FILTER MEMBERSHIP OF COATOMS IN A PARTITION LATTICE IS NP-COMPLETE

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Abstract. We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice, so the latter problem is NP-complete. We conclude with a discussion of the tractability of such filter membership problems.

1. Introduction

We show that 3SAT reduces to the problem of deciding whether all coatoms in a certain partition lattice are contained in the union of a collection of certain principal filters of the lattice. Thus the latter problem, which we call the *covered coatoms problem* (CCP), is NP-complete. To prove this, we show that NAE-3SAT reduces to CCP. We conclude with a discussion of the tractability of such a filter membership problem. In particular, we describe some first steps toward a polynomial-time algorithm for solving instances of CCP.

1.1. Definition of 3SAT.

Although there are other reasonable conventions, in this paper we will take the problem 3SAT to be defined as follows: an instance is specified by a finite set $V = \{v_0, v_1, \dots, v_{n-1}\}$ of *variables* and a formula ϕ in conjunctive normal form over V. To be more precise, we define a *literal* to be a variable or the negation of a variable, and we specify an instance of 3SAT by giving a formula ϕ that consists of a conjunction of clauses, where each clause is a disjunction of *exactly three distinct literals*. When ϕ is defined over the set $V = \{v_0, v_1, \dots, v_{n-1}\}$ of *variables*, we may denote the instance by $\phi(v_0, v_1, \dots, v_{n-1})$. Of course, we may assume that a variable and its negation do not both appear in the same clause, since such clauses are trivially satisfied.

A *solution* to the instance ϕ is a function $f : V \to \{0, 1\}$ that assigns the value 0 or 1 to each variable in such a way that the formula ϕ evaluates to true under this assignment. That is, $\phi(f(v_0), f(v_1), \dots, f(v_{n-1})) = 1$.

Example 1.1. Consider the following (non-satisfiable) instance of 3SAT: let $V = \{v_0, v_1, v_2\}$ be the set of variables that appear in the formula ϕ and let

$$\begin{aligned}\phi(v_0, v_1, v_2) = (v_0 \vee v_1 \vee v_2) \wedge (\neg v_0 \vee v_1 \vee v_2) \wedge (v_0 \vee \neg v_1 \vee v_2) \wedge (v_0 \vee v_1 \vee \neg v_2)\\ \wedge (\neg v_0 \vee \neg v_1 \vee v_2) \wedge (\neg v_0 \vee v_1 \vee \neg v_2) \wedge (v_0 \vee \neg v_1 \vee \neg v_2)\end{aligned}$$

A solution is a function $f : \{v_0, v_1, v_2\} \to \{0, 1\}$, such that $\phi(f(v_0), f(v_1), f(v_2)) = 1$. It's obvious that such a function does not exist since each of the eight possible choices for $(f(v_0), f(v_1), f(v_2))$ results in a failure of one of the clauses of ϕ. (If $(f(v_0), f(v_1), f(v_2)) = (0, 0, 0)$, then the first clause fails; if $(1, 0, 0)$, then the second fails, and so on.)

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1.2. **Definition of NAE-3SAT and NAEN-3SAT.** We take the problem NAE-3SAT to be defined as follows: an instance is specified by a finite set $V = \{v_0, v_1, \dots, v_{n-1}\}$ of *variables* and a ternary relation $\rho \subseteq V \times V \times V$, where we assume for every $(v_{i0}, v_{i1}, v_{i2}) \in \rho$ that v_{i0}, v_{i1}, and v_{i2} are distinct positive literals (i.e., non-negated variables). A *solution* to such an instance is a function $f : V \to \{0, 1\}$ satisfying the following: for all $(v_{i0}, v_{i1}, v_{i2}) \in \rho$ we have $(f(v_{i0}), f(v_{i1}), f(v_{i2})) \notin \{(0, 0, 0), (1, 1, 1)\}$.

A variation on this definition of NAE-3SAT allows negations of variables to appear in the tuples in ρ. Since this variation of the problem will be useful below, we give it a name—NAEN-3SAT (not-all-equal-with-negation 3SAT),

Example 1.2. Consider the following (non-satisfiable) instance of NAEN-3SAT: let $V = \{v_0, v_1, v_2\}$ and define

$$\rho(v_0, v_1, v_2) = \{(v_0, v_1, v_2), (\neg v_0, v_1, v_2), (v_0, \neg v_1, v_2), (v_0, v_1, \neg v_2)\}.$$

Then $\rho(v_0, v_1, v_2)$ is not satisfiable because in order to make it so, we would need $(v_0, v_1, v_2) \notin \{(0, 0, 0), (1, 1, 1)\}$ so that the first clause is NAE, and $(v_0, v_1, v_2) \notin \{(1, 0, 0), (0, 1, 1,)\}$ so that the second clause is NAE, and $(v_0, v_1, v_2) \notin \{(0, 1, 0), (1, 0, 1)\}$ so that the third clause is NAE, and $(v_0, v_1, v_2) \notin \{(0, 0, 1), (1, 1, 0)\}$.

