Lecture Notes on

Basic Structures: Sets, Relations and Functions

By

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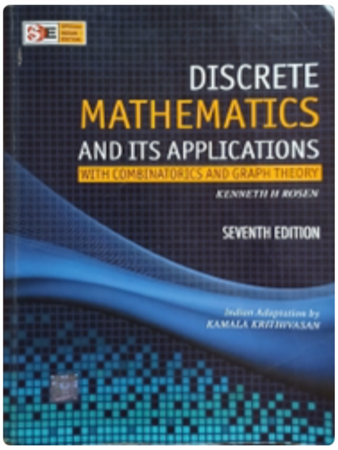
REFERENCE: "DISCRETE MATHEMATICS AND ITS APPLICATIONS WITH COMBINATORICS AND GRAPH THEORY" BY KENNETH H ROSEN, SEVENTH EDITION (Indian Adaption by KAMALA KRITHIVASAN)



SETS

Much of the Discrete mathematics — devoted to the study of Discrete structures used to represent discrete objects. SETS are one of the fundamental discrete structure.

Definition: (SETS)

A set is an unordered collection of well-defined objects.

The objects in a set are called the elements, or members of the set. A set is said to contain its elements. (element x in A is denoted by $x \in A$)

This read as x is a member of the set A or x belongs to A

unordered means: $\{1, 2, 3\}$ or $\{3, 1, 2\}$ or $\{2, 3, 1\}$
i.e. no specific order for elements/objects.

well defined : Set should be well defined in the sense that, it should not vary/change person to person.

eg: set of 5 most beutiful flowers is not a set.

Definition: Let S be a set if there are exactly 'n' distinct elements in S where $n \geq 0$, then S is finite set and we say n is cardinality of the set. denoted by $|S| = n$.
If there is no such an 'n', then the set S is infinite set, then $|S| = \infty$

Representation of sets:

① Roster form / Tabular form

List all the elements of the set, separated by comas and enclosed in curly braces { }.
eg: $\{1, 2, 3, 4\}$, $\{5, 10, 15, 20\}$

② Set Builder form

Describe the property of elements in the set, using a variable and condition.
eg: $S_1 = \{x \mid x$ is a natural number less than 6$\}$ here $|S_1| = 5$

This method mainly use when it is impossible to list out all elements.
eg: $S_2 = \{x \mid x$ is an even integer$\}$.
here $|S_2| = \infty$.

Some Standard notations.

$\mathbb{Z}$: Denote set of all integers.
$\{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{N}$: Set of natural numbers.
$\{1, 2, 3, 4, 5, \ldots\}$

$\mathbb{Q}$: Set of rational numbers.
$\{x : x = p/q \quad p, q \neq 0 \text{ in } \mathbb{Z}\}$

$\mathbb{R}$: Set of all real numbers.

$\mathbb{Z}^+$: +ve integers $= \mathbb{N}$

$\mathbb{Z}^-$: −ve integers $= \{-1, -2, -3, \ldots\}$

$\mathbb{Q}^+$: +ve rationals $x = p/q$, p, and $q \neq 0 \in \mathbb{Z}^+$

$\mathbb{Q}^-$: −ve rationals $x = p/q$, One of p or q is −ve, other +ve. integer.

$\mathbb{R}^+$: +ve real numbers.

Definition.

Two sets A & B are equal, that is
$A = B$ iff $\forall x\, (x \in A \longleftrightarrow x \in B)$

eg: $A = \{1, 2, 3\}$
$B = \{2, 3, 1\}$
Then $A = B$

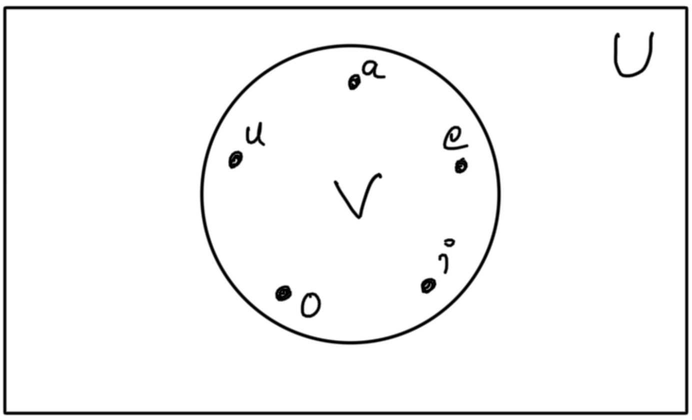
☆ it does not matter if an element of a

set is listed more than once.
so $\{1, 3, 3, 3, 5\ 5\}$ is the same as the set $\{1,3,5\}$.

