HW7

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1. Manhattan Walk

Let $P(a, b)$ be: "$a \leq 2b$". We prove $P(a, b)$ holds for all pairs $(a, b) \in S$ by structural induction.

Base Cases:
When $a = 3, b = 2$, $3 \leq 2 \cdot 2$. Thus, $P(3, 2)$ holds.

Inductive hypothesis:
Suppose $P(a, b)$ holds for arbitrary $a, b \in S$.

Inductive step:
Case 1: We show $P(a, b + 1)$ holds by inductive steps
By the definition of sets, we know $(a, b) \in S$. By inductive hypothesis, we have $x \leq 2 \cdot y$. By algebra, we have $x \leq 2 \cdot y + 2$ and thus $x \leq 2 \cdot (y + 1)$. This gives us $P(x, y + 1)$.
Case 2: We show $P(a + 3, b + 2)$ holds by inductive steps
By the definition of sets, we know $(a, b) \in S$. By induction hypothesis, we have $x \leq 2 \cdot y$. By algebra, we have $x + 3 \leq 2y + 4$ and thus we have $x + 3 \leq 2 \cdot (y + 2)$. This gives us $P(x + 3, y + 2)$.

Conclusion:
Therefore $P(a, b)$ holds for all pairs $(a, b) \in S$

2. What doesn't kill you makes you stronger

(a)
I spent 10 minutes trying.

(b)
When $a = 1.5, b = 2$, $3a + 2 = 6.5 \not< 3b = 6$

(c)
$f(n) - 1 < f(n)$, so if we could show $g(n) \leq f(n) - 1$, we could also prove $g(n) < f(n)$

(d)
Let $Q(n)$ be $g(n) \leq f(n) - 1$. We prove $Q(n)$ holds for all $n \geq 1$ by strong induction.

Base case: When $n = 1$, $g(1) = 2 \leq f(1) = 3$. Thus, $Q(1)$ holds.

Inductive hypothesis: Suppose $Q(1) \wedge Q(2) \wedge ... \wedge Q(n)$ holds for all $n \geq 1$, which is $g(n) \leq f(n) - 1$

Inductive step: We show $Q(n+1)$ holds which is $g(n+1) \leq f(n+1) - 1$
By inductive hypothesis, we know:

$$\begin{aligned}
g(n) &\leq f(n) - 1 && \text{Inductive hypothesis} \\
3 \cdot g(n) &\leq 3 \cdot (f(n) - 1) && \text{Algebra} \\
3 \cdot g(n) &\leq 3 \cdot f(n) - 3 && \text{Algebra} \\
3 \cdot g(n) + 2 &\leq 3 \cdot f(n) - 3 + 2 && \text{Algebra} \\
3 \cdot g(n) + 2 &\leq 3 \cdot f(n) - 1 && \text{Algebra} \\
g(n+1) &\leq f(n+1) - 1 && \text{Definition of g(n) and f(n)}
\end{aligned}$$

Thus we prove $Q(n)$

Conclusion: $Q(n)$ holds for all integers n 1 by the principle of strong induction.

3. Recursion – See: Recursion

(a)

Let S be a set of binary strings defined recursively as:
Basis Step: $1 \in S$ where
Recursive Step: if $x \in S$, then $x \cdot w \in S$ for arbitrary binary string w with $len(w) = 2$.
Justify: The basis step is 1 since 1 has the length of 1 and starts with 1. And concatenating any binary strings with length 2 to elements in the set could new element that satisfies the requirement.

(b)

Let T be a set of binary strings defined recursively as:
Basis Step: $a \in T$ where for arbitrary binary string a where $len(a) = 2$
Recursive Step: if $a \in T$ then $a \cdot b \in T$ for arbitrary binary string b where $len(b) = 3$
Justify: The basis step is a binary string with length 2 since 2 is equivalent to 2 $(mod 3)$. Then, concatenating arbitrary string with length 3 to elements in the set could produce new element that satisfies the requirement.

(c)

Let U be a set of binary strings defined recursively as:
Basis Step: $0 \in U$
Recursive Step: if $x \in U$, then $x \cdot y \in U$ for arbitrary binary string y where $\#_0(y) \equiv 0 \ (mod\ 2)$
Justify: The basis step is a binary string with only one 0 which has odd number of 0s. Then, concatenating any binary string that has even number of 0s results in a binary string with odd number of 0s.

4. Find The Bug

(a)
The purpose is to prove $Factorial(3) = 6$, so we must start from something that is clearly true and prove step by step to get the conclusion. However, the friend start the proof from the conclusion and conclude something that is clearly true, which is not correct.

(b) The statement is correct.
Since we know $Factorial(n) = n * Factorial(n-1)$ and $Factorial(0) = Factorial(1) = 1$, we can get $Factorial(3) = 3 * Factorial(2)$. By definition of Factorial(n), $Factorial(2) = 2 * Factorial(1)$ which equals to 1. Thus, we know $Factorial(2) = 2*1 = 2$ and $Factorial(3) = 3*Factorial(2) = 3*2 = 6$.

