FILTER MEMBERSHIP OF COATOMS IN A PARTITION LATTICE IS NP-COMPLETE

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ABSTRACT. We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice, so the latter problem is NP-complete. We conclude with a discussion of the tractability of such filter membership problems.

1. Introduction

We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice. Thus the latter problem, which we call the *covered coatoms problem* (CCP), is NP-complete. To prove this, we show that NAE-3SAT reduces to CCP. We conclude with a discussion of the tractability of such a filter membership problem. In particular, we describe some first steps toward a polynomial-time algorithm for solving instances of CCP.

1.1. **Reduction of 3SAT to NAE-3SAT.** The problem *not-all-equal satisfiability* (NAE-SAT) is a variation on SAT in which a formula is satisfied if and only if each clause contains at least one true literal and one false literal. Thus, a clause is satisfied if and only if it is not the case that all literals in that clause have the same value (hence, "not-all-equal"). The problem NAE-3SAT is the special case of NAE-SAT in which each cluase has exactly three literals.

We first demostrate the well known fact that NAE-3SAT is NP-complete. It is obviously in NP because a truth assignment is a certificate that can be confirmed or denied in polynomial-time. On the other hand, NAE-3SAT is NP-hard because 3SAT reduces to it, as we now verify.

Introducd two new variables, u and v. Given a clause $x_1 \vee x_2 \vee x_3$ in a 3SAT formula, replace x_i with y_i and $\neg x_i$ with $\neg y_i$ is such a way that $x_i = T$ if and only if $y_i \neq u$ and $x_i = F$ if and only if $y_i = u$. This results in the NAE-4SAT clause (y_1, y_2, y_3, u). This reduction works because (y_1, y_2, y_3, u) is satisfied if and only if at least one of the y_i's is different from u, which holds if and only if at least one member of $\{x_1, x_2, x_3\}$ is true. This shows that 3SAT is reducible to NAE-4SAT, so the latter is NP-hard.

To reduce NAE-4SAT to NAE-3SAT, we use our second reference variable, v. Given a clause (y_1, y_2, y_3, u) in an NAE-4SAT formula, we use v and $\neg v$ accordingly to reduce it to two 3SAT clauses

$$(y_1, y_2, v) \wedge (y_3, u, \neg v).$$

This reduction works because appropriate v and $\neg v$ in each clause will prevent all-three-true or all-three-false clauses.

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1.2. **Reduction of NAE-3SAT to CCP.** Consider the relational structure $\mathbb{B} = \langle \{0,1\}, R\rangle$ where R is the ternary relation $\{0,1\}^3 - \{(0,0,0),(1,1,1)\}$. That is,

$$R = \{(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0)\}.$$

The CSP associated with the relational structure $\mathbb{B}$ is denoted by CSP($\mathbb{B}$) and described as follows: An *instance* is a (finite) relational structure $\mathbb{A} = \langle A, S\rangle$ with a single ternary relation S, and CSP($\mathbb{B}$) is the following decision problem:

> **Problem.** Given an instance $\mathbb{A} = \langle A, S\rangle$, does there exist a (relational structure) homomorphism from $\mathbb{A}$ to $\mathbb{B}$?

In other words, does there exist a function $f\colon A \to \{0,1\}$ such that $(f(a), f(b), f(c)) \in R$ whenever $(a,b,c) \in S$?

The kernel of a function f with codomain $\{0,1\}$ has two equivalence classes—namely, $f^{-1}\{0\}$ and $f^{-1}\{1\}$. If one of these classes is empty, then f is constant in which case it cannot be a homomorphism into the relational structure $\langle\{0,1\}, R\rangle$ (since $(0,0,0) \notin R$ and $(1,1,1) \notin R$). Therefore, the kernel of every homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ has two nonempty blocks.

Now, given a partition of A into two blocks, $\pi = |A_1|A_2|$, there are exactly two functions of type $A \to \{0,1\}$ with kernel π. One is $f(x) = 0$ iff $x \in A_1$ and the other is $1 - f$. It is obvious that either both f and $1-f$ are homomorphisms or neither f nor $1-f$ is a homomorphism. Indeed, both are homomorphisms if and only if for all tuples $(a,b,c) \in S$ we have $\{a,b,c\} \nsubseteq A_1$ and $\{a,b,c\} \nsubseteq A_2$. Neither is a homomorphism if and only if there exists $(a,b,c) \in S$ with $\{a,b,c\} \subseteq A_1$ or $\{a,b,c\} \subseteq A_2$.

Now, for each tuple $s = (a,b,c) \in S$, we let $\operatorname{im}(s)$ (or simply $\operatorname{im} s$) denote the image of $\{0,1,2\}$ under s (viewing the sequence s as a function with domain the "index set" $\{0,1,2\}$ and codomain the set A). Furthermore, we let $\langle \operatorname{im} s\rangle$ denote the equivalence relation on A generated by $\operatorname{im} s$. Thus, if $s = (a,b,c)$, then

$$\langle \operatorname{im} s\rangle = \{(a,b),(b,a),(a,c),(c,a),(b,c),(c,b)\} \cup \{(x,x) : x \in A\}.$$

The partition corresponding to $\langle \operatorname{im} s\rangle$ is $\pi_{\langle \operatorname{im} s\rangle} = |a,b,c|x_1|x_2|\cdots$. It is clear that a function $f\colon A \to \{0,1\}$ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if for all $s \in S$ the relation $\langle \operatorname{im} s\rangle$ does not belong to the kernel of f. Therefore, a solution to the instance $\mathbb{A} = \langle A, S\rangle$ of CSP($\mathbb{B}$) exists if and only if there is at least one coatom in the lattice of equivalence relations of A that is not contained in the union $\bigcup_{s\in S} {\uparrow}\langle \operatorname{im} s\rangle$ of principal filters.

The *covered coatoms problem* (CCP) is the following: Given a set A and a list $s_1 = (a_1,b_1,c_1), s_2 = (a_2,b_2,c_2), \dots, s_n = (a_n,b_n,c_n)$ of triples with elements in A, decide whether all of the coatoms of the lattice $\prod_A$ of partitions of A are contained in the union $\bigcup_{i=1}^n {\uparrow}\langle \operatorname{im} s_i\rangle$ of principal filters.

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