→ Sets can be represented with venn diagrams.

To discuss venn diagram we take a set having all possible objects under consideration which we call the universal set, denote by U.

eg: Represent the set of vowels in english by venn diagram



⇒ A set is called empty set if it has no elements.
empty set denoted by ϕ or $\{\}$
$|\phi| = 0$.

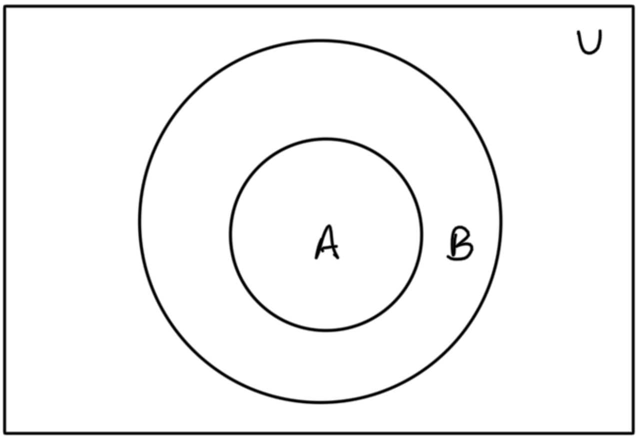
$|\phi|$ —

$\Rightarrow$ A set having single element is called singleton set.

Definition: (Subset)

set A is a subset of B if every element of A is also an element of B. denote it by $A \subseteq B$.

$A \subseteq B$ in logical expression is
$\forall x \, (x \in A \longrightarrow x \in B)$



venn diagram representing $A \subseteq B$.

Theorem: For every set S,
(i) $\phi \subseteq S$ and (ii) $S \subseteq S$.

proof: (i) $\phi \subseteq S$.

enough to show $\forall x \, (x \in \phi \rightarrow x \in S)$ is true.

as $x \in \phi$ is always false,

it follows $x \in \phi \rightarrow x \in S$ is always true. (by truth value of conditional statement)

Thus $\forall x \, (x \in \phi \rightarrow x \in S)$ is true.

This proves (i) (this is eg for vacuous proof.)

(ii) $S \subseteq S$

enough to P.T

$$\forall x \, (x \in S \rightarrow x \in S)$$

as $x \in S$ always true.

The conditional stmnt $x \in S \rightarrow x \in S$ is always true.

that is $\forall x \, (x \in S \rightarrow x \in S)$ true.

This proves (ii).

(note that this is an example for trivial proof.)

☆ A is proper subset of B if $A \subseteq B$ and $A \neq B$. we denote proper subset by $A \subset B$.

in logical expression $A \subset B$ is same as

$$\forall x \, (x \in A \rightarrow x \in B) \wedge \exists x \, (x \in B \wedge x \notin A)$$

☆ To show $A = B$ enough to S.T. $A \subseteq B$ and $B \subseteq A$.

Definition: Given a set S, the power set of S is the set of all subsets of the set S. The power set is denoted by $P(S)$.

eg: what is power set of $S = \{0, 1, 2\}$

$$P(S) = \{\phi, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$$

Q. $P(\phi) = \{\phi, \{\phi\}\}$

☆ $$|P(S)| = \begin{cases} 2^n & \text{if } |S| \geq 1 \\ 2 & \text{if } |S| = 0, \text{ i.e. } S = \phi \end{cases}$$

Russell's paradox

Let the domain be set of all sets.

define a set $S = \{x \mid x \notin x\}$

Then S is set of all sets which are not members of themselves.

Now the question is

"Is S is a member of S itself"

If S is a member of S, then by definition of S, $S \notin S$

a contradiction to assumption $S \in S$.

If S not a member of S, then by definition of S, $S \in S$.
a contradiction to assumption $S \notin S$.

This paradox shows inconsistency in sets/ set theory.

↓

This inconsistencies avoided by some axioms, such as creating axiom of "levels" of sets.
x can be a member of y, if y is a level higher than x.

