

# FILTER MEMBERSHIP OF COATOMS IN A PARTITION LATTICE IS NP-COMPLETE

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**ABSTRACT.** We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice, so the latter problem is NP-complete. We conclude with a discussion of the tractability of such filter membership problems.

## 1. INTRODUCTION

We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice. Thus the latter problem, which we call the *covered coatoms problem* (CCP), is NP-complete. To prove this, we show that NAE-3SAT reduces to CCP. We conclude with a discussion of the tractability of such a filter membership problem. In particular, we describe some first steps toward a polynomial-time algorithm for solving instances of CCP.

### 1.1. Reduction of 3SAT to NAE-3SAT.

(This section is under major revision.)

**1.2. Reduction of NAE-3SAT to CCP.** Consider the relational structure  $\mathbb{B} = \langle \{0, 1\}, R \rangle$  where  $R$  is the ternary relation  $\{0, 1\}^3 - \{(0, 0, 0), (1, 1, 1)\}$ . That is,

$$R = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0)\}.$$

The CSP associated with the relational structure  $\mathbb{B}$  is denoted by  $\text{CSP}(\mathbb{B})$  and described as follows: An *instance* is a (finite) relational structure  $\mathbb{A} = \langle A, S \rangle$  with a single ternary relation  $S$ , and  $\text{CSP}(\mathbb{B})$  is the following decision problem:

**Problem.** Given an instance  $\mathbb{A} = \langle A, S \rangle$ , does there exist a (relational structure) homomorphism from  $\mathbb{A}$  to  $\mathbb{B}$ ?

In other words, does there exist a function  $f: A \rightarrow \{0, 1\}$  such that  $(f(a), f(b), f(c)) \in R$  whenever  $(a, b, c) \in S$ ?

The kernel of a function  $f$  with codomain  $\{0, 1\}$  has two equivalence classes—namely,  $f^{-1}\{0\}$  and  $f^{-1}\{1\}$ . If one of these classes is empty, then  $f$  is constant in which case it cannot be a homomorphism into the relational structure  $\langle \{0, 1\}, R \rangle$  (since  $(0, 0, 0) \notin R$  and  $(1, 1, 1) \notin R$ ). Therefore, the kernel of every homomorphism  $f: \mathbb{A} \rightarrow \mathbb{B}$  has two nonempty blocks.

Now, given a partition of  $A$  into two blocks,  $\pi = |A_1|A_2|$ , there are exactly two functions of type  $A \rightarrow \{0, 1\}$  with kernel  $\pi$ . One is  $f(x) = 0$  iff  $x \in A_1$  and the other is  $1 - f$ . It is obvious that either both  $f$  and  $1 - f$  are homomorphisms or neither  $f$  nor  $1 - f$  is a homomorphism. Indeed, both are homomorphisms if and only if for all tuples  $(a, b, c) \in S$

we have  $\{a, b, c\} \not\subseteq A_1$  and  $\{a, b, c\} \not\subseteq A_2$ . Neither is a homomorphism if and only if there exists  $(a, b, c) \in S$  with  $\{a, b, c\} \subseteq A_1$  or  $\{a, b, c\} \subseteq A_2$ .

Now, for each tuple  $s = (a, b, c) \in S$ , we let  $\text{im}(s)$  (or simply  $\text{im } s$ ) denote the image of  $\{0, 1, 2\}$  under  $s$  (viewing the sequence  $s$  as a function with domain the “index set”  $\{0, 1, 2\}$  and codomain the set  $A$ ). Furthermore, we let  $\langle \text{im } s \rangle$  denote the equivalence relation on  $A$  generated by  $\text{im } s$ . Thus, if  $s = (a, b, c)$ , then

$$\langle \text{im } s \rangle = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\} \cup \{(x, x) : x \in A\}.$$

The partition corresponding to  $\langle \text{im } s \rangle$  is  $\pi_{\langle \text{im } s \rangle} = |a, b, c| x_1 |x_2| \cdots$ . It is clear that a function  $f: A \rightarrow \{0, 1\}$  is a homomorphism from  $\mathbb{A}$  to  $\mathbb{B}$  if and only if for all  $s \in S$  the relation  $\langle \text{im } s \rangle$  does not belong to the kernel of  $f$ . Therefore, a solution to the instance  $\mathbb{A} = \langle A, S \rangle$  of  $\text{CSP}(\mathbb{B})$  exists if and only if there is at least one coatom in the lattice of equivalence relations of  $A$  that is not contained in the union  $\bigcup_{s \in S} \uparrow \langle \text{im } s \rangle$  of principal filters.

The *covered coatoms problem* (CCP) is the following: Given a set  $A$  and a list  $s_1 = (a_1, b_1, c_1), s_2 = (a_2, b_2, c_2), \dots, s_n = (a_n, b_n, c_n)$  of triples with elements in  $A$ , decide whether all of the coatoms of the lattice  $\prod_A$  of partitions of  $A$  are contained in the union  $\bigcup_{i=1}^n \uparrow \langle \text{im } s_i \rangle$  of principal filters.

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