2. Reductions

2.1. **Reduction of NAEN-3SAT to NAE-3SAT.** An instance of NAEN-3SAT can be reduced (in polynomial time) to an equivalent instance of NAE-3SAT, as we now explain. Let $\rho(v_0, v_1, \dots, v_{n-1})$ be an instance of NAEN-3SAT over the variable set $V = \{v_0, v_1, \dots, v_{n-1}\}$. Introduce the variable sets $X = \{x_0, x_1, \dots, x_{n-1}\}$ and $Y = \{y_0, y_1, \dots, y_{n-1}\}$, where $x_i = v_i$ and $y_i = \neg v_i$. Introduce $Z = \{z, z'\}$ and $W = \{w, w', w''\}$. Let $U = X \cup Y \cup Z \cup W$. We will define an NAE-3SAT instance ρ' over the variable set U which has a solution if and only if ρ has a solution. We define this instance as follows: let $\rho' \subseteq U \times U \times U$ and for each triple in ρ, we assume that the same triple is contained in ρ' (but v_i is denoted by x_i and $\neg v_i$ is denoted by y_i). Next, for each i we let $(x_i, y_i, z) \in \rho'$ and $(x_i, y_i, z') \in \rho'$. Lastly, let

$$\{(z, z', w), (z, z', w'), (z, z', w')\} \subset \rho' \quad \text{and} \tag{2.1}$$

$$(w, w', w'') \in \rho'. \tag{2.2}$$

Because of (2.2) a solution to the NAEN-3SAT instance ρ' assigns at least one 0 and at least one 1 to the variables in W. This and (2.1) imply that a solution must assign distinct values to z and z', and this in turn implies that, for each i, the variables x_i and y_i must be assigned distinct values. Thus, y_i plays the role of $\neg x_i$, as desired.

2.2. **Reduction of 3SAT to NAEN-3SAT.** In this section we describe a (polynomial-time) reduction from a 3SAT instance to an NAEN-3SAT instance. Because of the reduction described in Section 2.1 above, this will prove that a 3SAT instance can be reduced to an NAE-3SAT instance.

We will now decribe a procedure that takes an arbitrary instance ϕ of 3SAT and transforms it into an instance R of NAEN-3SAT such that ϕ has a solution iff R has a solution.

Let $V = \{v_0, v_1, \dots, v_{n-1}\}$ be the set of variables and denote the given 3SAT instance as follows:

$$\phi = (x_0 \vee y_0 \vee z_0) \wedge (x_1 \vee y_1 \vee z_1) \wedge \cdots (x_{p-1} \vee y_{p-1} \vee z_{p-1}),$$

where, for each $0 \le i < p$, the literals x_i, y_i, and z_i each belong to the set

$$V \cup \neg V := \{v_0, v_1, \dots, v_{n-1}\} \cup \{\neg v_0, \neg v_1, \dots, \neg v_{n-1}\}.$$

We transform ϕ into an NAEN-3SAT instance as follows: for each $0 \le i < p$, introduce new variables w_i and w_i', and map the clause $x_i \vee y_i \vee z_i$ to the set

$$S_i = \{(x_i, y_i, w_i), (y_i, z_i, w_i'), (w_i, w_i', 1)\}.$$

If S_i is a subset of a relation $R \subseteq (V \cup \neg V)^3$, and if R is an NAEN-3SAT instance over the variable set $W := V \cup \{w_i, w_i' \mid 0 \le i < p\}$, then a solution to R cannot assign both w_i and w_i' the value 1 (otherwise $(w_i, w_i', 1)$ would be mapped to $(1, 1, 1)$). Now, consider the NAEN-3SAT instance given by the relation $R = S_0 \cup S_1 \cup \cdots \cup S_{p-1}$. An assignment $f : V \to \{0, 1\}$ is a solution to ϕ iff for all $0 \le i < p$ we have $(f(x_i), f(y_i), f(z_i)) \neq (0, 0, 0)$. Let $\tilde{f} : W \to \{0, 1\}$ be define by

$$\tilde{f}(u) = \begin{cases} f(u), & \text{if } u \in V \cup \neg V, \\ 1 - \delta, & \text{if } u = w_i \text{ and } f(x_i) = f(y_i) = \delta, \\ 1 - \delta, & \text{if } u = w_i' \text{ and } f(z_i) = f(y_i) = \delta. \end{cases}$$

