MATH/STAT395: Probability II Winter 2022

Exercises from the lectures

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1 Joint distribution of discrete random variables

Exercise 1. Roll two fair dice with **4 faces**, denote

(i) S the sum of the two dice

(ii) Y the indicator variable that you get a pair

1. Record which outcomes lead to different values of S, Y
2. Compute the corresponding joint probability mass function of S, Y
3. Read the proba to get a sum of 4 while not having a pair
4. Compute probability to get a sum higher or equal than 5 with pairs
5. Score is the sum of the dice, doubled if it is a pair. What is the average score?
6. Compute the marginal p.m.f. of Y from the joint p.m.f.

Solution

1.

		Y	
		0	1
	2		(1,1)
	3	(1, 2) (2, 1)	
	4	(1, 3) (3, 1)	(2, 2)
S	5	(1, 4) (2, 3) (3, 2) (4, 1)	
	6	(2, 4) (4, 2)	(3, 3)
	7	(3, 4) (4, 3)	
	8		(4, 4)

2.

		Y	
		0	1
	2	0	1/16
	3	1/8	0
	4	1/8	1/16
S	5	1/4	0
	6	1/8	1/16
	7	1/8	0
	8	0	1/16

3. $\mathbb{P}(S = 4, Y = 0) = 1/8$
4. $\mathbb{P}(S \geq 5, Y = 1) = \mathbb{P}(S = 5, Y = 1) + \ldots + \mathbb{P}(S = 8, Y = 1) = 2/16$

5. The score is $g(S,Y) = S(Y+1)$.

   The average score reads

$$\begin{aligned}\mathbb{E}[g(S,Y)] &= \sum_{s=2}^{8}\sum_{y=0}^{1} s(y+1)p(s,y)\\ &= \sum_{s=2}^{8} sp(s,0) + 2\sum_{s=2}^{8} sp(s,1) = \frac{3+4+2\times 5+6+7}{8} + 2\times\frac{2+4+6+8}{16}\\ &= 25/4 = 6.25\end{aligned}$$

6. Sum the columns of $p(s,y)$, so you get $\mathbb{P}(Y=1) = 4/16$ and $\mathbb{P}(Y=0) = 12/16$. Note that it is exactly the p.m.f. you would have computed in the first place for Y (you would simply not have taken into account the result of the sum)

Exercise 2. Roll repeatedly a pair of dice.

Denote N the number of rolls until the sum of the dice is 2 or a 6

1. What is the distribution of N?
2. Denote X the sum you finally get (2 or 6),

   are X and N independent?

Solution Let S_i be the sum of the two dice at the i$^{\text{th}}$ roll.

We have $\mathbb{P}(S_i \in \{2,6\}) = 1/36 + 5/36 = 1/6$ and so $N \sim \mathrm{Geom}(1/6)$

$\mathbb{P}(N=n, X=6) = \mathbb{P}(S_1 \notin \{2,6\}, \dots, S_{n-1} \notin \{2,6\}, S_n = 6) = \left(\frac{5}{6}\right)^{n-1}\frac{5}{36}$ Therefore $\mathbb{P}(X=6) = \sum_{n=1}^{+\infty}\left(\frac{5}{6}\right)^{n-1}\frac{5}{36} = \frac{5/36}{1-5/6} = 5/6$

So $\mathbb{P}(N=n, X=6) = \left(\frac{5}{6}\right)^{n-1}\frac{1}{6}\frac{5}{6} = \mathbb{P}(N=n)\mathbb{P}(X=6)$

Same argument shows $\mathbb{P}(N=n, X=2) = \mathbb{P}(N=n)\mathbb{P}(X=2)$

$\rightarrow$ N and X are independent.

Exercise 3.

1. Define 2 independent geometric variables

   - X the number of days until your friend Lulu sends you a letter, with an average waiting time of $1/\lambda$ days $(0 < \lambda < 1, X \geq 1)$
   - Y the number of days until your friend Barry sends you a letter, with an average waiting time of $1/\mu$ days $(0 < \mu < 1, Y \geq 1)$

   Let D be the time before either sends you a letter. What is the cumulative distribution function of D?

