A CSP PRIMER

Let Γ be a set of relation symbols. We assume that there is a function $r\colon \Gamma \to \mathbb{Z}^+$ giving the arity of each member of Γ. We seldom mention the r. By a Γ-*structure* we mean a relational structure $\mathbb{A} = \langle A, \Gamma^{\mathbb{A}} \rangle$ in which $\Gamma^{\mathbb{A}} = \langle \gamma^{\mathbb{A}} : \gamma \in \Gamma \rangle$ and, for every $\gamma \in \Gamma$, $\gamma^{\mathbb{A}} \in \mathrm{Rel}_{r(\gamma)}(A)$. If $\Gamma_0 \subseteq \Gamma$ then $\mathbb{A}|_{\Gamma_0} = \langle A, \Gamma_0^{\mathbb{A}} \rangle$.

Definition 1. Let $\mathbb{A}$ be a Γ-structure. $\mathrm{CSP}(\mathbb{A})$ is the following problem.
Instance: A finite subset Γ_0 of Γ and a Γ_0-structure $\mathbb{B} = \langle B, \Gamma_0^{\mathbb{B}} \rangle$.
Query: Is $\mathrm{Hom}(\mathbb{B}, \mathbb{A}|_{\Gamma_0}) \neq \emptyset$?

When the subset Γ_0 is clear from $\mathbb{B}$, we often do not mention it explicitly.

We say that the Γ-structure $\mathbb{A}$ is *tractable* if $\mathrm{CSP}(\mathbb{A})$ lies in P. Sometimes an apparently weaker notion presents itself. We say that $\mathbb{A}$ is *locally tractable* if, for every finite subset Γ_0 of Γ, $\mathrm{CSP}(\mathbb{A}|_{\Gamma_0})$ is tractable. Obviously, every tractable structure is locally tractable. I believe the converse is open.

Clones and Relational Clones. Let A be a finite set. Recall the Galois correspondence between $\mathrm{Op}(A)$ and $\mathrm{Rel}(A)$ induced by the preservation relation (see pages 89–90 in my book). In the book, I call the two polarities $\mathcal{F}$ and $\mathcal{R}$. In this document, I use Pm and Inv instead. Thus

$$\mathrm{Pm}(\Gamma) = \{\, f \in \mathrm{Op}(A) : f \mathrel{|\!\!:} \Gamma \,\}$$
$$\mathrm{Inv}(F) = \{\, \gamma \in \mathrm{Rel}(A) : F \mathrel{|\!\!:} \gamma \,\}\,.$$

$\mathrm{Pm}(\Gamma)$ are the *polymorphisms* of Γ and $\mathrm{Inv}(F)$ are the relations *invariant* under F.

Under this Galois connection, the closed sets of operations are the clones. I don't talk much about the closed sets of relations in the book. Here is a brief discussion. By definition, if Γ is a set of relations, then its closure, $\overline{\Gamma} = \mathrm{Inv}(\mathrm{Pm}(\Gamma))$. We shall call this the *relational clone* generated by Γ. This definition does not help us understand how the members of the closure are built from the members of Γ.

A *primitive-positive formula* is a formula with free variables $x_1, \dots, x_n$ of the form

$$\exists y_1 \, \exists y_2 \, \dots, \exists y_m (\text{conjunction of atomic formulas})$$

An atomic formula is either an equality between variables or an expression of the form $(z_1, \dots, z_t) \in \alpha$, where the z's are variables and α is a relation. Each primitive-positive formula defines a relation consisting of those values that, when substituted for the x's, make the formula true.

For example, the formula $\exists y ((x_1, y) \in \alpha \wedge (y, x_2) \in \beta \wedge x_1 = y)$ defines the binary relation $(\alpha \cap \delta) \circ \beta$, where $\delta = \{\, (x, x) : x \in A \,\}$. Note that δ lives in every relational clone.

Theorem 2 (Bodnarčuk et al.; Geiger, 1968)**.** *Let Γ be a set of relations on a finite set A. Then $\overline{\Gamma}$ consists precisely of those relations that are pp-definable from Γ.*

Theorem 3. *Let Γ be a set of relation symbols, and $\Delta \subseteq \Gamma$.*

(1) $\mathrm{CSP}\langle A, \Delta^A \rangle \leq_p \mathrm{CSP}\langle A, \Gamma^A \rangle$.
(2) $\mathrm{CSP}\langle A, \Gamma \rangle \equiv_p \mathrm{CSP}\langle A, \overline{\Gamma} \rangle$

Definition 4. Let $\mathbb{A} = \langle A, \Gamma^{\mathbb{A}} \rangle$ be a relational structure. Then $\mathrm{Alg}(\mathbb{A}) = \langle A, \mathrm{Pm}(\Gamma^{\mathbb{A}}) \rangle$. We will typically denote this algebra as $\mathbf{A}$.

It follows from Theorem **??** that if $\mathrm{Alg}(\mathbb{A}) = \mathrm{Alg}(\mathbb{B})$ then $\mathrm{CSP}(\mathbb{A}) \equiv_p \mathrm{CSP}(\mathbb{B})$. It is because of this relationship that algebras determine the complexity of the corresponding constraint satisfaction problem.

