

# Homework 3: Predicate Logic

**VERSION 2:** We corrected a typo in Problem 5, line 10 (the prior steps should be 5,9 not 4,8).

Due date: Wednesday April 20th at 10 PM

If you work with others (and you should!), remember to follow the collaboration policy outlined in the [syllabus](#).

In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.

We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the [grading guidelines](#) on the assignments page for more information on what we're looking for. **Be sure to read the guidelines on inference proofs on the assignments page. The requirements on some rules are looser now that we're nearing the end of the training wheels phase!**

**This assignment has 5 pages, be sure you keep scrolling!**

## 1. Onion Debate

For those who are not interested in roller-skating, Robbie also owns a glorious Walla Walla sweet onion farm. Restrictions imposed by the Department of Agriculture require that only people who bring a shallot with them may enter this farm.

You let your domain of discourse be all people and plants in the farm.

The predicates Shallot, WallaWalla, Human are true if and only if the input is a shallot, a Walla Walla sweet onion, or a human respectively. It is a sunny day and Robbie is hosting a debate about whether Walla Walla sweet onions are better than shallots. The predicate SweetOnionLover( $x$ ) means “ $x$  is a sweet onion lover” and similarly for ShallotLover( $x$ ). Finally EnjoyingSun is true if and only if the input human/plant is enjoying the sun.

### 1.1. Round One[12 points]

Translate the following observations into English. Your translations should take advantage of “restricting the domain” to make more natural translations when possible, but you should not otherwise simplify the expression before translating.

These requirements mean that

- You must not use variable names in your English translation (e.g., don't say “for every  $x$ ...”)
- For every quantified variable where one or more of the predicates can be interpreted as a domain restriction, you must use at least one of them to make your translation more natural. So with a domain of discourse of all integers,  $\forall x([Even(x) \wedge Prime(x)] \rightarrow IsEqual(x, 2))$  could be translated as “For every even integer, if it is prime it is equal to 2” or “Every prime and even integer is equal to 2” but could not be translated as “For every integer, if the integer is prime and even then it is equal to 2.”
- If the sentence does not have domain restriction, you may use “everyone” or “someone” to refer to an arbitrary element of the domain.

(a)  $\exists x (WallaWalla(x) \wedge \neg SweetOnionLover(x) \wedge EnjoyingSun(x))$

(b)  $\forall x (ShallotLover(x) \vee Shallot(x) \rightarrow \neg EnjoyingSun(x)) \wedge \forall x (SweetOnionLover(x) \rightarrow EnjoyingSun(x))$

(c)  $\neg \exists x (Human(x) \wedge EnjoyingSun(x) \wedge ShallotLover(x))$

## 1.2. Round Two [4 points]

You realize that the first sentence is false. State the negation of (a) in English. You should simplify the negation so that the English sentence is natural.

## 2. Become a Domain Expert [10 points]

For the following statements, translate them into predicate logic (specifying and defining any predicates you use). Then provide a domain of discourse where the statement is true and another domain of discourse where the statement is false.

Also include 1-2 sentences for each domain for why the statement has the truth value it does.

If you wish to make extra assumptions about the world (like “there is a cat named Garfield who is fat and unhappy”) you may do so.

- (a) Every  $x$  that speaks gets a treat.
- (b) There is an  $x$  such that for all  $y$ ,  $x$  is less than or equal to  $y$ .

## 3. Nested Quantifiers [15 points]

Fix your domain of discourse to be all mammals. Use the predicates  $\text{Cat}(x)$ ,  $\text{Dog}(x)$ ,  $\text{Human}(x)$  to say  $x$  is a cat, dog, or human respectively. You can also use the predicates  $\text{Happy}(x)$ ,  $\text{Sad}(x)$ ,  $\text{Fluffy}(x)$  to mean  $x$  is happy, sad, or fluffy respectively. Finally, the predicate  $\text{IsPetOf}(x, y)$  means  $y$  is the pet of  $x$  (note the order – the pet goes second).

In this problem, an example of something you might give for a “scenario” might be “There is at least one cat who is happy and at least one cat who is not happy.”

- (a) Your friend tried to translate “Every cat is happy or fluffy” and got

$$\forall x(\text{cat}(x) \wedge [\text{happy}(x) \vee \text{fluffy}(x)]).$$

The translation is incorrect. Give a correct translation, and describe a scenario (i.e. facts about mammals) in which your translation and their translation evaluate to different truth values.

- (b) Your friend tried to translate “There is a cat who is the pet of all humans” and got

$$\exists x \forall y ([\text{cat}(x) \wedge \text{human}(y)] \rightarrow \text{pet}(y, x)).$$

The translation is incorrect. Give a correct translation, and describe a scenario (i.e. facts about widgets) in which your translation and their translation evaluate to different truth values.

- (c) Translate the sentence “For every mammal  $x$ , there is a mammal  $y$  such that for every mammal  $z$ :  $y$  is a cat and  $x$  is the pet of  $z$ ” into predicate logic.

## 4. There is an implication [8 points]

Implications are uncommon under existential quantifiers. Consider this expression (which we’ll call “the original expression”):  $\exists x(P(x) \rightarrow Q(x))$

- (a) Suppose that  $P(x)$  is not always true (i.e. there is an element in the domain for which  $P(x)$  is false). Explain why the original expression is true in this case. (1-2 sentences should suffice. If you prefer, you may give a formal proof instead).