2. In terms of the named distributions we have covered, what is the distribution of D? Include any parameter values.

3. Derive the p.m.f. of I defined by

$$I = \begin{cases} 0 & \text{if Lulu's letter arrive stricly before Barry's letter} \\ 1 & \text{if Lulu's letter arrive the same day as Barry's letter} \\ 2 & \text{if Lulu's letter arrive strictly after Barry's letter} \end{cases}$$

4. Are I and D independent?

Solution

1. Note that $D = \min(X, Y)$. Thus,

$$\begin{aligned}\mathbb{P}(D \le d) &= 1 - \mathbb{P}(X > d, Y > d) \\ &= 1 - \mathbb{P}(X > d) \cdot \mathbb{P}(Y > d) \\ &= 1 - (1-\lambda)^d \cdot (1-\mu)^d \\ &= 1 - ((1-\lambda)(1-\mu))^d\end{aligned}$$

2. D is an $\mathrm{Geom}(1 - (1-\lambda)(1-\mu))$ variable.

3. First, note that the joint p.m.f of X, Y is

$$f_{X,Y}(x, y) = \lambda(1-\lambda)^{x-1}\mu(1-\mu)^{y-1}$$

$$\begin{aligned}\mathbb{P}(I = 0) &= \mathbb{P}(X < Y) \\ &= \sum_{y=1}^{\infty}\sum_{x=1}^{y-1} f_{X,Y}(x, y) \\ &= \sum_{y=1}^{\infty}\sum_{x=1}^{y-1} \lambda(1-\lambda)^{x-1}\mu(1-\mu)^{y-1} \\ &= \sum_{y=1}^{\infty} \mu(1-\mu)^{y-1}(1 - (1-\lambda)^{y-1}) \\ &= 1 - \frac{\mu}{1 - (1-\mu)(1-\lambda)} \\ &= \frac{\lambda(1-\mu)}{\lambda + \mu - \lambda\mu}\end{aligned}$$

Similarly, $\mathbb{P}(I = 2) = 1 - \frac{\lambda}{1-(1-\mu)(1-\lambda)} = \frac{\mu(1-\lambda)}{\lambda+\mu-\lambda\mu}$ and we get $\mathbb{P}(I = 1) = 1 - \mathbb{P}(I = 0) - \mathbb{P}(I = 2) = \frac{\mu\lambda}{1-(1-\mu)(1-\lambda)} = \frac{\mu\lambda}{\lambda+\mu-\lambda\mu}$.

4. Note that $\mathbb{P}(I = 0, D \le d) = \mathbb{P}(X \le d, Y > X)$.

$$\begin{aligned}\mathbb{P}(X \le d, Y > X) &= \sum_{x=1}^{d}\sum_{y=x+1}^{\infty} \lambda(1-\lambda)^{x-1}\mu(1-\mu)^{y-1} \\ &= \sum_{x=1}^{d} \lambda(1-\lambda)^{x-1} \sum_{y=x+1}^{\infty} \mu(1-\mu)^{y-1} \\ &= \sum_{x=1}^{d} \lambda(1-\lambda)^{x-1}(1-\mu)^{x} \\ &= \frac{\lambda(1-\mu)(1 - (1-\lambda)^d(1-\mu)^d)}{1 - (1-\lambda)(1-\mu)} \\ &= P(I = 0) \cdot \mathbb{P}(D \le d)\end{aligned}$$

The computations for the case $I = 2, D \leq d$ are analogous such that $\mathbb{P}(I = 2, D \leq d) = \mathbb{P}(I = 2)\mathbb{P}(D \leq d)$ and finally

$$\begin{aligned}\mathbb{P}(I = 1, D \leq d) &= \mathbb{P}(I = 1|D \leq d)\mathbb{P}(D \leq d)\\ &= [1 - \mathbb{P}(I = 0|D \leq d) - \mathbb{P}(I = 2|D \leq d)]\mathbb{P}(D \leq d)\\ &= [1 - \mathbb{P}(I = 0) - \mathbb{P}(I = 2)]\mathbb{P}(D \leq d)\\ &= \mathbb{P}(I = 1)\mathbb{P}(D \leq d)\end{aligned}$$

Thus, I and D are independent.

2 Multinomial random variables

Exercise 4. Roll a fair die 100 times. Find the probability that among the 100 rolls, we observe exactly 22 ones, 17 fives.

