FILTER MEMBERSHIP OF COATOMS IN A PARTITION LATTICE IS NP-COMPLETE

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ABSTRACT. We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice, so the latter problem is NP-complete. We conclude with a discussion of the tractability of such filter membership problems.

1. INTRODUCTION

We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice. Thus the latter problem, which we call the *covered coatoms problem* (CCP), is NP-complete. To prove this, we show that NAE-3SAT reduces to CCP. We conclude with a discussion of the tractability of such a filter membership problem. In particular, we describe some first steps toward a polynomial-time algorithm for solving instances of CCP.

1.1. **Definition of 3SAT.** Although there are other reasonable conventions, in this paper we will take the problem 3SAT to be defined as follows: an instance is specified by a finite set $V = \{v_0, v_1, \dots, v_{n-1}\}$ of *variables* and a formula ϕ in conjunctive normal form over V. To be more precise, we define a *literal* to be a variable or the negation of a variable, and we specify an instance of 3SAT by giving a formula ϕ that consists of a conjunction of clauses, where each clause is a disjunction of *exactly three distinct literals*. When ϕ is defined over the set $V = \{v_0, v_1, \dots, v_{n-1}\}$ of *variables*, we may denote the instance by $\phi(v_0, v_1, \dots, v_{n-1})$. Of course, we may assume that a variable and its negation do not both appear in the same clause, since such clauses are trivially satisfied.

A *solution* to the instance ϕ is a function $f : V \to \{0, 1\}$ that assigns the value 0 or 1 to each variable in such a way that the formula ϕ evaluates to true under this assignment. That is, $\phi(f(v_0), f(v_1), \dots, f(v_{n-1})) = 1$.

Example 1.1. Consider the following (non-satisfiable) instance of 3SAT: let $V = \{v_0, v_1, v_2\}$ be the set of variables that appear in the formula ϕ and let

$$\begin{aligned}\phi(v_0, v_1, v_2) = (v_0 \vee v_1 \vee v_2) \wedge (\neg v_0 \vee v_1 \vee v_2) \wedge (v_0 \vee \neg v_1 \vee v_2) \wedge (v_0 \vee v_1 \vee \neg v_2)\\ \wedge (\neg v_0 \vee \neg v_1 \vee v_2) \wedge (\neg v_0 \vee v_1 \vee \neg v_2) \wedge (v_0 \vee \neg v_1 \vee \neg v_2)\end{aligned}$$

A solution is a function $f : \{v_0, v_1, v_2\} \to \{0, 1\}$, such that $\phi(f(v_0), f(v_1), f(v_2)) = 1$. It's obvious that such a function does not exist since each of the eight possible choices for $(f(v_0), f(v_1), f(v_2))$ results in a failure of one of the clauses of ϕ. (If $(f(v_0), f(v_1), f(v_2)) = (0, 0, 0)$, then the first clause fails; if $(1, 0, 0)$, then the second fails, and so on.)

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1.2. **Definition of NAE-3SAT and naen!-3SAT.** We take the problem NAE-3SAT to be defined as follows: an instance is specified by a finite set $V = \{v_0, v_1, \dots, v_{n-1}\}$ of *variables* and a ternary relation $\rho \subseteq V \times V \times V$, where we assume for every $(v_{i0}, v_{i1}, v_{i2}) \in \rho$ that v_{i0}, v_{i1}, and v_{i2} are distinct positive literals (i.e., non-negated variables). A *solution* to such an instance is a function $f : V \to \{0, 1\}$ satisfying the following: for all $(v_{i0}, v_{i1}, v_{i2}) \in \rho$ we have $(f(v_{i0}), f(v_{i1}), f(v_{i2})) \notin \{(0, 0, 0), (1, 1, 1)\}$.

A variation on this definition of NAE-3SAT allows negations of variables to appear in the tuples in ρ. Since this variation of the problem will be useful below, we give it a name— **naen!**-3SAT (not-all-equal-with-negation 3SAT),

Example 1.2. Consider the following (non-satisfiable) instance of **naen!**-3SAT: let $V = \{v_0, v_1, v_2\}$ and define

$$\rho(v_0, v_1, v_2) = \{(v_0, v_1, v_2), (\neg v_0, v_1, v_2), (v_0, \neg v_1, v_2), (v_0, v_1, \neg v_2)\}.$$

Then $\rho(v_0, v_1, v_2)$ is not satisfiable because in order to make it so, we would need $(v_0, v_1, v_2) \notin \{(0, 0, 0), (1, 1, 1)\}$ so that the first clause is NAE, and $(v_0, v_1, v_2) \notin \{(1, 0, 0), (0, 1, 1,)\}$ so that the second clause is NAE, and $(v_0, v_1, v_2) \notin \{(0, 1, 0), (1, 0, 1)\}$ so that the third clause is NAE, and $(v_0, v_1, v_2) \notin \{(0, 0, 1), (1, 1, 0)\}$.

