

# Constraint Satisfaction Problems, Graph Theory, and Universal Algebra

Cliff Bergman

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# What is a CSP?

Informally, a **C**onstraint **S**atisfaction **P**roblem consists of

- a list of variables ranging over a finite domain and
- a set of constraints on those variables.

**Problem:** can we assign values to all the variables so that all of the constraints are satisfied?

# Examples

A system of linear equations is a CSP

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Also, a system of nonlinear equations is a CSP

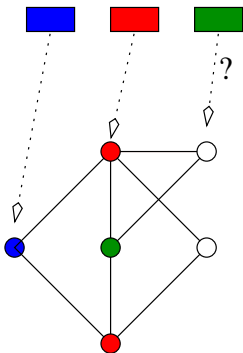
$$a_{11}x_1^2x_3 + a_{12}x_2x_3x_7 + \cdots + a_{1n}x_4x_n^3 = b_1$$

$$a_{21}x_2x_5 + a_{22}x_2 + \cdots + a_{2n}x_4^3 = b_2$$

$$\vdots$$

$$a_{m1}x_3x_5x_8 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

For a fixed  $k$ , determining whether a graph is  $k$ -colorable is a CSP



Given a propositional formula  $\phi(x_1, \dots, x_n)$ , determine whether  $\phi$  is satisfiable

$$\phi(x, y, z) = (x \vee y \vee z') \wedge (x' \vee y \vee z')$$

then

$$\phi(0, 0, 1) = 1$$

# Algorithms

There is an efficient algorithm (Gaussian elimination) for solving any linear system. That is

There is an algorithm that accepts as **input** a linear system and decides whether that system has a solution. The running time of the algorithm is bounded above by  $f(s)$  where  $f$  is a polynomial and  $s$  is the size of the system.

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The particular system is an **instance** of the **problem**  
LINEAR SYSTEM



Similarly

There is an algorithm that accepts as input a graph and decides whether the graph is 2-colorable.

The running time is bounded by  $f(s)$  where  $f$  is a polynomial and  $s$  is the size of the graph.

The graph is an instance of the problem  
2-COLORABILITY.

There is an algorithm that accepts as input a formula,  $\phi = \phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$ , in which each  $\phi_i$  is biconjunctive, and decides whether  $\phi$  is satisfiable.

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We say these algorithms run in **polynomial time**.

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Thus these problems are solvable in **nondeterministic polynomial time**.

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It is possible for  $X \leq_p Y \leq_p X$ . In that case, write  $X \equiv_p Y$ .

$\mathbb{P}$  is the class of all problems solvable in polynomial time.  
Its members are called **tractable**.

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- $\mathbb{P} \subseteq \mathbb{NP}$
- Both  $\mathbb{P}$  and  $\mathbb{NP}$  are downsets, i.e.,  
 $Y \in \mathbb{P} \ \& \ X \leq_p Y \implies X \in \mathbb{P}$

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The maximal members of  $\mathbb{NP}$  are called  **$\mathbb{NP}$ -complete**.

3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are known to be  $\mathbb{NP}$ -complete.

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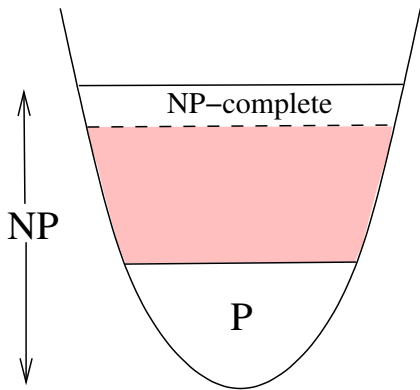
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**Theorem (Ladner, 1975 )**

*If  $\mathbb{P} \neq \text{NP}$  then there are problems in  $\text{NP} - \mathbb{P}$  that are not NP-complete.*





If  $P \neq NP$  then  
the pink area is  
nonempty.

# Formal Definition of CSP

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An  $n$ -ary relation on  $D$  is a subset of  $D^n$

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$\text{Rel}_n(D)$  denotes the set of all  $n$ -ary relations on  $D$

$$\text{Rel}(D) = \bigcup_{n>0} \text{Rel}_n(D)$$

Let  $D$  be a finite set and  $\Delta \subseteq \text{Rel}(D)$

$\text{CSP}(\langle D, \Delta \rangle)$  is the problem:

**instance:** A finite set  $V = \{v_1, \dots, v_n\}$  of **variables** and a finite set  $\{C_1, \dots, C_m\}$  of **constraints**

Each constraint  $C_i$  is a pair  $(\langle x_{i1}, \dots, x_{ip_i} \rangle, \delta_i)$  in which  $x_{i1}, \dots, x_{ip_i} \in V$  and  $\delta_i \in \Delta$

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**Question:** Does there exist a mapping  $f: V \rightarrow D$  such that for all  $i \leq m$ ,  $\langle f(x_{i1}), \dots, f(x_{ip_i}) \rangle \in \delta_i$ ?