Solution Denote X_1, X_5 the number of times you get a 1 or a 5 resp. among 100 rolls. We have $\mathbb{P}(\text{"face is 1"}) = \mathbb{P}(\text{"face is 5"}) = 1/6$. We could model $X_1, X_2, X_3, X_4, X_5, X_6$ as a multinomial but that can be simplified. Denote $Y = X_2 + X_3 + X_4 + X_6$ the number of times you get any other face. We have $\mathbb{P}(\text{"face is not 1 or 5"}) = 4/6 = 2/3$. Then $(X_1, X_5, Y) \sim \text{Multinom}(100, 3, 1/6, 1/6, 2/3)$. So

$$\mathbb{P}(X_1 = 22, X_5 = 17, Y = 100 - (22 + 17)) = \frac{100!}{22!17!61!}\left(\frac{1}{6}\right)^{22}\left(\frac{1}{6}\right)^{17}\left(\frac{2}{3}\right)^{61} \approx 0.0037$$

3 Joint distribution of continuous random variables

Exercise 5. Throw a dart uniformly at random on a disk of radius 2
What is the probability that the dart is in the central disk of radius one?

Solution Denote $D_r = \{(x, y) : x^2 + y^2 < r^2\}$ a disk of radius r. Then, $(X, Y) \sim \text{Unif}(D_2)$.

$$\mathbb{P}((X, Y) \in D_1) = \frac{\pi 1^2}{\pi 2^2} = \frac{1}{4}$$

Exercise 6. Throw a dart uniformly at random on a square of edge size 2 centered on 0
Assume your score is equal to the square distance to the center
What is your average score?

Solution $(X, Y) \sim \text{Unif}(S)$ with $S = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$.
Score is $g(x, y) = x^2 + y^2$
Average score:

$$\mathbb{E}[g(X, Y)] = \frac{1}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x, y)\mathbf{1}_S(x, y)dxdy = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(x^2 + y^2)dxdy = 2/3$$

Exercise 7. Consider a disk of radius r, $D_r = \{(x, y) : x^2 + y^2 \le r^2\}$ and $(X, Y) \sim \text{Unif}(D_r)$.
What is the marginal p.d.f. of X?

Solution Joint p.d.f. is $f_{X,Y}(x, y) = \frac{1}{\pi r^2}\mathbf{1}_{D_r}(x, y)$ where $D_r = \{(x, y) : x^2 + y^2 \le r^2\}$
Marginal density is then $f_X(x) = 0$ for $|x| > r$, and for $|x| \le r$,

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y)dy = \frac{1}{\pi r^2}\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} dy = \frac{2}{\pi r^2}\sqrt{r^2 - x^2}$$

Exercise 8 (Shooting an arrow)**.** Consider X, Y with p.d.f.

$$f(x, y) = \frac{1}{\lambda}\frac{e^x}{\sqrt{y+1}}\mathbf{1}_W(x, y)$$

for $\lambda{=}2(\sqrt{2} - 1)(e - e^{-1})$
where $W{=}\{(x, y) : -1{\le}x{\le}1, 0{\le}y{\le}1\}$.

1. Are X, Y independent?
2. What consequences it had when computing the probability to get the target $T{=}\{(x, y){:}-0.1{\le}x{\le}0.1, 0.4{\le}y{\le}0.6\}$?

Solution

1. Note that $\mathbf{1}_W(x, y) = \mathbf{1}_{[-1,1]}(x)\mathbf{1}_{[0,1]}(y)$,

   then one has $f_X(x) = \frac{1}{e-e^{-1}}e^x\mathbf{1}_{[-1,1]}(x)$, $f_Y(y) = \frac{1}{2(\sqrt{2}-1)\sqrt{y+1}}\mathbf{1}_{[0,1]}(y)$

   So X, Y are independent.
2. $\mathbb{P}((X, Y) \in T) = \mathbb{P}(X \in [-0.1, 0.1])\mathbb{P}(Y \in [0.4, 0.6])$ where $\mathbb{P}(X \in [-0.1, 0.1])$, $\mathbb{P}(Y \in [0.4, 0.6])$ can be computed from f_X, f_Y respectively.

Exercise 9. Let $(X, Y) \sim \text{Unif}(D)$ with D a disk centered at 0 with radius r_0 Let (R, Θ) be the polar coordinates of (X, Y) such that

$$X = R\cos(\Theta), Y = R\sin(\Theta)$$

1. Find the joint and marginal p.d.f. of R and Θ
2. Are R and Θ independent?

Solution

1. Since (X, Y) is in $D = \{(x, y) : x^2 + y^2 \le r_0^2\}$ we have that $0 \le R \le r_0$. Moreover $\Theta \in [0, 2\pi)$. So we want to compute $F_{R,\Theta}(u, v) = \mathbb{P}(R \le u, \Theta \le v)$ for $u \in [0, r_0]$,$v \in [0, 2\pi)$. Let $r(x, y)$, $\theta(x, y)$ be the polar coordinates (r, θ) of a point (x, y). Then the set $A_{u,v} = \{(x, y) : r(x, y) \le u, \theta(x, y) \le v\}$ is a circular sector of radius u bounded by the angles 0 and v