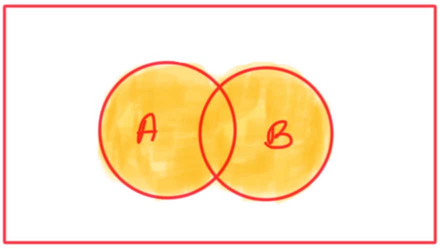
SET OPERATIONS

operations which combine two or more sets.

① Union

Let A & B be two sets.
union of A & B denoted by $A \cup B$ and it is defined by

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

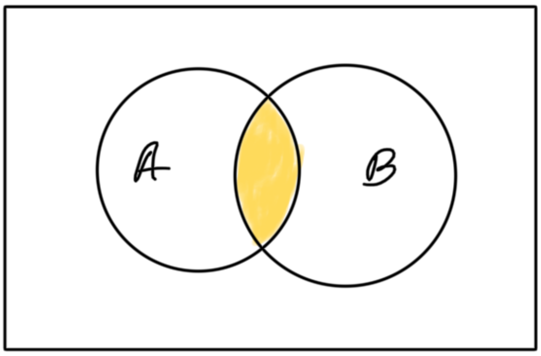


venn diagram: $A \cup B$ is shaded

② Intersection

Let A & B be two sets.
Intersection of A & B denoted by $A \cap B$ and it is defined by

$$A \cap B = \{ x \mid x \in A \wedge x \in B \}$$



venn diagram: $A \cap B$ is shaded

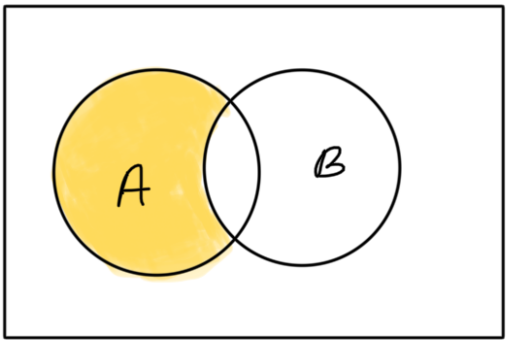
Remark: Two sets A and B are said to be disjoint if $A \cap B = \phi$

Remark 2: $|A \cup B| = |A| + |B| - |A \cap B|$

③ Difference

Let A & B be two sets.
difference of A & B denoted by $A - B$ and it is defined by

$$A - B = \{ x \mid x \in A \wedge x \notin B \}$$

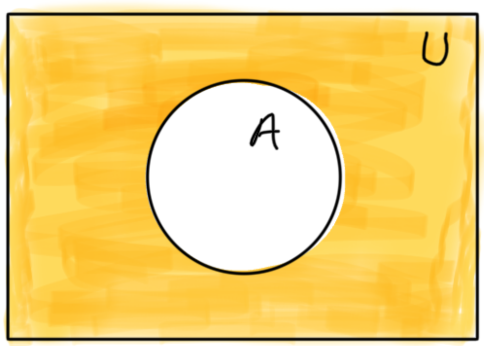


venn diagram : $A - B$ is shaded

④ Complement.

Let U be universal set. The complement of a set A (or complement of A w.r.t. U) is denoted by $\overline{A}$ (A^c), and is defined by

$$\overline{A} = U - A$$



venn diagram : $\overline{A}$ is shaded.

Set Identities

Identity	Name
$A \cup \phi = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \phi = \phi$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	Demorgans laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \phi$	Complement laws

Q. prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

proof: To P.T. $\overline{A \cap B} = \bar{A} \cup \bar{B}$

we prove $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ &

$\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

Let $x \in \overline{A \cap B}$

$\Rightarrow x \notin A \cap B$ by def. of complement.

$\Rightarrow \neg((x \in A) \wedge (x \in B))$

by def of $\cap$.

$\Rightarrow \neg(x \in A) \vee \neg(x \in B)$

by demorgans law of $\neg$

$\Rightarrow x \notin A \vee x \notin B$

by def. of $\neg$.

$\Rightarrow x \in \bar{A} \vee x \in \bar{B}$

by def. of compliment.

$\Rightarrow x \in \bar{A} \cup \bar{B}$

Thus $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ — (1)

on other hand let $x \in \bar{A} \cup \bar{B}$

$\Rightarrow x \in \bar{A}$ or $x \in \bar{B}$

by def. of $\cup$.