The core of a structure.

Lemma 5. *Let S be a finite semigroup, and $a \in S$. Then there is $k > 0$ such that $a^{2k} = a^k$.*

Proof. Since S is finite, there are $m > n > 0$ such that $a^m = a^n$. Let $r = m - n$. Then it is easy to argue by induction on k that for all $k \geq n$, $a^k = a^{k+r}$. Pick $k = jr > n$ for a sufficiently large j. Then $a^{2k} = a^{k+jr} = a^k$. □

Let A be a structure (either algebraic or relational). A substructure B is a *retract* of A if there is a morphism $h\colon A \to B$ such that $h{\restriction}_B$ is the identity on B. The map h is called a *retraction.* Note that if h is a retraction map, then $h \in \mathrm{End}(A)$ and $h \circ h = h$. Conversely, if h is an idempotent endomorphism of A then $h(A)$ is a retract of A.

Lemma 6. *Let $\mathbb{A}$ be a relational structure and suppose that $\mathbb{B}$ is a retract of $\mathbb{A}$. Then $\mathrm{CSP}(\mathbb{A}) \equiv_p \mathrm{CSP}(\mathbb{B})$.*

Proof. Let h be the retraction of A onto B. Then $g \in \mathrm{Hom}(\mathbb{C}, \mathbb{A}) \implies h \circ g \in \mathrm{Hom}(\mathbb{C}, \mathbb{B})$. Conversely, if $g \in \mathrm{Hom}(\mathbb{C}, \mathbb{B})$ then $g \in \mathrm{Hom}(\mathbb{C}, \mathbb{A})$. □

A finite structure A is called *a core* if it has no proper retracts.

Lemma 7. *A structure A is a core if and only if every endomorphism is an automorphism.*

Proof. Let A be a core and suppose $h \in \mathrm{End}(A)$. By Lemma 5, there is an integer k such that $g = h^k$ is idempotent. So $g(A)$ is a retract of A. Therefore, by the assumption, $g(A) = A$. So g, hence h, is a permutation of A, i.e., $h \in \mathrm{Aut}(A)$. □

Let A be a finite structure. Then *a core of* A is a minimal retract of A. Note that if B is a core of A, then B is itself a core.

Theorem 8. *Let A be a finite structure, B_1 and B_2 cores of A. Then $B_1 \cong B_2$.*

Proof. Let h_i be the retraction onto B_i, for $i = 1, 2$. Define $g_1 = h_1{\restriction}_{B_2}$ and $g_2 = h_2{\restriction}_{B_1}$. Then $g_1 \circ g_2 \in \mathrm{End}(B_1)$. Since B_1 is a core of A we have $g_1 \circ g_2 \in \mathrm{Aut}(B_1)$. By finiteness, there is a positive integer n such that $(g_1 \circ g_2)^n = \mathrm{id}_{B_1}$. Hence $(g_1 \circ g_2)^{n-1} \circ g_1 = g_2^{-1}$ so g_2 is an isomorphism of B_1 with B_2. $\square$

$$\begin{tikzcd}
 & A \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
B_1 \arrow[rr, shift left, "g_2"] & & B_2 \arrow[ll, shift left, "g_1"]
\end{tikzcd}$$

Reducing to Idempotence. Start with a Γ-structure $\mathbb{A}$. Assume $\mathbb{A}$ is a core. thus $\mathrm{End}(\mathbb{A}) = \mathrm{Aut}(\mathbb{A})$. Let $A = \{a_1, a_2, \dots, a_k\}$. Define

$$\rho^{\mathbb{A}} = \{\, (g(a_1), \dots, g(a_k)) : g \in \mathrm{Aut}(\mathbb{A}) \,\} \in \mathrm{Rel}_k(A) \text{ and}$$
$$\delta^A = \{\, (x, x) : x \in A \,\} \in \mathrm{Rel}_2(A).$$

Lemma 9. *Both $\rho^{\mathbb{A}}$ and δ^A are members of $\overline{\Gamma^A}$ (the relational clone generated by Γ^A).*

Proof. Recall that $\overline{\Gamma^A} = \mathrm{Inv}(\mathrm{Pm}(\Gamma))$. So we want to show that if $f \in \mathrm{Pm}(\Gamma)$ then f preserves $\rho^{\mathbb{A}}$ and δ^A. For δ^A this is trivial. In fact, every operation preserves δ^A. Assume that f is n-ary and

$$\big(g_i(a_1), \dots, g_i(a_k)\big) \in \rho^{\mathbb{A}}, \text{ for } i = 1, \dots, k.$$

Then

$$\big(f(g_1(a_1), \dots, g_n(a_1)), \dots (f(g_1(a_k), \dots, g_n(a_k))\big) = (h(a_1), \dots, h(a_k)) \in \rho^{\mathbb{A}}$$

where $h = f[g_1, \dots, g_n] \in \mathrm{Pm}_1(\mathbb{A}) = \mathrm{Aut}(\mathbb{A})$. $\square$