2.3. **Reduction of NAE-3SAT to CCP.** Consider the relational structure $\mathbb{B} = \langle \{0, 1\}, R \rangle$ where R is the ternary relation $\{0, 1\}^3 - \{(0, 0, 0), (1, 1, 1)\}$. That is,

$$R = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0)\}.$$

The CSP associated with the relational structure $\mathbb{B}$ is denoted by $\mathrm{CSP}(\mathbb{B})$ and described as follows: An *instance* is a (finite) relational structure $\mathbb{A} = \langle A, S \rangle$ with a single ternary relation S, and $\mathrm{CSP}(\mathbb{B})$ is the following decision problem:

> **Problem.** Given an instance $\mathbb{A} = \langle A, S \rangle$, does there exist a (relational structure) homomorphism from $\mathbb{A}$ to $\mathbb{B}$?

In other words, does there exist a function $f \colon A \to \{0, 1\}$ such that $(f(a), f(b), f(c)) \in R$ whenever $(a, b, c) \in S$?

The kernel of a function f with codomain $\{0, 1\}$ has two equivalence classes—namely, $f^{-1}\{0\}$ and $f^{-1}\{1\}$. If one of these classes is empty, then f is constant in which case it cannot be a homomorphism into the relational structure $\langle \{0, 1\}, R \rangle$ (since $(0, 0, 0) \notin R$ and $(1, 1, 1) \notin R$). Therefore, the kernel of every homomorphism $f \colon \mathbb{A} \to \mathbb{B}$ has two nonempty blocks.

Now, given a partition of A into two blocks, $\pi = |A_1|A_2|$, there are exactly two functions of type $A \to \{0, 1\}$ with kernel π. One is $f(x) = 0$ iff $x \in A_1$ and the other is $1 - f$. It is obvious that either both f and $1 - f$ are homomorphisms or neither f nor $1 - f$ is a homomorphism. Indeed, both are homomorphisms if and only if for all tuples $(a, b, c) \in S$ we have $\{a, b, c\} \nsubseteq A_1$ and $\{a, b, c\} \nsubseteq A_2$. Neither is a homomorphism if and only if there exists $(a, b, c) \in S$ with $\{a, b, c\} \subseteq A_1$ or $\{a, b, c\} \subseteq A_2$.

Now, for each tuple $s = (a, b, c) \in S$, we let $\mathrm{im}(s)$ (or simply $\mathrm{im}\, s$) denote the image of $\{0, 1, 2\}$ under s (viewing the sequence s as a function with domain the "index set" $\{0, 1, 2\}$ and codomain the set A). Furthermore, we let $\langle \mathrm{im}\, s \rangle$ denote the equivalence relation on A generated by $\mathrm{im}\, s$. Thus, if $s = (a, b, c)$, then

$$\langle \mathrm{im}\, s \rangle = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\} \cup \{(x, x) : x \in A\}.$$

The partition corresponding to $\langle \mathrm{im}\, s \rangle$ is $\pi_{\langle \mathrm{im}\, s \rangle} = |a, b, c|x_1|x_2| \cdots$. It is clear that a function $f \colon A \to \{0, 1\}$ is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if for all $s \in S$ the relation

$\langle \operatorname{im} s\rangle$ does not belong to the kernel of f. Therefore, a solution to the instance $\mathbb{A} = \langle A, S\rangle$ of CSP($\mathbb{B}$) exists if and only if there is at least one coatom in the lattice of equivalence relations of A that is not contained in the union $\bigcup_{s\in S} {\uparrow}\langle \operatorname{im} s\rangle$ of principal filters.

The *covered coatoms problem* (CCP) is the following: Given a set A and a list $s_1 = (a_1, b_1, c_1), s_2 = (a_2, b_2, c_2), \dots, s_n = (a_n, b_n, c_n)$ of triples with elements in A, decide whether all of the coatoms of the lattice $\prod_A$ of partitions of A are contained in the union $\bigcup_{i=1}^{n} {\uparrow}\langle \operatorname{im} s_i\rangle$ of principal filters.

3. Algorithms

We now discuss the tractability of the filter membership problem described in Section 2.3. Specifically, we describe some first steps toward a polynomial-time algorithm for solving instances of CCP.

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