Extra Credit: Exponentially increasing fun

(a)

(b)

5. Induction Divides

Let $P(n)$ be $5|(6^n - 1)$ for all $n \in \mathbb{N}$. This means we need to prove that for all $n \in \mathbb{N}$ there exists some integer z that satisfies $z = \frac{6^n-1}{5}$.
We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base case: When $n = 0$, $\frac{6^0-1}{5} = \frac{0}{5} = 0$ so we get an integer z that is 0. Thus, P(0) holds.
Inductive Hypothesis. Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$ which satisfies the situation $5|(6^k - 1)$, meaning that there exists an integer z that equals to $\frac{6^k-1}{5}$ by the definition of divides. We can manipulate the equation to get another expression by algebra from step 1 to 5:

$$
\begin{aligned}
z &= \frac{6^k - 1}{5} && (1)\\
asdfqwer &= \frac{6^{k-1} \cdot 6 - 1}{5} && (2)\\
&= \frac{6^{k-1} \cdot (5+1) - 1}{5} && (3)\\
&= \frac{6^{k-1} \cdot 5}{5} + \frac{6^{k-1} - 1}{5} && (4)\\
&= 6^{k-1} + \frac{6^{k-1} - 1}{5} && (5)
\end{aligned}
$$

Inductive Step: We show when $P(k+1)$, there is an integer z where $z = 6^k + \frac{6^k-1}{5}$

For $P(k+1)$:

$$
\begin{aligned}
z &= \frac{6^{k+1} - 1}{5} && \text{Definition of divides} && (6)\\
&= \frac{6^k \cdot 6 - 1}{5} && \text{By algebra} && (7)\\
&= \frac{6^k \cdot (5+1) - 1}{5} && \text{By Algebra} && (8)\\
&= \frac{6^k \cdot 5}{5} + \frac{6^k - 1}{5} && \text{By Algebra} && (9)\\
&= 6^k + \frac{6^k - 1}{5} && \text{inductive hypothesis} && (10)
\end{aligned}
$$

From step 6 to 9 we know 6^k is an integer and $\frac{6^k-1}{5}$ is integer too, $6^k + \frac{6^k-1}{5}$ is also an integer. Step

10 is inductive hypothesis which proves P(k+1)
Conclusion: P(n) holds for all $n \in \mathbb{N}$ by the principle of induction.

6. Induction Code

For $n \in \mathbb{N}$, let $P(n)$ be $Mystery(n) = 7 \cdot 2^n + 3 \cdot (-1)^{n+4}$
Base Case:
For $n = 0$, $P(0) = 7 \cdot 2^0 + 3 \cdot (-1)^4 = 7 + 3 = 10$ and this is shown in the code.
When $n = 1$, $P(1) = 7 \cdot 2^1 + 3 \cdot (-1)^{1+5} = 14 - 3 = 11$ and the result also proven in the code segment.
Inductive Hypothesis: Suppose P(k) holds for an arbitrary integer $k \geq 0$
Inductive Step: We show P(k+1)

$$\begin{aligned}
Mystery(k+1) &\equiv Mystery(k) + 2 \cdot Mystery(k-1) && \text{Definition of Mystery} \quad (11)\\
&\equiv 7 \cdot 2^k + 3 \cdot (-1)^{k+4} + 2 \cdot [7 \cdot 2^{k-1} + 3 \cdot (-1)^{n-1+4}] && \text{Math from the code} \quad (12)\\
&\equiv 7 \cdot 2^k + 2 \cdot 7 \cdot 2^{k-1} + 3 \cdot (-1)^{k+4} + 2 \cdot 3 \cdot (-1)^{k+3} && \text{By Algebra} \quad (13)\\
&\equiv 7 \cdot 2^k + 7 \cdot 2^k + 3 \cdot (-1)^{k+4} + 2 \cdot 3 \cdot (-1)^{k+3} && \text{By Algebra} \quad (14)\\
&\equiv 2 \cdot 7 \cdot 2^k + 3 \cdot (-1)^{k+4} + 2 \cdot 3 \cdot (-1)^{k+3} && \text{By Algebra} \quad (15)\\
&\equiv 7 \cdot 2^{k+1} + 3 \cdot (-1)^{k+4} + 2 \cdot 3 \cdot (-1)^{k+3} && \text{By Algebra} \quad (16)\\
&\equiv 7 \cdot 2^{k+1} + 3 \cdot (-1)^{k+3} \cdot (-1 + 2) && \text{By Algebra} \quad (17)\\
&\equiv 7 \cdot 2^{k+1} + 3 \cdot (-1)^{k+3} \cdot (1) && \text{By Algebra} \quad (18)\\
&\equiv 7 \cdot 2^{k+1} + 3 \cdot (-1)^{k+3} \cdot (-1)^2 && \text{By Algebra} \quad (19)\\
&\equiv 7 \cdot 2^{k+1} + 3 \cdot (-1)^{(k+1)+4} && \text{Inductive Hypothesis} \quad (20)
\end{aligned}$$

This proves $P(k+1)$

Conclusion: P(n): "$Mystery(n) = 7 \cdot 2^n + 3 \cdot (-1)^{n+4}$" is true for all integers $n \geq 0$

7. feedback

(a) 9 hours

(b) extra credit question

(c) No