$\Rightarrow x \notin A$ or $x \notin B$.

by def. of complement.

$\Rightarrow \neg(x \in A) \vee \neg(x \in B)$

$\Rightarrow \neg((x \in A) \wedge (x \in B))$

by demorgans law of $\neg$

$\Rightarrow \quad \neg (x \in A \cap B)$

by def. of $\cap$

$\Rightarrow \quad x \in \overline{A \cap B}$

by def. of complement

This shows $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ — (2)

from (1) & (2) we get

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

eg: Use set builder notation and logical equivalences to show that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Ans. $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$

by def. of complement.

$= \{x \mid \neg (x \in (A \cap B))\}$

by def. of $\notin$

$= \{x \mid \neg (x \in A \wedge x \in B)\}$

by def. of $\cap$

$= \{x \mid \neg (x \in A) \vee \neg (x \in B)\}$

by demorgan's law of $\wedge$

$= \{x \mid x \notin A \vee x \notin B\}$

by def of $\notin$

$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$

by def. of compliment

$$= \{x \mid x \in \overline{A} \cup \overline{B}\}$$
by def. of union

$$= \overline{A} \cup \overline{B}$$
by meaning of setbuilder notation

Q. P.T the distributive law
i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

proof.

Suppose $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in B \cup C$ by def. of $\cap$

$\Rightarrow x \in A$ and $x \in B$ or $x \in C$
by def. of $\cup$.

$\Rightarrow x \in A$ and $x \in B$ or $x \in A$ and $x \in C$

$\Rightarrow x \in A \cap B$ or $x \in A \cap C$
by def of $\cap$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$
by def of $\cup$.

$\Rightarrow A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ — (1)

On other hand, Let $x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in (A \cap B)$ or $x \in A \cap C$
by def of $\cup$.

$\Rightarrow x \in A$ and $x \in B$ or $x \in A$ and $x \in C$
by def of $\cap$

$\Rightarrow x \in A$ and $x \in B$ or $x \in C$

$\Rightarrow x \in A$ and $x \in B \cup C$

by def of $\cup$

$\Rightarrow x \in A \cap (B \cup C)$

by def of $\cap$

Thus $(A \cap B) \cup (A \cap) \subseteq A \cap (B \cup C)$ — (2)

Hence (1) & (2) gives

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Membership table:

The set identies can also be proved by using membership table which is equivalent to logical truth table.
To indicate an element is in a set, a 1 is used; to indicate an element is not in a set, a 0 is used.

eg: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof:

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	0	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Q. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

Ans. $\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$

by demorgan's law-1

$= \overline{A} \cap (\overline{B} \cup \overline{C})$

by demorgan's law-2.

$= (\overline{B} \cup \overline{C}) \cap \overline{A}$

commutativity of $\cap$.

$= (\overline{C} \cup \overline{B}) \cap \overline{A}$

commutativity of $\cup$.

Generalised union & Intersection

☆ The union of collection of sets is the set that contains those elements that are members of atleast one set in the collection.

Notation:

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

☆ The intersection of collection of sets is the set that contains those elements that are members of all the sets in the collection.

Notation:

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

eg: for $i = 1, 2, 3, \ldots$

define $A_i = \{i, i+1, i+2, \ldots\}$

find $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$

Ans: $A_1 = \{1, 2, 3, \ldots\}$

$A_2 = \{2, 3, 4, \ldots\}$

$A_3 = \{3, 4, 5, \ldots\}$

$\vdots$

Thus $\bigcup_{i=1}^{n} A_i = A_1 = \{1, 2, 3, \ldots\}$

and

$\bigcap_{i=1}^{n} A_i = A_n = \{n, n+1, \ldots\}$

☆ infinite union

$$A_1 \cup A_2 \cup A_3 \cup \cdots = \bigcup_{i=1}^{\infty} A_i \quad \text{and}$$

$$A_1 \cap A_2 \cap A_3 \cap \cdots = \bigcap_{i=1}^{\infty} A_i$$

eg:

Let $A_i = \{1, 2, 3, \ldots, i\}$
for $i = 1, 2, 3, \ldots,$ Then

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \{1, 2, 3, \ldots, i\} = \mathbb{Z}^+$$

and

$$\bigcap_{i=1}^{\infty} A_i = \bigcap_{i=1}^{\infty} \{1, 2, \ldots, i\} = \{1\}$$

Relations and functions.