- (b) Suppose that  $P(x)$  is always true (i.e.  $\forall x P(x)$ ). There is a simpler statement which conveys the meaning of the original expression (i.e. is equivalent to it for all domains and predicates. By simpler, we mean “uses fewer symbols”). Give that expression, and briefly (1-2 sentences) explain why it works.
- (c) When we do domain restriction incorrectly we’ll often get an expression like the original expression – it doesn’t usually mean what it might look like at first glance. Ponder, based on the last two parts, why it’s very uncommon to write the original expression. You do not have to write anything for this part, simply ponder. [0 points]

## 5. Spoof [12 points]

This problem will be our first “spoof.” A spoof is something that looks like a proof, but isn’t (because there are one or more significant mistakes made along the way).

**Theorem?:** Given  $\neg c, \neg b \rightarrow c, c \wedge \neg a \vee b$  can you conclude F? (Concluding F would mean that the givens [or more formally, the conjunction of all the givens] is a contradiction).

”Spoof”:

1.	$\neg b \rightarrow c$	Given
2.	$\neg c \rightarrow \neg \neg b$	Contrapositive (1)
3.	$\neg c \rightarrow b$	Double Negation (2)
4.	$\neg c$	Given
5.	$b$	Modus Ponens (3,4)
6.	$c \wedge \neg a \vee b$	Given
7.	$c \wedge (\neg a \vee b)$	Associativity (6)
8.	$c \wedge (b \vee \neg a)$	Commutativity (7)
9.	$c \wedge (b \rightarrow a)$	Law of Implication (8)
10.	$c \wedge a$	Modus Ponens (5,9)
11.	$\neg c \wedge c \wedge a$	Intro $\wedge$ (4,10)
12.	$c \wedge \neg c \wedge a$	Commutativity (11)
13.	$F \wedge a$	Negation (12)
14.	$a \wedge F$	Commutativity (13)
15.	$F$	Domination (14)

There are (at least) three significant errors in the spoof above. List three significant errors. For each error, state the line number where the error first appears and why it’s incorrect (like “step 3, Modus Ponens needs  $p \rightarrow q$  and  $p$ , we have  $p \rightarrow q$  and  $q$ ”). 1 sentence was enough for each of our explanations.

We tried to write a pristine spoof (i.e., we tried to make it so there are exactly three errors and nothing else wrong), but there may be (unintentional) insignificant errors along with the (intentional) significant ones. An insignificant error would be something like listing the wrong number in a prior step or forgetting to mention that commutativity was used along with another rule, or using the wrong name for a rule – something that could be fixed quickly.

A significant error is one that could lead to a false conclusion, or which requires inserting multiple steps to correct.

## 6. Inference Spoof [22 points]

**Theorem:** Given  $\neg s \rightarrow (p \wedge q)$ ,  $s \rightarrow r$ , and  $(r \wedge p) \rightarrow q$ , prove  $q$ .

“Spoof:”

1.	$\neg s \rightarrow (p \wedge q)$	Given
2.1.	$\neg s$	Assumption
2.2.	$p \wedge q$	MP: 2.1 1
2.3.	$p$	Elim of $\wedge$ : 2.2
2.4.	$q$	Elim of $\wedge$ : 2.2
2.	$\neg s \rightarrow q$	Direct Proof Rule
3.	$s \rightarrow r$	Given
4.	$(r \wedge p) \rightarrow q$	Given
5.1.	$s$	Assumption
5.2.	$r$	MP: 5.1, 3
5.3.	$r \wedge p$	Intro $\wedge$ : 5.2, 2.3
5.4.	$q$	MP: 5.3, 4
5.	$s \rightarrow q$	Direct Proof Rule
6.	$(s \rightarrow q) \wedge (\neg s \rightarrow q)$	Intro $\wedge$ : 5, 2
7.	$(\neg s \vee q) \wedge (\neg \neg s \vee q)$	Law of Implication
8.	$(\neg s \vee q) \wedge (s \vee q)$	Double Negation
9.	$((\neg s \vee q) \wedge s) \vee ((\neg s \vee q) \wedge q)$	Distributivity
10.	$((\neg s \vee q) \wedge s) \vee (q \wedge (\neg s \vee q))$	Commutativity
11.	$((\neg s \vee q) \wedge s) \vee (q \wedge (q \vee \neg s))$	Commutativity
12.	$((\neg s \vee q) \wedge s) \vee q$	Absorption
13.	$q \vee ((\neg s \vee q) \wedge s)$	Commutativity
14.	$(q \vee (\neg s \vee q)) \wedge (q \vee s)$	Distributivity
15.	$(q \vee (q \vee \neg s)) \wedge (q \vee s)$	Commutativity
16.	$q \wedge (q \vee s)$	Absorption
17.	$q$	Absorption

- (a) There are (at least) two significant errors in this proof. Indicate which lines contain the errors and, for each one, explain why that line is incorrect. We only needed about 1 sentence to explain each incorrect line. [8 points]
- (b) Is the conclusion of the “spoof” correct (that is, is the “Theorem” true)? If it is incorrect, describe propositions  $p, q, r, s$  such that the givens are true, but the claim is false. If the conclusion is correct, briefly explain how to correct any errors in lines 1–6 (you’ll explain errors in 6–18 in part c). [4 points]
- (c) Give a correct proof of what is claimed in lines 6–18, i.e., that from  $(s \rightarrow q) \wedge (\neg s \rightarrow q)$ , we can infer that  $q$  is true. [10 points]

## 7. Feedback [1 point]

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?

### Office Hours Feedback

We're trying to balance the timing and modes of office hours – have you found yourself wanting more in-person office hours? Or more zoom ones? More on a certain day of the week? If so, please let us know!