2. Reductions

2.1. **Reduction of naen!-3SAT to NAE-3SAT.** An instance of **naen!**-3SAT can be reduced (in polynomial time) to an equivalent instance of NAE-3SAT, as we now explain. Let $\rho(v_0, v_1, \dots, v_{n-1})$ be an instance of **naen!**-3SAT over the variable set $V = \{v_0, v_1, \dots, v_{n-1}\}$. Introduce the variable sets $X = \{x_0, x_1, \dots, x_{n-1}\}$ and $Y = \{y_0, y_1, \dots, y_{n-1}\}$, where $x_i = v_i$ and $y_i = \neg v_i$. Introduce $Z = \{z, z'\}$ and $W = \{w, w', w''\}$. Let $U = X \cup Y \cup Z \cup W$. We will define an NAE-3SAT instance ρ' over the variable set U which has a solution if and only if ρ has a solution. We define this instance as follows: let $\rho' \subseteq U \times U \times U$ and for each triple in ρ, we assume that the same triple is contained in ρ' (but v_i is denoted by x_i and $\neg v_i$ is denoted by y_i). Next, for each i we let $(x_i, y_i, z) \in \rho'$ and $(x_i, y_i, z') \in \rho'$. Lastly, let

$$\{(z, z', w), (z, z', w'), (z, z', w')\} \subset \rho' \quad \text{and} \tag{2.1}$$

$$(w, w', w'') \in \rho'. \tag{2.2}$$

Because of (**??**) a solution to the **naen!**-3SAT instance ρ' assigns at least one 0 and at least one 1 to the variables in W. This and (**??**) imply that a solution must assign distinct values to z and z', and this in turn implies that, for each i, the variables x_i and y_i must be assigned distinct values. Thus, y_i plays the role of $\neg x_i$, as desired.

2.2. **Reduction of 3SAT to naen!-3SAT.** In this section we describe a (polynomial-time) reduction from a 3SAT instance to an **naen!**-3SAT instance. Because of the reduction described in Section **??** above, this will prove that a 3SAT instance can be reduced to an NAE-3SAT instance.

We will now decribe a procedure that takes an arbitrary instance ϕ of 3SAT and transforms it into an instance R of **naen!**-3SAT such that ϕ has a solution iff R has a solution.

Let $V = \{v_0, v_1, \dots, v_{n-1}\}$ be the set of variables and denote the given 3SAT instance as follows:

$$\phi = (x_0 \vee y_0 \vee z_0) \wedge (x_1 \vee y_1 \vee z_1) \wedge \cdots (x_{p-1} \vee y_{p-1} \vee z_{p-1}),$$

where, for each $0 \le i < p$, the literals x_i, y_i, and z_i each belong to the set

$$V \cup \neg V := \{v_0, v_1, \ldots, v_{n-1}\} \cup \{\neg v_0, \neg v_1, \ldots, \neg v_{n-1}\}.$$

We transform ϕ into an **naen!**-3SAT instance as follows: for each $0 \le i < p$, introduce new variables w_i and w_i', and map the clause $x_i \vee y_i \vee z_i$ to the set

$$S_i = \{(x_i, y_i, w_i), (y_i, z_i, w_i'), (w_i, w_i', 1)\}.$$

If S_i is a subset of a relation $R \subseteq (V \cup \neg V)^3$, and if R is an **naen!**-3SAT instance over the variable set $W := V \cup \{w_i, w_i' \mid 0 \le i < p\}$, then a solution to R cannot assign both w_i and w_i' the value 1 (otherwise $(w_i, w_i', 1)$ would be mapped to $(1, 1, 1)$). Now, consider the **naen!**-3SAT instance given by the relation $R = S_0 \cup S_1 \cup \cdots \cup S_{p-1}$. An assignment $f : V \to \{0, 1\}$ is a solution to ϕ iff for all $0 \le i < p$ we have $(f(x_i), f(y_i), f(z_i)) \ne (0, 0, 0)$. Let $\tilde{f} : W \to \{0, 1\}$ be define by

$$\tilde{f}(u) = \begin{cases} f(u), & \text{if } u \in V \cup \neg V, \\ 1 - \delta, & \text{if } u = w_i \text{ and } f(x_i) = f(y_i) = \delta, \\ 1 - \delta, & \text{if } u = w_i' \text{ and } f(z_i) = f(y_i) = \delta. \end{cases}$$