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$\text{CSP}(\langle D, \Delta \rangle)$  always lies in  $\text{NP}$ .

# Example: Linear Equations over $\mathbb{F}_2$

$$D = \{0, 1\}$$

$\Delta$  consists of all relations

$$\delta_{n,\mathbf{a}}^b = \{ \langle x_1, \dots, x_n \rangle \in D^n : a_1 x_1 + \dots + a_n x_n = b \}$$

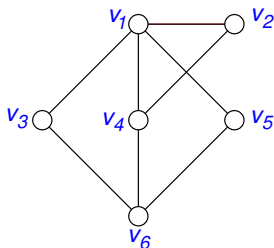
Here,  $\mathbf{a} = \langle a_1, a_2, \dots, a_n \rangle \in D^n$ ,  $b \in D$ .

Then  $\text{CSP}(\langle D, \Delta \rangle)$  is LINEAR SYSTEM

# Example: 3-colorability

$$D = \{r, g, b\}, \quad \Delta = \{\kappa_3\}$$
$$\kappa_3 = \{ (x, y) \in D : x \neq y \}$$

Then  $\text{CSP}(\langle D, \Delta \rangle)$  is the 3-colorability problem



$$V = \{v_1, \dots, v_6\}$$

$$\langle v_1, v_2 \rangle \in \kappa$$

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$$\langle v_1, v_4 \rangle \in \kappa$$

$$\langle v_2, v_4 \rangle \in \kappa$$

$$\vdots$$

$$\langle v_5, v_6 \rangle \in \kappa$$



# Example: 2-SAT

$$D = \{0, 1\}$$

For a bijunctive clause  $\phi(x, y)$ ,

$$\delta_\phi = \{ \langle a, b \rangle \in D^2 : \phi(a, b) = 1 \}$$

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0	1	0
1	0	1
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$$\Delta = \{ \delta_{x \vee y}, \delta_{x \vee y'}, \delta_{x' \vee y}, \delta_{x' \vee y'} \}$$

# Example: Horn-SAT

Horn formula:

$$(x_0 \wedge x_1 \wedge \dots \wedge x_{i-1} \wedge x_{i+1} \wedge \dots \wedge x_{n-1}) \rightarrow x_i$$

Equivalently:

$$x'_0 \vee x'_1 \vee \dots \vee x'_{i-1} \vee x_i \vee \dots \vee x'_{n-1}$$

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Horn-SAT is  $\text{CSP}\langle D, \Gamma \rangle$

$$D = \{0, 1\}, \quad \Gamma = \{ \gamma_i^n : 0 \leq i < n \}$$

# Schaefer's Dichotomy

## Theorem (Schaefer, 1978 )

*Let  $D = \{0, 1\}$ . There are six families  $\Delta_0, \dots, \Delta_5$  such that*

$$\text{CSP}(\langle D, \Delta \rangle) \in \mathbb{P} \iff \Delta \subseteq \Delta_i, \text{ some } i < 6$$

*Otherwise  $\text{CSP}(\langle D, \Delta \rangle)$  is  $\text{NP}$ -complete.*



# The six families

$$\Delta_0 = \{ \delta : \langle 0, 0, \dots, 0 \rangle \in \delta \} \text{ ("All False")}$$

$$\Delta_1 = \{ \delta : \langle 1, 1, \dots, 1 \rangle \in \delta \} \text{ ("All True")}$$

$$\Delta_2 = \{ \delta_{x \vee y}, \delta_{x \vee y'}, \delta_{x' \vee y}, \delta_{x' \vee y'} \} \text{ (bijunctive)}$$

$$\Delta_3 = \Gamma \text{ (Horn)}$$

$$\Delta_4 = \Gamma^\partial \text{ (dual-Horn)}$$

$$\Delta_5 \text{ (affine, i.e., linear system over } \mathbb{F}_2 \text{)}$$

# Two Motivating Questions

## 1 Dichotomy Conjecture

Every  $\text{CSP}(\langle D, \Delta \rangle)$  either lies in  $\mathbb{P}$  or is  $\text{NP}$ -complete.

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Characterize those CSPs that lie in  $\mathbb{P}$ .

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## 2 Tractability Problem

Characterize those CSPs that lie in  $\mathbb{P}$ .

What would a characterization look like? What language could we use?