Definition: Let A and B be two sets. The cartesian product of A and B is denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.

i.e. $A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$

eg: $A = \{1, 2\}$ & $B = \{a, b, c\}$

Then

$$A \times B = \{ (1, a), (1, b), (1, c), (2, a), (2, b), (2, c) \}$$

Remark: in general $A \times B \neq B \times A$.
If $A = \phi$ or $B = \phi$ then $A \times B = \phi$.

Remark 2: $|A \times B| = |A| \cdot |B|$.

Definition: The cartesian product of n sets
$A_1 \times A_2 \times \cdots \times A_n = \{ (a_1, a_2, \cdots, a_n) \mid a_i \in A_i$
for $i = 1, 2, \cdots n \}$.
called an n-tuple.

Remark: $A \times B \times C \neq (A \times B) \times C$
A^2 denote $A \times A$.
A^3 denote $A \times A \times A$.
A^n denote $\underbrace{A \times A \times \cdots \times A}_{n \text{ times } A}$.

Relation: A subset R of the cartesian product $A \times B$ is called a relation from the set A to the set B.

☆ Note that elements of R are ordered pairs, with first element belong to A and 2nd belong to B.

☆ If $R \subseteq A \times B$, then we say R is a relation from A to B and we denote it by $R: A \mapsto B$.

Moreover if $(a,b) \in R$, then we say element a R-Related b, and denote it by aRb. If $R \subseteq A \times A$, then we say R is a Relation on A.

Q: R be the relation on $A = \{0, 1, 2, 3\}$ given by $aRb \longleftrightarrow a \leq b$. write R explicitely.

$$R = \{ (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2,3), (3,3) \}$$

Types of Relations

(i) Reflexive

A Relation R on a set A is called reflexive if $(a,a) \in R$ for every $a \in A$.

i.e. R is reflexive if $\forall a \in A, (a,a) \in R$.

eg (1) The "divides" relation on set of +ve integers is reflexive, because $a|a$ ($x|y$ denote x divides y) $\forall a \in \mathbb{Z}^+$.

but "divides" is not reflexive

on set of all integers $\mathbb{Z}$. because 0 does not divide 0.

$$\boxed{\text{Reflexive : } aRa \text{ or } (a,a) \in R}$$

② Symmetric Relation.

A relation R on a set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$; for all $a, b \in A$.

i.e. If $(a,b) \in R$ then $(b,a) \in R$

$$\boxed{\text{Symmetric : } (a,b) \in R \longrightarrow (b,a) \in R}$$

③ Antisymmetric

If $(a,b) \in R$, and $(b,a) \in R$ then $a = b$.

$$\boxed{\text{antisymmetric : } (a,b) \in R \text{ \& } (b,a) \in R \longrightarrow a = b}$$

eg: ② The "divides" Relation on set of +ve integers is not symmetric

since $1 | 2$ ✓ but $2 \nmid 1$

It is anti-symmetric.

since if $a|b$ and $b|a$ then $a = b$.

Thus eg ① & ②

Reflexive $\not\Longrightarrow$ Symmetric

Anti-symmetric $\not\Rightarrow$ symmetric

eg:

$R = \{(1,1)\ (2,2)\ (1,2)\}$

R on $\{1, 2\}$

R is reflexive

R is not symmetric as $1R2$ but 2 not related to 1

R is anti-symmetric.

This eg. shows reflexive can be anti-symmetric

eg: R on $\{1,2\}$ given by

$R = \{(1,1), (2,2), (1,2), (2,1)\}$

R is reflexive

R is symmetric

R not antisymmetric.

Thus symmetric $\not\Rightarrow$ anti-symmetric

(4) Transitive.

A relation R on a set A is called transitive if $(a,b) \in R$, $(b,c) \in R$ then $(a,c) \in R$.

eg: "divides" relation on set of +ve integers is transitive as if $a|b$ and $b|c$

Then $b = ak_1$ and $c = bk_2$.

$\Rightarrow \quad c = a k_1 k_2$. (use $b = a k_1$ on c)

$\Rightarrow \quad a \mid c$.

