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HW6

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BB

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1. Put One Foot In Front of the Other

(a)

P(n) should be a predicate but "$n^3 + 17n - 4$" is not a predicate. For this problem, the predicate P(n) should be "$cleverMethodName\ (int\ n) = n^3 + 17n - 4$".

(b)

For this problem, the predicate P(n) is wrong. Because we will show Q(n) holds for all n by induction on n, we are trying to show $\forall n(Q(n) \rightarrow Q(n+1))$ for every $n \in \mathbb{N}$. "For all" statement is the domain of discourse. Thus, the predicate should be "$cleverFibonacci\ (int\ n) = F_n$".

(c)

In the inductive step, we should not use G(a) and G(b) to represent Fibonacci number since they are predicates instead of fibonacci numbers. It should be written as "$F_{k+1} = F_k + f_{k-1}$" because of the definition of Fibonacci numbers. Then, we have $F_{k+1} \leq 2^k + 2^{k-1} < 2^k + 2^k = 2^{k+1}$, which proves the G(k+1).

2. Proof by contradiction

(a)

Claim: $\sqrt{17}$ is irrational

Proof:

By definition of rational, there are integers s, t such that $t \neq 0$ and $\sqrt{17} = s/t$

Let $p = \frac{s}{gcd(s,t)}$, $q = \frac{t}{gcd(s,t}$. By the fundamental theorem of arithmetic, we have divided out all common factors of s, t and so p, q have no more common prime factors. Therefore the $gcd(p, q) = 1$.

$\sqrt{17} = \frac{p}{q}$

$17 = \frac{p^2}{q^2}$

$17 \cdot q^2 = p^2$

Thus, we know that 17 divides p$^2 and 17$ — p^2 by the definition of divides.

By the fact mentioned in the specification, we could know $17|p$ or $17|p$ which is $17|p$.

By the definition of divides again, we could know there is some integer k satisfies $k = \frac{p}{17}$ and we could $17k = p$ by algebra.

Replacing p in the equation $17 = \frac{p^2}{q^2}$, we could get $17 = \frac{(17k)^2}{q^2}$.

By algebra, we could get $17q^2 = (17k)^2$.

Divides both sides 17, we could get $q^2 = 17k^2$.

Thus, we could know 17 divides q^2 and $17|q^2$ by the definition of divides.

This means $17|q$ by the fact mentioned in the specification.

Since 17 divides both q and p, 17 is a common factor of p and q which contradicts assumption that p and q have no common factor greater than 1. So, there is no such p and q, which proves that $\sqrt{17}$ is irrational.

3. Alligator Eats The Bigger One

Let $P(n)$ be "$2^n + 1 \leq 2^{n+1} - 2^{n-1}$". We will prove $P(n)$ holds for all integers $n \geq 1$ by strong induction on n.
Base case: When $n = 1$, $P(n)$ would be: $2^1 + 1 = 3 \leq 2^2 - 2^0 = 4 - 1 = 3$. Thus, $P(1)$ holds.
Inductive hypothesis: $P(1) \wedge ... \wedge P(k)$ holds for $k \geq 1$, which is $2^k + 1 \leq 2^{k+1} - 2^{k-1}$
Inductive step: We show $P(k+1):\ 2^{k+1} + 1 \leq 2^{k+2} - 2^k$

$$\begin{aligned} 2^{k+1} + 1 &= 2 \cdot 2^k + 1 && \text{Algebra} \\ &\leq 2 \cdot (2^{k+1} - 2^{k-1}) - 1 && \text{Inductive hypothesis} \\ & 2^{k+2} - 2^k - 1 && \text{Algebra} \\ &\leq 2^{k+2} - 2^k. && \text{Algebra} \end{aligned}$$

So, we prove P(k+1)
Conclusion: $P(n)$ holds for all integers $n \geq 1$ by the principle of strong induction.