# Why is 2-SAT tractable, but 3-SAT is not?

$$\text{2-SAT: } \Delta_2 = \{\delta_{x \vee y}, \delta_{x \vee y'}, \delta_{x' \vee y}, \delta_{x' \vee y'}\}$$

$$\text{3-SAT: } \Lambda = \{\lambda_0, \lambda_1, \dots, \lambda_7\}$$

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$$M(x, y, z) = \begin{cases} 0 & \text{if at least 2 of } x, y, z \text{ equal 0} \\ 1 & \text{otherwise} \end{cases}$$

“Majority Operation”

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$$M(x, y, z) = \begin{cases} 0 & \text{if at least 2 of } x, y, z \text{ equal 0} \\ 1 & \text{otherwise} \end{cases}$$

“Majority Operation”

$M$  preserves each  $\delta \in \Delta_2$ :

$$\begin{array}{lll} \langle a_1, & b_1 \rangle & \in \delta \\ \langle a_2, & b_2 \rangle & \in \delta \\ \langle a_3, & b_3 \rangle & \in \delta \text{ implies} \\ \langle M(a_1, a_2, a_3), & M(b_1, b_2, b_3) \rangle & \in \delta \end{array}$$

But  $M$  fails to preserve each  $\lambda \in \Lambda$

For example, with  $\lambda = \lambda_{x \vee y \vee z'} = \{0, 1\}^3 - \{\langle 001 \rangle\}$

$$\langle 1, 0, 0 \rangle \in \lambda$$

$$\langle 0, 0, 1 \rangle \in \lambda$$

$$\langle 0, 1, 1 \rangle \in \lambda \text{ but}$$

$$\langle 0, 0, 1 \rangle \notin \lambda$$



# Polymorphisms

## Definition

Let  $\delta \in \text{Rel}_k(D)$  and  $f: D^n \rightarrow D$ . We say  **$f$  preserves  $\delta$**  if

$$(a_{11}, \dots, a_{1k}), \dots, (a_{n1}, \dots, a_{nk}) \in \delta \implies \\ (f(a_{11}, \dots, a_{n1}), \dots, f(a_{1k}, \dots, a_{nk})) \in \delta$$

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$$\begin{array}{ccccccc} a_{11} & a_{12} & \dots & a_{1k} & \in & \delta \\ a_{21} & a_{22} & \dots & a_{2k} & \in & \delta \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} & \in & \delta \\ \downarrow f & \downarrow f & & \downarrow f & & \\ \star & \star & \dots & \star & \in & \delta \end{array}$$

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Important point:  $\langle D, \text{Pol}(\Delta) \rangle$  is an algebraic structure

## Theorem

*Let  $\Gamma, \Delta \subseteq \text{Rel}(D)$ . Then*

$$\text{Pol}(\Gamma) \subseteq \text{Pol}(\Delta) \implies \text{CSP}(\Delta) \leq_p \text{CSP}(\Gamma).$$

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Thus, the richer the algebraic structure, the easier the corresponding CSP



Schaefer proved that on  $D = \{0, 1\}$ , there are 4 key polymorphisms:

$M(x, y, z)$  (majority)

$$x \wedge y$$

$$x \vee y$$

$$P(x, y, z) = x \oplus y \oplus z = x - y + z$$

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Unfortunately, things become much more complicated when  $|D| > 2$ .

One can go back and forth between relational and algebraic structures

$$\begin{array}{ccc} \textbf{Relational} & & \textbf{Algebraic} \\ \langle D, \Delta \rangle & \longrightarrow & \langle D, \text{Pol}(\Delta) \rangle \\ \langle D, \text{Inv}(F) \rangle & \longleftarrow & \langle D, F \rangle \end{array}$$

$$\text{CSP}\langle D, \Delta \rangle \equiv_p \text{CSP}\langle D, \text{Inv}(\text{Pol}(\Delta)) \rangle$$

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Perhaps the expressive power of algebra can be used to classify CSPs.

