MATH/STAT395: Probability II	Winter 2022

Challenge Problem 1

Due **Feb 9th** by 11:59pm

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Upload **detailed solutions** to the questions below to Gradescope **as a pdf file**. **Answer each exercise on a separate page**. No credit will be given without **clear and complete** mathematical explanations.

Exercise 1 (10 pts)**.**

1. Let $Z_1, \dots Z_{25}$ be independent $\mathrm{Unif}[0,1]$ random variables. Let Y be the 13th largest of the 25 random variables. Find the probability density function of Y.

Exercise 2 (10pts)**.** Let $X_1, X_2, ...$ be a sequence of independent and identically distributed continuous random variables. Let $T \geq 2$ be such that

$$X_1 \geq X_2 \geq .... \geq X_{T-1} < X_T$$

That is, T is the point at which the sequence stops decreasing. Compute $E[T]$.

Hint: Use the exchangeability of $X_1, X_2 \dots$.

Exercise 1 (10 pts).

1. Let $Z_1, \ldots Z_{25}$ be independent Unif$[0,1]$ random variables. Let Y be the 13th largest of the 25 random variables. Find the probability density function of Y.

Since $Z_1, \ldots Z_{25}$ are independent, identically distributed random variables. (Unif$[0,1]$)

If we denote Y be the 13th largest of the 25 random variables. there will be 12 random variables less than or equal to Y.

And thus $Y \sim Beta(k, n-k+1)$ where $k=13$, $n=25$

$$Y \sim Beta(13, 13)$$

Therefore,

the probability density function of Y is

$$\boxed{Y = \begin{cases} \frac{25!}{12!\,12!} \cdot y^{12} \cdot (1-y)^{12} & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}}$$

Exercise 2 (10pts)**.** Let $X_1, X_2, ...$ be a sequence of independent and identically distributed continuous random variables. Let $T \geq 2$ be such that

$$X_1 \geq X_2 \geq .... \geq X_{T-1} < X_T$$

That is, T is the point at which the sequence stops decreasing. Compute $E[T]$.

Hint: Use the exchangeability of $X_1, X_2 \ldots$.

Since $X_1 \ldots X_{T-1}$ are i.i.d. continuous random variables.

So the possible number of permutation is $(T-1)!$ So

$P(X_1 \geq X_2 \geq \ldots \geq X_{T-1}) = \frac{1}{(T-1)!}$

If we requires $X_{T-1} < X_T$ and also $X_1 \geq X_2 \geq \ldots \geq X_{T-1}$.

$P(X_T > X_{T-1} \mid X_1 \geq X_2 \geq \ldots \geq X_{T-1}) = 1 - \frac{1}{T} = \frac{T-1}{T}$

Therefore, $P(X_1 \geq X_2 \geq \ldots \geq X_{T-1} < X_T) = \frac{1}{(T-1)!} \cdot \frac{T-1}{T} = \frac{1}{(T-2)!\,T}$

$$E(T) = \sum_{T=2}^{\infty} T \cdot \frac{1}{(T-2)! \cdot T}$$

$$= \sum_{T=2}^{\infty} \frac{1}{(T-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{1}{x!}$$

$$\boxed{= e}$$