(5) Equivalence Relation-

A relation which is Reflexive, Symmetric and Transitive together is called equivalence relation.

Ex! : find more examples for each type of relations-!

Definition

Let R be a relation from a set A to a set B and S be a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs – (a, c), where $a \in A$, $c \in C$, and for which there exist an element $b \in B$, such that $(a, b) \in R$ and $(b, c) \in S$. we denote composite of R and S by $S \circ R$.

Remark: Let R be a relation on A.

Then $\quad R^2 = R \circ R$.

$\quad R^3 = R^2 \circ R \quad = R \circ R \circ R$.

Thus, $R^{n+1} = R^n \circ R$.

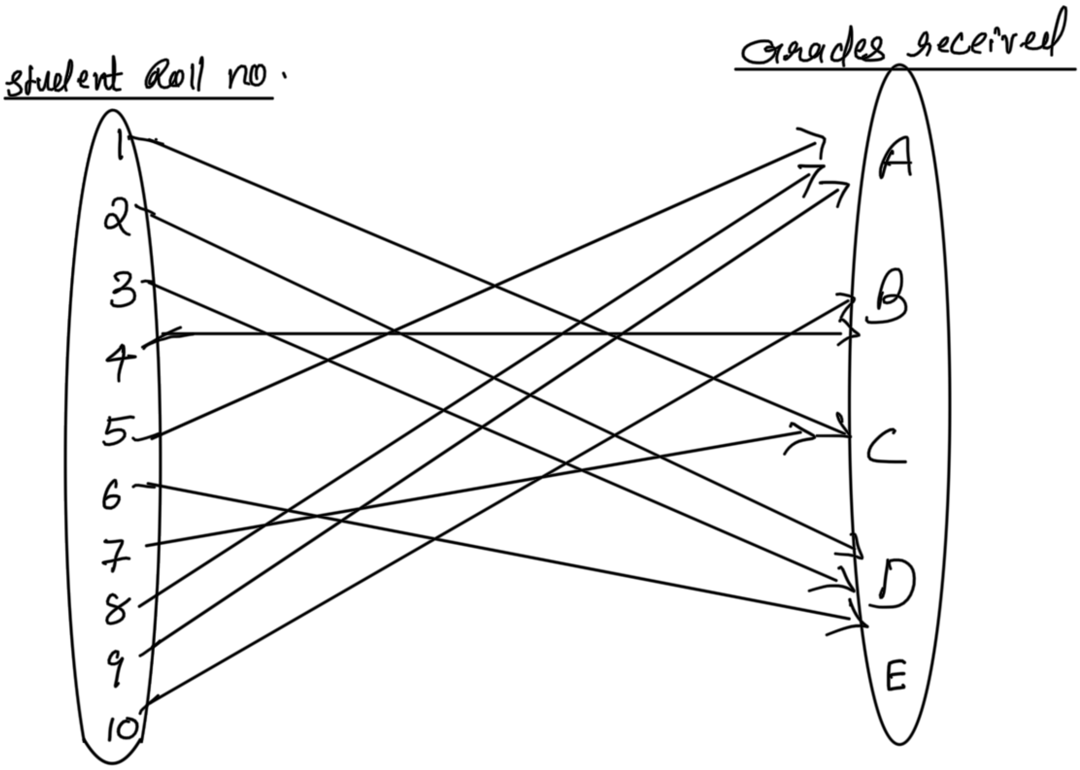
Functions

Consider the tasks that

1. Assigning grades (A or B or C or D or E) of Discrete mathematics course to 10 students with roll number 1 to 10.

2. Analyzing how many students received same grades.

In case -1



1. each students will get exactly one grade.

   that is a student will not get 2 different grades

2. different students can get same grade.

3. There may be some grade which is not assigned for any student.

4. A particular grade can assign to any number of students.

5. All students should assign a grade.

In case - 2 :

Grades.	Students received
A	5, 8, 9
B	4, 10
C	1, 7
D	2, 3, 6
E	

1. Any grade can receive for any number of students 0 to 10.

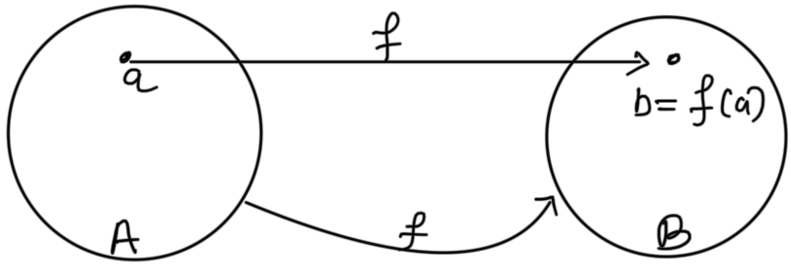
when we observe task - 1 we can see that it has more properties and more restrictions, so we should handle the situation carefully. In mathematics we handle such things with the help of special kind of relations called functions.