4. Running Times

(a)

Let $P(n)$ be: "$T(n) \leq 20n$". We will show $P(n)$ for all $n \in \mathbb{N}$ by strong induction on n.
Base case: when n = 1, $P(1)$: "$T(1) = 1 \cdot 5 = 5$". Thus, P(1) holds.
Inductive hypothesis: $P(1) \wedge ... \wedge P(k)$ holds for arbitrary integer $k \geq 1$
Inductive step: Show $P(k+1)$ holds, which is $T(k+1) \leq 20(k+1)$
case 1: when $k+1 \leq 4$, $T(k+1) = 5(k+1)$

$$\begin{aligned}
T(k+1) &= 5(k+1) && \text{Definition of T} \\
&= 5k+5 && \text{By algebra} \\
&\leq 20k+5 && \text{Inductive hypothesis} \\
&\leq 20k+20 && \text{By Algebra} \\
&= 20(k+1) && \text{inductive hypothesis}
\end{aligned}$$

case 2: when $k+1 > 4$, $T(k+1) = T(\lfloor\frac{k+1}{2}\rfloor) + T(\lfloor\frac{k+1}{4}\rfloor) + 5(k+1)$
By the definition of floor function, we could know $\frac{k+1}{2}\rfloor$ and $\lfloor\frac{k+1}{4}\rfloor$ integers and are smaller than $\frac{k+1}{2}$ and $\frac{k+1}{4}$ respectively. Thus, by the inductive hypothesis, we have $P(k+1)$ holds. So, $T(\lfloor\frac{k+1}{2}\rfloor) \leq 20 \cdot (\lfloor\frac{k+1}{2}\rfloor)$ which is less than or equal to $20 \cdot \frac{k+1}{2}$ and $T(\lfloor\frac{k+1}{4}\rfloor) \leq 20 \cdot (\lfloor\frac{k+1}{4}\rfloor)$ which is less than or equal to $20 \cdot \frac{k+1}{4}$.
Combine all information mentioned above, we have:

$$\begin{aligned}
T(k+1) &= T(\lfloor\frac{k+1}{2}\rfloor) + T(\lfloor\frac{k+1}{4}\rfloor) + 5(k+1) && \text{Definition of T} \\
&\leq 20 \cdot \frac{k+1}{2} + 20 \cdot \frac{k+1}{4} + 5(k+1) && \text{Inductive hypothesis} \\
&= 10(k+1) + 5(k+1) + 5(k+1) && \text{algebra} \\
&= 20(k+1) && \text{algebra}
\end{aligned}$$

This shows $P(k+1)$.
Conclusion: We prove $P(k+1)$ holds for arbitrary integer $k \geq 1$.

5. These are pretty long str1111ngs

Let P(n) be "The set S contains the string 1^n by concatenating strings $1^4 \in S$ and $1^7 \in S$". We prove P(n) holds for every integer $n \geq 18$.

Base case:
When $n = 18$, the set S contains the string 1^{18} which could be obtained by concatenating one 1^4 and two 1^7.
When $n = 19$, the set S contains the string 1^{19} which could be obtained by concatenating three 1^4 and one 1^7.
When $n = 20$, the set S contains the string 1^{20} which could be obtained by concatenating five 1^4.
When $n = 21$, the set S contains the string 1^{21} which could be obtained by concatenating three 1^7.
Thus, P(18), P(19), P(20), P(21) hold.

Inductive hypothesis:
Suppose $P(18) \wedge P(19) \wedge P(20) \wedge P(21) \wedge ... \wedge P(k)$, for an arbitrary $k \geq 21$

Inductive Step: We show the set S contains string 1^{k+1}
By inductive hypothesis, we could get string 1^{k-3} with string 1^4 and 1^7. Concatenating another 1^4 string could give us the string 1^{k+1}. Thus, $P(k+1)$ holds by strong induction.

Conclusion: For every integer $n \geq 18$, the set S contains the string 1^n by the principle of strong induction.

6. Chess

(a)
Let P(n) be "rook moving game where the rook starts at a coordinate in the form of (n, n), the player that goes second can win the game". We prove P(n) holds for every integer $n > 0$ by strong induction.

Base case:
When n = 1, the initial position of the rook is (1, 1). Since rook can move any distance on a straight line and in this game it is only allowed to move downward or to the left, the total distance players need to take to get to (0, 0) is $1 + 1 = 2 \cdot 1$ which shows total number of units is even and the total number of turns is also even. Thus, when the first player move downward or to the left by 1 unit, the second player could move in another direction to (0, 0) and win the game. Thus, P(1) holds.

Inductive hypothesis:
Suppose $P(1) \wedge ... \wedge P(k)$ hold for an arbitrary integer $k \geq 1$.

Inductive step: We show $P(k+1)$ holds
When $n = k + 1$, the rook is at position (k+1, k+1). Thus the total number of units two players need to take to reach (0, 0) is $k + 1 + k + 1 = 2 \cdot (k + 1)$ which is an even number.
Case 1: When the first player moves downward by a units ($where\ 1 \leq a \leq k + 1$), now it's my turn, I could move a units to the left. Thus, the rook will be at the position (k+1-a, k+1-a) (where $k + 1 - a \geq 0$). By inductive hypothesis, we know P(k+1-a, k+1-a) holds.