2.3. **Reduction of NAE-3SAT to CCP.** Consider the relational structure $\mathbb{B} = \langle \{0, 1\}, R \rangle$ where R is the ternary relation $\{0, 1\}^3 - \{(0, 0, 0), (1, 1, 1)\}$. That is,

$$R = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0)\}.$$

The CSP associated with the relational structure $\mathbb{B}$ is denoted by CSP($\mathbb{B}$) and described as follows: An *instance* is a (finite) relational structure $\mathbb{A} = \langle A, S \rangle$ with a single ternary relation S, and CSP($\mathbb{B}$) is the following decision problem:

> **Problem.** Given an instance $\mathbb{A} = \langle A, S \rangle$, does there exist a (relational structure) homomorphism from $\mathbb{A}$ to $\mathbb{B}$?

In other words, does there exist a function $f \colon A \to \{0, 1\}$ such that $(f(a), f(b), f(c)) \in R$ whenever $(a, b, c) \in S$?

The kernel of a function f with codomain $\{0, 1\}$ has two equivalence classes—namely, $f^{-1}\{0\}$ and $f^{-1}\{1\}$. If one of these classes is empty, then f is constant in which case it cannot be a homomorphism into the relational structure $\langle \{0, 1\}, R \rangle$ (since $(0, 0, 0) \notin R$ and $(1, 1, 1) \notin R$). Therefore, the kernel of every homomorphism $f \colon \mathbb{A} \to \mathbb{B}$ has two nonempty blocks.

Now, given a partition of A into two blocks, $\pi = |A_1|A_2|$, there are exactly two functions of type $A \to \{0, 1\}$ with kernel π. One is $f(x) = 0$ iff $x \in A_1$ and the other is $1 - f$. It is obvious that either both f and $1 - f$ are homomorphisms or neither f nor $1 - f$ is a homomorphism. Indeed, both are homomorphisms if and only if for all tuples $(a, b, c) \in S$ we have $\{a, b, c\} \nsubseteq A_1$ and $\{a, b, c\} \nsubseteq A_2$. Neither is a homomorphism if and only if there exists $(a, b, c) \in S$ with $\{a, b, c\} \subseteq A_1$ or $\{a, b, c\} \subseteq A_2$.

Now, for each tuple $s = (a, b, c) \in S$, we let $\operatorname{im}(s)$ (or simply $\operatorname{im} s$) denote the image of $\{0, 1, 2\}$ under s (viewing the sequence s as a function with domain the "index set" $\{0, 1, 2\}$ and codomain the set A). Furthermore, we let $\langle \operatorname{im} s \rangle$ denote the equivalence relation on A generated by $\operatorname{im} s$. Thus, if $s = (a, b, c)$, then

$$\langle \operatorname{im} s \rangle = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\} \cup \{(x, x) : x \in A\}.$$

The partition corresponding to $\langle \operatorname{im} s \rangle$ is $\pi_{\langle \operatorname{im} s \rangle} = |a, b, c|x_1|x_2| \cdots$. It is clear that a function $f \colon A \to \{0, 1\}$ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if for all $s \in S$ the relation

$\langle \operatorname{im} s\rangle$ does not belong to the kernel of f. Therefore, a solution to the instance $\mathbb{A} = \langle A, S\rangle$ of CSP($\mathbb{B}$) exists if and only if there is at least one coatom in the lattice of equivalence relations of A that is not contained in the union $\bigcup_{s\in S} {\uparrow}\langle \operatorname{im} s\rangle$ of principal filters.

The *covered coatoms problem* (CCP) is the following: Given a set A and a list $s_1 = (a_1, b_1, c_1), s_2 = (a_2, b_2, c_2), \dots, s_n = (a_n, b_n, c_n)$ of triples with elements in A, decide whether all of the coatoms of the lattice $\prod_A$ of partitions of A are contained in the union $\bigcup_{i=1}^{n} {\uparrow}\langle \operatorname{im} s_i\rangle$ of principal filters.

3. Algorithms

We now discuss the tractability of the filter membership problem described in Section **??**. Specifically, we describe some first steps toward a polynomial-time algorithm for solving instances of CCP.

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