Definition: Let A and B be non-empty sets. A function $f: A \to B$ is an assignment of exactly one element of B to each element of A.

⇒ we write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B we write $f: A \to B$. Functions are sometimes also called mappings or transformations.

Different ways of defining functions.

1. By diagramatic way:



(Refer what we have done for the

mark assigning task discussed just above)

2. For function $f: A \to B$ one can give formula as $f(x) = x+1$

3. Computer program to specify a function

4. A function is a relation from A to B that contains one, and only one ordered pair (a, b) for every element $a \in A$, defines a function $f: A \to B$.

In this case we denote $f(a) = b$ and we say that b is image of a & a is called preimage of b

Definition:

Let $f: A \longrightarrow B$ be a function

Then A is called domain of f.

B is called codomain of f.

The range or image of f is the set

$$R(f) = \{ f(a) : a \in A \}$$

☆ if $f : A \to B$, we say f maps A to B.

☆ Two functions $f : A \to B$ and $g : C \to D$ are equal if $A = C$, $B = D$ and $f(a) = g(a)$ for every a in common domain.
In this case we denote $f = g$.

Example: find domain, codomain, and range of the function that assigns marks to students in task-1

Ans. Domain = $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
i.e. students roll numbers.
Codomain = $\{A, B, C, D, E\}$
all possible grade list.
Range = $\{A, B, C, D\}$
grades received by any student.

Example 2: Let R be the relation given by ordered pairs (Rajni, 18), (Anu, 19), (Kiran, 20), (Abid, 19), (Tessa, 18) where each pair represent a graduate student and the age of this student. what is the function that this relation determines.

Ans. The relation defines a function f such that $f(\text{Raju}) = 18$, $f(\text{Anu}) = 18$
$f(\text{Kiran}) = 20$, $f(\text{Abid}) = 19$ and $f(\text{Tessa}) = 18$

Here the domain is the set

$\{\text{Raju, Anu, Kiran, Abid, Tessa}\}$

Codomain is the set of possible ages of graduate students, for eg: you can take

$$\text{Co-domain} = \{15, 16, 17, \dots, 100\} \text{ or}$$
$$= \{1, 2, \dots, 120\} \text{ or}$$
$$= \{18, 19, 20, 21, 22, 23, 24, 25\}$$

finally the range is

$$= \{18, 19, 20\}$$

Remark: we know that a function is a relation, so that the domain will be set of all first component of ordered pair elements of R. In this case, Co-Domain = Range and it is the set of all second components of ordered pair elements of R.

Example: Let f be the function that assigns the last two bits of a bit string of length 2 or greater to that string.

for eg: $f(11010) = 10$.

Domain of f: set of all bit strings of length ≥ 2.

Codomain = Range $= \{00, 01, 10, 11\}$

Example: Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$.

Domain of $f = \mathbb{Z}$.

Co-domain $= \mathbb{Z}$

Range $= \{0, 1, 4, 9, \dots\}$

= set of all perfect squares.

Definition: (Addition and multiplication of real valued functions.)

Let f_1 and f_2 be functions from A to $\mathbb{R}$. then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x) \cdot f_2(x).$$

Example: Let f_1 and f_2 be functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f_1(x) = x^2$ and $f_2(x) = x - x^2$. what are the functions $f_1 + f_2$ and $f_1 \cdot f_2$?

