Eigenvalues, Eigenvectors & Diagonalisation

Eigenvalues & Eigenvectors

Let A be a square matrix of order n .

A non-zero column vector $u \in \mathbb{R}^n$ is called an eigenvector of A if $Au = \lambda u$ for some scalar λ .

The scalar λ is called an eigenvalue of A and u is said to be an eigenvector of A associated with the eigenvalue λ .

Basic Properties

Let A be a square matrix of order n .

1. If λ is an eigenvalue of A , λ^n is an eigenvalue of A^n .
2. If λ is an eigenvalue of A , $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} .
3. A and A^T have the same eigenvalues.
4. A is diagonalisable if and only if A has n linearly independent eigenvectors.
5. If A is symmetric, its eigenvalues are guaranteed to be real numbers and not complex numbers.
6. If A is symmetric, eigenvectors from different eigenspaces of A are always orthogonal to each other.

Some Proofs

Prove that if λ is an eigenvalue of A , λ^n is an eigenvalue of A^n .

$Au = \lambda u$

Assume $A^k u = \lambda^k u$ is true. Then $A^{k+1} u = A A^k u = A \lambda^k u = \lambda^k A u = \lambda^k \lambda u = \lambda^{k+1} u$.

By mathematical induction, we proved that if λ is an eigenvalue of A , λ^n is an eigenvalue of A^n .

Prove that if λ is an eigenvalue of A , $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} .

$$Au = \lambda u$$
$$u = \lambda A^{-1} u$$
$$\frac{1}{\lambda} u = A^{-1} u$$
$$A^{-1} u = \frac{1}{\lambda} u$$

Thus, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} .

Prove that A and A^T have the same eigenvalues.

$$det(\lambda I - A) = 0$$
$$det((\lambda I - A)^T) = 0$$
$$det(\lambda I - A^T) = 0$$

Thus, if λ is an eigenvalue of A , λ is also an eigenvalue of A^T .

Diagonalisation

All vectors in this explanation are column vectors.

Let A be a diagonalisable square matrix of order n .

Let $P = (p_1, p_2, ..., p_n)$ where $p_1, p_2, ..., p_n$ are basis vectors for the eigenspace of A .

Let $D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$ where $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues associated with $p_1, p_2, ..., p_n$ respectively.

The goal of diagonalisation is to get a diagonal matrices due to the useful properties of diagonal matrices. We know that the product of A and its eigenvectors is a scalar of the eigenvectors.

$$AP = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$
$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = P^{-1} A P$$
$$D = P^{-1} A P$$

Orthogonal Diagonalisation

All vectors in this explanation are column vectors.

A square matrix A is called orthogonally diagonalisable if there exists an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

A square matrix A is orthogonally diagonalisable if and only if it is symmetric.