The area of $A_{u,v}$ is $\frac{1}{2}u^2 v$ so we get

$$F_{R,\Theta}(u,v) = \mathbb{P}((X,Y) \in A_{u,v}) = \frac{\frac{1}{2}u^2 v}{r_0^2 \pi}$$

So we get the joint p.d.f. of R, Θ for $0 \le r \le r_0$ and $0 \le \theta \le 2\pi$

$$f_{R,\Theta}(r,\theta) = \frac{\partial^2}{\partial u \partial v} F_{R,\Theta}(u,v) = \frac{r}{r_0^2 \pi}$$

We deduce

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta) d\theta = \frac{2r}{r_0^2} \quad f_\Theta(\theta) = \int_0^{r_0} f_{R,\Theta}(r,\theta) dr = \frac{1}{2\pi}$$

2.

$$f_{R,\Theta}(r,\theta) = f_R(r) f_\Theta(\theta)$$

so R and Θ are independent !

Exercise 10. Let X, Y be two independent $\text{Exp}(\lambda)$ r.v.
Find the joint p.d.f. of $U = X + Y$ and $V = \frac{X}{X+Y}$. Are U, V independent?

Solution p.d.f. of X is $f_X(x) = e^{-\lambda} \mathbf{1}_{(0,+\infty)}(x)$, same for Y Since X, Y are independent, they are jointly continuous with joint p.d.f. $f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)} \mathbf{1}_{(0,+\infty)^2}(x,y)$. We have that $g((0,+\infty)^2)$=$(0,+\infty) \times (0,1)$. For $u, v \in (0,+\infty) \times (0,1)$ and $(x,y) \in (0,+\infty) \times (0,+\infty)$,

$$u = x + y, \quad v = \frac{x}{x+y} \iff x = uv, \quad y = (1-v)u,$$

Therefore the inverse mapping is

$$\gamma(u,v) = (uv, (1-v)u)$$

Denoting $\alpha(u,v) = uv$, $\beta(u,v) = (1-v)u$, the determinant of the Jacobian is

$$\det(J_\gamma(u,v)) = \det \begin{pmatrix} \frac{\partial \alpha}{\partial u} & \frac{\partial \alpha}{\partial v} \\ \frac{\partial \beta}{\partial u} & \frac{\partial \beta}{\partial v} \end{pmatrix} = \det \begin{pmatrix} v & u \\ 1-v & -u \end{pmatrix} = (v-1)u - uv = -u$$

Applying the formula

$$f_{U,V}(u,v) = f_{X,Y}(\gamma(u,v)) |\det(J(u,v))| \mathbf{1}_{g(S)}(u,v) = \lambda^2 u e^{-\lambda u} \mathbf{1}_{(0,+\infty)}(u) \mathbf{1}_{(0,1)}(v)$$

We got

$$f_{U,V}(u,v) = \lambda^2 u e^{-\lambda u} \mathbf{1}_{(0,+\infty)}(u) \mathbf{1}_{(0,1)}(v)$$

Clearly, U, V are independent since the join is the product of one function depending exclusively on u and another function depending exclusively on v

Exercise 11. Let $B_1, \dots B_{m+n} \sim \text{Ber}(p)$ be $m + n$ independent random variables.
Denote $S_1 = \sum_{i=1}^m B_i$ and $S_2 = \sum_{i=m+1}^n B_i$,

1. Are S_1 and S_2 are independent.
2. Are $Z = S_1 + S_2$ and S_1 independent?

Solution

1. By Lemma, yes:

   Lemma 3.1. *Let $X_1, \dots, X_{m+n}$ be $m+n$ independent r.v. (discrete or continuous).*
   Let $g : \mathbb{R}^m \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$.
   Then $Y = g(X_1, \dots, X_m)$ and $Z = h(X_{m+1}, \dots, X_{m+n})$ are independent.

2. $\mathbb{P}(S_1 = 1, Z = 0) = 0 \neq \mathbb{P}(S_1 = 1)\mathbb{P}(Z = 0) > 0$ so no

Exercise 12. n players have each one coin with p_i the probability for player i to get a tail
At each round they all toss the coin, independently, they repeat it until one player gets a tail (the winner)
What is the distribution of the number of rounds before the game ends?

Note: When asked what is the distribution of some r.v., we are asking you to recognize the r.v. among one of the classical ones.