Now, let $\Theta = \{\, \theta_a : a \in A \,\}$ be a set of new unary relation symbols. Let us define two new structures

$$\mathbb{A}^+ = \langle A, \Gamma^A \cup \Theta^A \rangle \quad \text{where } \theta_a^{\mathbb{A}^+} = \{a\}$$
$$\mathbb{A}' = \langle A, \Gamma^A \cup \{\delta^A, \rho^{\mathbb{A}}\} \rangle.$$

Thus $\mathbb{A}^+$ is a $(\Gamma \cup \Theta)$-structure and $\mathbb{A}'$ is a $(\Gamma \cup \{\delta, \rho\})$-structure. We shall show that

$$\mathrm{CSP}(\mathbb{A}') \leq_p \mathrm{CSP}(\mathbb{A}) \leq_p \mathrm{CSP}(\mathbb{A}^+) \leq_p \mathrm{CSP}(\mathbb{A}') \tag{1}$$

from which it will follow that $\mathbb{A}$ and $\mathbb{A}^+$ yield polynomially equivalent constraint satisfaction problems.

The first reduction follows from the fact that δ^A and ρ^A lie in the relational clone generated by Γ^A. The second reduction is trivially true. So only the third needs work.

Consider an instance $\left[\Gamma_0^{\mathbb{B}} \cup \Theta^{\mathbb{B}}, \mathbb{B}\right]$ of $\mathrm{CSP}(\mathbb{A}^+)$. We shall construct, in polynomial time, an instance $\left[\Gamma_0^B \cup \{\delta^{\mathbb{B}'}, \rho^{\mathbb{B}'}\}, \mathbb{B}'\right]$ such that

$$\mathrm{Hom}(\mathbb{B}, \mathbb{A}^+|_{\Gamma_0 \cup \Theta}) \neq \emptyset \iff \mathrm{Hom}(\mathbb{B}', \mathbb{A}'|_{\Gamma_0, \cup\{\delta,\rho\}}) \neq \emptyset. \tag{2}$$

First, let $Y = \{y_1, \dots, y_k\}$ be a set disjoint from B. (Recall that we assumed at the outset that $A = \{a_1, \dots, a_k\}$.) Set $B' = B \cup Y$ and $\rho^{\mathbb{B}'} = \{(y_1, \dots, y_k)\}$. Now, for each $i \leq k$ recall that θ_{a_i} is a unary relation symbol. Thus $\theta_{a_i}^{\mathbb{B}}$ must be a unary relation on B, say $\theta_{a_i}^{\mathbb{B}} = \{b_1, \dots, b_m\}$. Define $\pi_i = \{(b_1, y_i), (b_2, y_i), \dots, (b_m, y_i)\}$. Finally, define $\delta^{\mathbb{B}'} = \cup_i \pi_i$. Note that $\delta^{\mathbb{B}'}$ is a binary relation, as it should be.

Now we must verify the equivalence in (2). Suppose first that $f \in \mathrm{Hom}(\mathbb{B}, \mathbb{A}^+)$. Extend f to a mapping f' on B' by defining $f'(y_i) = a_i$ for $i \leq k$. Since f preserves the relations in Γ_0, so does f'. For ρ,

$$(y_1, \dots, y_k) \in \rho^{\mathbb{B}'} \implies (f'(y_1), \dots, f'(y_k)) = (a_1, \dots, a_k) \in \rho^{\mathbb{A}}$$

as desired. Finally to see that f' preserves δ,

$$(b, y_i) \in \delta^{\mathbb{B}'} \implies b \in \theta_{a_i}^{\mathbb{B}} \implies f(b) \in \theta_{a_i}^{\mathbb{A}} \implies$$
$$f'(b) = f(b) = a_i \implies (f'(b), f'(y_i)) \in \delta^A.$$

Thus f' is a homomorphism from $\mathbb{B}'$ to $\mathbb{A}'$.

Conversely, let $f \in \mathrm{Hom}(\mathbb{B}', \mathbb{A}')$. Since f preserves ρ and $\rho^{\mathbb{B}'} = \{(y_1, \dots, y_k)\}$, we must have $(f(y_1), \dots, f(y_k)) \in \rho^{\mathbb{A}}$. Hence, there is $g \in \mathrm{Aut}(\mathbb{A})$ such that $f(y_i) = g(a_i)$, for $i = 1, \dots, k$. Let $h = g^{-1} \circ f$. Thus $h(y_i) = a_i$, for $i \leq k$. We shall show that h is a homomorphism from $\mathbb{B}$ to $\mathbb{A}^+$.

Since both f and g preserve Γ_0, so does h. To show that h preserves Θ, let $i \leq k$ and $b \in \theta_{a_i}$. Then $(b, y_i) \in \delta^{\mathbb{B}'}$, so $(f(b), f(y_i)) \in \delta^A$ by assumption. Therefore $f(b) = f(y_i)$. Thus $h(b) = h(y_i) = a_i$, which means that h preserves θ_{a_i}.

Putting all of this together, we have proved the following theorem.

Theorem 10. *If $\mathbb{A}$ is a finite core, then* $\mathrm{CSP}(\mathbb{A}) \equiv_p \mathrm{CSP}(\mathbb{A}^+)$.