NOTES ON DOWEK'S "PROOFS AND ALGORITHMS"

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Abstract. This document contains notes on the book [Dow11], *Proofs and Algorithms*, by by Gilles Dowek. We exerpt the main definitions and theorems, attempt to solve some exercises, and fix any typos or problems with the original text that we may find.

Part 1. Proofs

1. 1 Predicate Logic

First let's recall an elementary definition that plays a central role in this section. Let E be a set. A *binary relation* on E is a subset of $E \times E$. If $\leq$ is a binary relation on E that satisfies properties (1)–(3) below, then we call $\leq$ a *partial order* and we call the pair $\langle E, \leq \rangle$ a *partially ordered set* or *poset*.

(1) $(\forall x)\,(x \in E \longrightarrow x \leq x)$;
(2) $(\forall x, y)\,(x \in E \bigwedge y \in E \bigwedge x \leq y \bigwedge y \leq x \longrightarrow x = y)$
(3) $(\forall x, y, z)\,(x \in E \bigwedge y \in E \bigwedge z \in E \bigwedge x \leq y \bigwedge y \leq z \longrightarrow x \leq z)$

When (1) holds we say "$\leq$ is reflexive"; when (2) holds we say "$\leq$ is antisymmetric"; when (3) holds we say "$\leq$ is transitive".

1.1. Inductive Definitions.

1.1.1. *The Fixed Point Theorem.*

Definition 1.2 (Weakly complete partial order) An partial order relation $\leq$ on a set E is said to be *weakly complete* if each increasing sequence has a limit.[1] In this case, we call the pair $\langle E, \leq \rangle$ a *weakly complete poset*.

Definition 1.3 (Increasing function) Let $\langle E, \leq \rangle$ be a poset and f a function from E to E. The function f is increasing if $x \leq y \Rightarrow fx \leq fy$.

Definition 1.4 (Continuous function) Let $\langle E, \leq \rangle$ be a weakly complete poset and f an increasing function from E to E. The function f is continuous if for every increasing sequence $(u_i)_i$ we have $\lim_i(fu_i) = f(\lim_i u_i)$.

Proposition 1.1 (First fixed point theorem) Let $\langle E, \leq \rangle$ be a weakly complete poset that has a least element m. Let f be a function from E to E. If f is continuous then $p = \lim_i(f^i m)$ is the least fixed point of f.

Definition 1.5 (Strongly complete ordering) A partial order $\leq$ on a set E is *strongly complete*

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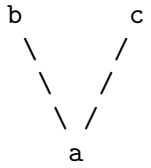
[1] A *sequence* is a function whose domain is the set $\mathbb{N} = \{0, 1, \dots\}$ of natural numbers, so Definition 1.3 can be used to infer what we mean by "increasing sequence" in Definition 1.2, although, the domain of a sequence need not be $\mathbb{N}$. So, more precisely, an *increasing sequence* is an *poset homomorphism* $u : \langle \mathbb{N}, \leq \rangle \to \langle E, \leq \rangle$; that is, if $n \leq m$ in $\mathbb{N}$, then $u_n \leq u_m$ in E.

if every subset A of E has a least upper bound, denoted by $\sup A$. In this case, we call the pair $\langle E, \leq \rangle$ a *strongly complete poset*.[2]

Exercise 1.1 Show that every strongly complete partial order is also weakly complete. (done)

Is the poset shown below weakly complete? Is it strongly complete?



(Answers: yes; no.)

Proposition 1.2 If $\langle E, \leq \rangle$ is *strongly complete*, then every subset A of E has a greatest lower bound, $\inf A$.

Proof. Since $\leq$ is strongly complete, $\sup A \in E$ exists. Similarly, the subset $B = \{y \in E \mid (\forall x \in A)\, y \leq x\}$ has a least upper bound, say, l. Note

- $(\forall y \in B)\, y \leq l$
- $\big((\forall y \in B)\, y \leq l'\big) \;\Rightarrow\; l \leq l'$.

Then l is the greatest lower bound of A. Why?

- l is a lower bound of A since if $x \in A$, then x is an upper bound of B and $l \leq x$.
- l is the greatest lower bound since if y is a lower bound of A, y is in B and $y \leq l$.

Proposition 1.3 (Second fixed point theorem) Let $\langle E, \leq \rangle$ be a strongly complete poset. Let f be a function from E to E. If f is increasing then $p = \inf\{c \mid fc \leq c\}$ is the least fixed point of f.

Proof. Let C be the set $\{c \mid fc \leq c\}$ and c be an element of C. Then $p \leq c$ since p is $\inf C$. Since f is increasing $fp \leq fc$. Note $fc \leq c$ because c is an element of C.

1.1.2. *Inductive Definitions.*

1.1.3. *Structural Induction.*

1.1.4. *Derivations.*

1.1.5. *The Reflexive-Transitive Closure of a Relation.*

1.2. **Languages.**

1.2.1. *Languages Without Variables.*

1.2.2. *Variables.*

1.2.3. *Many-Sorted Languages.*

1.2.4. *Substitution.*

1.2.5. *Articulation.*

1.3. **The Languages of Predicate Logic.**

1.4. **Proofs.**

[2]Notice that this definition does not require E have a least element and also does not require A be nonempty. This is not a problem. In fact, it is implicit in this definition that E contain a least element—namely, $\sup \emptyset$—as well as a greatest element—namely, $\sup E$.

1.5. **Examples of Theories.**

1.6. **Variations on the Principle of the Excluded Middle.**

1.6.1. *Double Negation.*

1.6.2. *Multi-conclusion Sequents.*

References

[Dow11] Gilles Dowek. *Proofs and algorithms*. Undergraduate Topics in Computer Science. Springer, London, 2011. An introduction to logic and computability, Translated from the French by Maribel Fernandez. URL: http://dx.doi.org/10.1007/978-0-85729-121-9, doi:10.1007/978-0-85729-121-9.

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