Case 1: When the first player moves to the left by a units ($where\ 1 \leq a \leq k + 1$), now it's my turn, I could move a units downward. Thus, the rook will be at the position (k+1-a, k+1-a) (where $k + 1 - a \geq 0$). By inductive hypothesis, we know P(k+1-a, k+1-a) holds.
Thus, P(k+1) holds.

Conclusion:
Therefore, we have prove $P(n)$ holds for arbitrary integer $n \geq 1$ by the principle of strong induction.

(b) When my friend moves to location (a, b), if $a > b$, I would move to the left by $(a - b)$ units to (b, b) position, and if $a < b$, I would move downward by $(b - a)$ units to (a, a) position. By this strategy, every time I could move to a position on the diagnosis so eventually I could reach (0, 0).

7. The Apple Doesn't Fall Far From The... Tree

The question asks us to prove: For all JTree that have $d-1$ copies of data then they have d copies of null. This is similar in meaning to the statement that all JTree have one more copy of null than data.
Thus, let P(T) be "Tree T has one more copies of null than copies of data". We prove P(T) for all trees T by structural induction on T.

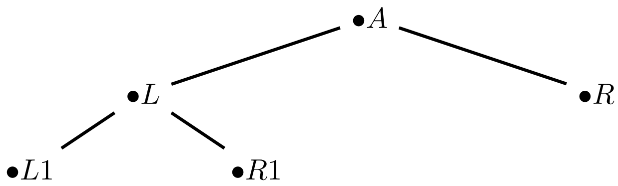
Base case($T = \bullet$):
By definition of JTree, null is Tree that has 1 copies of null and 0 copies of data. Thus, $P(\bullet)$ holds.

Inductive hypothesis:
Suppose $P(L)$ and $P(R)$ hold for some arbitrary JTree L, R.

Inductive step: We show that $P(Tree(\bullet, L, R))$ holds.
By the inductive hypothesis, we could know the left JTree and right JTree both have one more copy of null than their copies of data. Thus, we could assume the left tree have m copies of null and $m-1$ copies of data and right tree have n copes of null and $n-1$ copies of data. Since the root also contains one data by the definition of JTree, the total number of data for the Tree $T(\bullet, L, R)$ is $n-1+m-1+1 = n+m-1$ and the total number of null is $n+m$. Since n and m are integers, $n+m$ is also an integer and there is some integer d that equals $n+m$. Thus, we have the number of copies for the Tree $T(\bullet, L, R)$ is d and the number of data for it is $d-1$. Thus, we prove that $P(Tree(\bullet, L, R))$ which is Tree $T(\bullet, L, R)$ has one more copies of null than copies of data.

Conclusion:
Thus, $P(T)$ holds for all trees T by structural induction.

8. Find. The. Bug.

(a) 

From the graph above, we let $A, L, L1, R1, R$ be $\bullet$. Thus, we know $L1, R1, R$ are trees and we could denote $T1$ to be $Tree(L, L1, R1)$ and $T2$ to be $Tree(R)$. Therefore, since $T1, T2$ are trees, we have T to be $Tree(A, T1, T2)$.

By definition of leaves:

$$\begin{aligned}
leaves(T) &= leaves(T1) + leaves(T2) \\
&= leaves(L1) + leaves(R1) + leaves(T2) \\
&= leaves(L1) + leaves(R1) + leaves(R) \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

by definition of height:

$$\begin{aligned}
height(T) &= 1 + max\{height(T1), height(T2)\} \\
&= 1 + max\{1 + max\{height(L1), height(R1)\}, 0\} \\
&= 1 + max\{1 + 0, 0\} \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

For the tree above, the height is 2 and the number of leaf is 3. This contradicts to the claim that the number of leaf of a tree is 2^{height} which is 4 in this case but the number of leaf in this case is 3.

(b) The biggest flaw in the proof is 4A. It only assumes T1 and T2 are trees of the same height k. When one of T1 and T2 have a smaller height, the height of the T $Tree(\bullet, T1, T2)$ would still be $k + 1$, but this would lead to mistake in 4E when it calculates the leaves of the tree T.

9. feedback

(a) 9 hours

(b) question 6, and question 7

(c) No