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$$\phi(x_1, \dots, x_n) = (\exists y_1)(\exists y_2) \cdots (\exists y_m) (\delta_1(z_{1_1}, \dots, z_{1_k}) \wedge \dots \wedge \delta_t(z_{t_1}, \dots, z_{t_j}))$$

Here  $\delta_1 \dots, \delta_t \in \Delta$  and every  $z_{ij} \in \{x_1, \dots, x_n, y_1, \dots, y_m\}$

# Algebraic Facts

Let **A** and **B** be algebras

**B** a subalgebra of **A**  $\implies \text{CSP}(\mathbf{B}) \leq_p \text{CSP}(\mathbf{A})$ .

**B** a homomorphic image of **A**  $\implies \text{CSP}(\mathbf{B}) \leq_p \text{CSP}(\mathbf{A})$ .

$\text{CSP}(\mathbf{A}^n) \equiv_p \text{CSP}(\mathbf{A})$

Theorem (Bulatov, Jeavons, Krokhin, 2000 )

*If  $\langle D, \Delta \rangle$  is a core and every polymorphism is essentially unary, then  $\text{CSP}(\Delta)$  is  $\text{NP}$ -complete.*

$f$  is **essentially unary** if  $f(x_1, \dots, x_n) = g(x_j)$  for some unary  $g$  and some  $j \leq n$ .



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### Corollary

*3-COLORABILITY, NONLINEAR SYSTEM, and 3-SAT are  $\text{NP}$ -complete.*

## Informal reformulation of the dichotomy conjecture

If  $\mathbf{A}$  has some kind of decent algebraic structure then  $\text{CSP}(\mathbf{A}) \in \mathbb{P}$  otherwise  $\text{CSP}(\mathbf{A})$  is  $\text{NP}$ -complete.

## Definition

Let  $n > 1$ . An  $n$ -ary operation  $f$  is called a **weak near-unanimity operation** if

$$\begin{aligned} f(x, x, \dots, x) &= x \text{ and} \\ f(y, x, x, x, \dots, x) &= f(x, y, x, x, \dots, x) = \dots \\ &= f(x, x, \dots, x, y) \end{aligned}$$

Note: no essentially unary operation is WNU

Theorem (Bulatov, Larose, Zádori, McKenzie, Maróti )

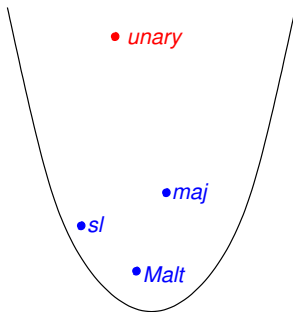
*If  $\Delta$  is a core and  $\text{Pol}(\Delta)$  has no WNU operation then  $\text{CSP}(\Delta)$  is  $\text{NP}$ -complete.*

# Reformulated Dichotomy Conjecture

Let  $\Delta$  be a core. Then  $\text{CSP}(\Delta)$  is tractable if and only if it has a WNU polymorphism. Otherwise, it is  $\text{NP}$ -complete.

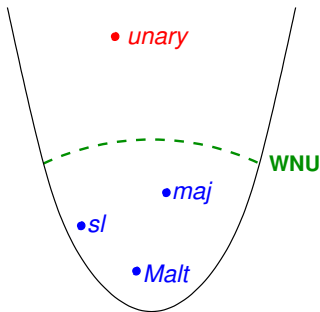
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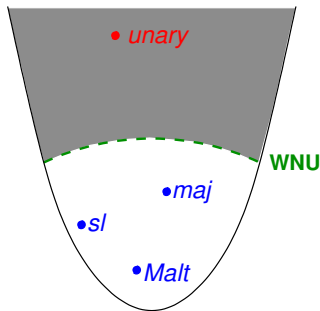
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## Supporting Examples

- 2-SAT, 2-COLORABILITY, LINEAR SYSTEM have a WNU.

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## Supporting Examples

- 2-SAT, 2-COLORABILITY, LINEAR SYSTEM have a WNU.
- Let  $\mathbf{A}$  be an abelian group,  $n = |\mathbf{A}|$ . Choose integers  $k, l$  with  $kl \equiv 1 \pmod{n}$ . Then

$$f(x_1, \dots, x_k) = l(x_1 + \dots + x_k)$$

is a WNU operation.

# Two General Techniques for Tractable Algorithms

## Method 1

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Examples of cube operations:

$$P(x, y, z) = x - y + z$$

$$M(x, y, z) = \text{majority}$$

Essentially a generalization of Gaussian elimination.

Algebras with a cube operation possess “few subpowers”. This algebraic property is used to prove that the algorithm terminates in polynomial time.

## Method 2

If  $\text{Pol}(\Delta)$  contains WNU operations  $v(x, y, z)$  and  $w(x, y, z, u)$  satisfying  $v(y, x, x) = w(y, x, x, x)$ , then  $\text{CSP}(\Delta) \in \mathbb{P}$ .

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Examples: majority, semilattice

Algebras with these operations have a property called “congruence meet-semidistributivity.”

# Current State of Affairs

The two general techniques do not cover all cases of a WNU. What to do next?

Two possible directions:

- 1 Find a completely new algorithm.
- 2 Combine the two existing algorithms.

I am exploring the second approach.