det(A) = \det(BA).$$

### Determinant of 2x2 Matrix (Example 2.5.4.1)

Given matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , we can calculate the determinant using this formula:

$$\det(A) = ad - bc$$

### Determinant of 3x3 Matrix (Remark 2.5.5)

Given matrix  $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ , we can calculate the determinant using this formula:

$$\det(A) = aei + bfg + cdh - ceg - afh - bdi$$

### Determinants of Transpose of Square Matrices

The determinant of a square matrix is equal to the determinant of its transpose.

$$\det(A^T) = \det(A)$$

### Determinants of Triangular Matrices (Theorem 2.5.8)

If  $A$  is an  $n \times n$  triangular matrix, then  $\det(A) = a_{11} \times a_{22} \times \dots \times a_{nn}$ .

$$\text{Example: } A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \text{ or } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

This also means that the reduced row-echelon form of an invertible matrix is the identity matrix.

This also means that a row-echelon form of a singular matrix must have at least 1 zero row.

### Identical Row or Columns (Theorem 2.5.12)

The determinant of a square matrix with 2 identical rows or columns is 0.

The following matrices have zero determinant:

- $\begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}$
- $\begin{pmatrix} 1 & 2 & 4 \\ -1 & 10 & 4 \\ 1 & 2 & 4 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -3 & -3 & 9 \\ 2 & 4 & 4 & 0 \\ 0 & -2 & -2 & -1 \end{pmatrix}$

### Determinants & Elementary Row Operations (Theorem 2.5.15)

Let  $A$  be an  $n \times n$  matrix,  $E$  be an  $n \times n$  elementary matrix and  $B = EA$ .

Elementary Matrix (E)	Elementary Row Operation	Effect on Determinant
$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{pmatrix} A$	Scale a row by $k$ .	$\det(B) = k \times \det(A)$
$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A$	Swap 2 rows.	$\det(B) = -\det(A)$
$B = \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$	Add the multiple of one row to another.	$\det(B) = \det(A)$

### Invertible Matrices (Theorem 2.5.19)

A square matrix  $A$  is invertible if and only if  $\det(A) \neq 0$ .

### Adjoint (Theorem 2.5.25)

If  $A$  is an invertible matrix, then  $A \times \text{adj}(A) = \det(A) \times I$  and  $A \times \left[ \frac{1}{\det(A)} \text{adj}(A) \right] = I$ .