Solution Let X_i be the number of tosses player i does before getting a tail,
we model it as $X_i \sim \mathrm{Geom}(p_i)$, $p_i \in (0, 1)$. The game ends when any player gets a tail so the number of tosses before the game ends is $Y = \min(X_1, \dots, X_n)$. For $k \in \mathbb{N}$, $1 - F_{X_i}(k) = \mathbb{P}(X_i > k) = (1 - p_i)^k$. By previous lemma, $1 - F_Y(k) = \mathbb{P}(Y > k) = \prod_{i=1}^n (1 - p_i)^k$. Then denoting $(1 - r) = \prod_{i=1}^n (1 - p_i)$,

$$\mathbb{P}(Y = k) = \mathbb{P}(Y > k - 1) - \mathbb{P}(Y > k) = (1 - r)^{k-1} - (1 - r)^k = (1 - r)^{k-1} r$$

So we recognize $Y \sim \mathrm{Geom}(r)$

Exercise 13. I'm sitting on a bench in Dalvìkurbyggd watching whales.
On average, a whale shows up every 5min.
What is p.d.f. of the time I wait for seeing n whales?
I'm assuming that the times that I wait to see each new whale are independent

Solution Let T_i be the time that I wait to see the ith whale after seeing the i-1th one. Model $T_i \sim \mathrm{Exp}(\lambda)$ with $\lambda = 5$ with the T_i independent. The total time that I'm waiting to see the nth whale is then $X_n = T_1 + \ldots + T_n$. Let's start with $n = 2$, for $x \geq 0$ (otherwise clearly $f_{X_2}(x) = 0$)

$$\begin{aligned} f_{X_2}(x) &= \int_{-\infty}^{\infty} f_{T_1}(t) f_{T_2}(x - t) dt = \int_0^x f_{T_1}(t) f_{T_2}(x - t) dt \\ &= \int_0^x \lambda e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^2 x e^{-\lambda x} \end{aligned}$$

So $f_{X_2}(x) = \lambda^2 x e^{-\lambda x} \mathbf{1}_{[0,+\infty)}$. Now for $n = 3$, $X_3 = X_2 + T_3$ with X_2 and T_1 independent

$$\begin{aligned} f_{X_3}(x) &= \int_{-\infty}^{\infty} f_{X_2}(t) f_{T_3}(x - t) dt = \int_0^x f_{X_2}(t) f_{T_3}(x - t) dt \\ &= \int_0^x \lambda^2 t e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^3 e^{-\lambda x} \int_0^x t dt = \lambda^3 \frac{x^2}{2} e^{-\lambda x} \end{aligned}$$

Let's conjecture that for $k \in \mathbb{N}$, $f_{X_k}(x) = \lambda^k \frac{x^{k-1}}{(k-1)!} e^{-\lambda x}$ and assume it's true for some $k \geq 1$, then

$$\begin{aligned} f_{X_{k+1}}(x) &= \int_{-\infty}^{\infty} f_{X_k}(t) f_{T_{k+1}}(x-t)dt = \int_0^x f_{X_k}(t) f_{T_{k+1}}(x-t)dt \\ &= \int_0^x \lambda^k \frac{t^{k-1}}{(k-1)!} e^{-\lambda t} \lambda e^{-\lambda(x-t)} dt = \lambda^{k+1} e^{-\lambda x} \int_0^x \frac{t^{k-1}}{(k-1)!} dt = \lambda^{k+1} \frac{x^k}{k!} e^{-\lambda x} \end{aligned}$$

So we have shown that

$$f_{X_n}(x) = \lambda^n \frac{x^{n-1}}{(n-1)!} e^{-\lambda x}$$

Exercise 14. 1. Suppose that X and Y are independent random variables with density functions

$$f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} 4ye^{-2y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

Find the density function of $X + Y$.

Solution

1. Both X and Y have probability densities that are zero for negative values, so the same is true for $X + Y$. Using the convolution formula, for $z \geq 0$, we get

$$\begin{aligned} f_{X+Y}(z) &= \int_0^z f_X(x) f_Y(z-x) dx \\ &= \int_0^z 2e^{-2x} 4(z-x) e^{-2(z-x)} \\ &= \int_0^z 8(z-x) e^{-2z} dx \\ &= 8e^{-2z} \int_0^z (z-x) dx = 4z^2 e^{-2x} \end{aligned}$$

Thus,

$$f_{X+Y}(z) = \begin{cases} 4z^2 e^{-2z} & z \geq 0 \\ 0 & z < 0 \end{cases}$$