MATH/STAT395: Probability II Winter 2022

Exercises from the lectures

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1 Joint distribution of discrete random variables

Exercise 1. Roll two fair dice with **4 faces**, denote

(i) S the sum of the two dice

(ii) Y the indicator variable that you get a pair

1. Record which outcomes lead to different values of S, Y
2. Compute the corresponding joint probability mass function of S, Y
3. Read the proba to get a sum of 4 while not having a pair
4. Compute probability to get a sum higher or equal than 5 with pairs
5. Score is the sum of the dice, doubled if it is a pair. What is the average score?
6. Compute the marginal p.m.f. of Y from the joint p.m.f.

Exercise 2. Roll repeatedly a pair of dice.
Denote N the number of rolls until the sum of the dice is 2 or a 6

1. What is the distribution of N?
2. Denote X the sum you finally get (2 or 6),
   are X and N independent?

Exercise 3. 1. Define 2 independent geometric variables

- X the number of days until your friend Lulu sends you a letter, with an average waiting time of $1/\lambda$ days $(0 < \lambda < 1, X \geq 1)$
- Y the number of days until your friend Barry sends you a letter, with an average waiting time of $1/\mu$ days $(0 < \mu < 1, Y \geq 1)$

Let D be the time before either sends you a letter. What is the cumulative distribution function of D?

2. In terms of the named distributions we have covered, what is the distribution of D? Include any parameter values.
3. Derive the p.m.f. of I defined by

$$I = \begin{cases} 0 & \text{if Lulu's letter arrive stricly before Barry's letter} \\ 1 & \text{if Lulu's letter arrive the same day as Barry's letter} \\ 2 & \text{if Lulu's letter arrive strictly after Barry's letter} \end{cases}$$

4. Are I and D independent?

2 Multinomial random variables

Exercise 4. Roll a fair die 100 times. Find the probability that among the 100 rolls, we observe exactly 22 ones, 17 fives.

3 Joint distribution of continuous random variables

Exercise 5. Throw a dart uniformly at random on a disk of radius 2
What is the probability that the dart is in the central disk of radius one?

Exercise 6. Throw a dart uniformly at random on a square of edge size 2 centered on 0
Assume your score is equal to the square distance to the center
What is your average score?

Exercise 7. Consider a disk of radius r, $D_r = \{(x, y) : x^2 + y^2 \leq r^2\}$ and $(X, Y) \sim \text{Unif}(D_r)$.
What is the marginal p.d.f. of X?

Exercise 8 (Shooting an arrow)**.** Consider X, Y with p.d.f.

$$f(x, y) = \frac{1}{\lambda} \frac{e^x}{\sqrt{y+1}} \mathbf{1}_W(x, y)$$

for $\lambda{=}2(\sqrt{2} - 1)(e - e^{-1})$
where $W{=}\{(x, y) : -1{\leq}x{\leq}1, 0{\leq}y{\leq}1\}$.

1. Are X, Y independent?
2. What consequences it had when computing the probability to get the target $T{=}\{(x, y){:}-0.1{\leq}x{\leq}0.1, 0.4{\leq}y{\leq}0.6\}$?

Exercise 9. Let $(X, Y) \sim \text{Unif}(D)$ with D a disk centered at 0 with radius r_0 Let (R, Θ) be the polar coordinates of (X, Y) such that

$$X = R\cos(\Theta), Y = R\sin(\Theta)$$

1. Find the joint and marginal p.d.f. of R and Θ
2. Are R and Θ independent?

Exercise 10. Let X, Y be two independent $\text{Exp}(\lambda)$ r.v.
Find the joint p.d.f. of $U = X + Y$ and $V = \frac{X}{X+Y}$. Are U, V independent?

Exercise 11. Let $B_1, \ldots B_{m+n} \sim \text{Ber}(p)$ be $m + n$ independent random variables.
Denote $S_1 = \sum_{i=1}^{m} B_i$ and $S_2 = \sum_{i=m+1}^{n} B_i$,

1. Are S_1 and S_2 are independent.
2. Are $Z = S_1 + S_2$ and S_1 independent?

Exercise 12. n players have each one coin with p_i the probability for player i to get a tail
At each round they all toss the coin, independently, they repeat it until one player gets a tail (the winner)
What is the distribution of the number of rounds before the game ends?
Note: When asked what is the distribution of some r.v., we are asking you to recognize the r.v. among one of the classical ones.

Exercise 13. I'm sitting on a bench in Dalvìkurbyggd watching whales.
On average, a whale shows up every 5min.
What is p.d.f. of the time I wait for seeing n whales?
I'm assuming that the times that I wait to see each new whale are independent

Exercise 14. 1. Suppose that X and Y are independent random variables with density functions

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 4ye^{-2y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Find the density function of $X + Y$.