Ans: From the definition of sum and product of functions, we have

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$= x^2 + (x - x^2)$$

$$= x$$

and $(f_1 f_2)(x) = f_1(x) f_2(x)$

$$= x^2 (x - x^2)$$

$$= \underline{\underline{x^3 - x^4}}$$

Definition: Let $f: A \longrightarrow B$ be a function and $S \subseteq A$. The image of S under f is denoted by $f(S)$ and defined by

$$f(S) = \{ f(s) : s \in S \}$$

Example: Let $A = \{a, b, c, d, e\}$ and $B = \{1, 2, 3, 4\}$ with $f(a) = 2$, $f(b) = 1$, $f(c) = 4$, $f(d) = 1$ and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

⇒ One-to-one and Onto Functions:

Definition: A function f is said to be one to one, or injective, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be an injection if it is one to one.

i.e. f is 1-1 means

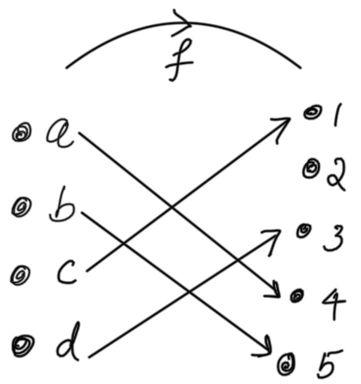
$$\forall\, a,b \in D \left(f(a) = f(b) \longrightarrow a = b \right)$$

This is equivalent to

$$\forall\, a,b \in D \left(a \neq b \longrightarrow f(a) \neq f(b) \right)$$

Example: Let $f : \{a, b, c, d\} \longrightarrow \{1, 2, 3, 4, 5\}$ given by $f(a) = 4$, $f(b) = 5$, $f(c) = 1$ and $f(d) = 3$. Check whether f is one-to-one or not.

Sol^n: f is one-to-one because f takes on different values at the four elements of it's domain. This is illustrated in following figure:



Example: Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one?

Sl^n : $f(x) = x^2$ not one-to-one

as $f(1) = f(-1) = 1$ but

$1 \neq -1$

Remark: The function $f(x) = x^2$ with it's domain restricted to $\mathbb{Z}^+$ is one-to-one.

Let $f: A \to B$ and $C \subseteq A$ then the restriction of f to C is the funtion $f|_C : C \to B$ s.t. $f|_C(x) = f(x)$ for all $x \in C$.

Example: Is $f: \mathbb{R} \to \mathbb{R}$. given by

$$f(x) = x+1, \text{ one-to-one ?}$$

Sl^n : If $f(x) = f(y)$ for any $x, y \in \mathbb{R}$

we have $x+1 = y+1$

$$\Rightarrow x = y.$$

This shows that $\underline{\underline{f \text{ is } 1\text{-}1}}$.

Definition: Let f be a function with domain and codomain are subsets of real numbers.

1. f is increasing if
$$\forall x \forall y \, (x < y \rightarrow f(x) \leq f(y))$$
2. f is strictly increasing if
$$\forall x \forall y \, (x < y \rightarrow f(x) < f(y))$$
3. f is decreasing if
$$\forall x \forall y \, (x < y \rightarrow f(x) \geq f(y))$$
4. f is strictly decreasing if
$$\forall x \forall y \, (x < y \rightarrow f(x) > f(y))$$

Remark: From these definition it follows that a function which is strictly increasing or strictly decreasing must be always one-to-one. However, a function increasing but not strictly increasing or decreasing but not strictly decreasing is not necessarily one-to-one.

Onto functions.

Definition: A function $f: A \to B$ is called onto or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjection if it is onto.

Remark: f is onto if $\forall y \, \exists x \, (f(x) = y)$

Domain of x = Domain of function

Domain of y = Codomain of function.

Remark 2: A function f is onto if and only if

Codomain of f = Range of f

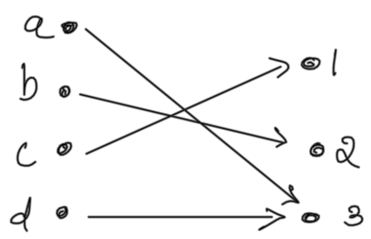
eg: ① $f: \{a, b, c, d\} \longrightarrow \{1, 2, 3\}$

$f(a) = 3$ $f(b) = 2$, $f(c) = 1$ and

$f(d) = 3$. is f is onto. ?

Sln: Because all 3 elements of the codomain are images of elements in the domain, we see that f is onto.

Following figure shows this fact more clearly.



Remark: If codomain $= \{1, 2, 3, 4\}$, then the above function f is not onto.

eg 2: $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$.

is f is onto ?

Ans: f is not onto, as

There is no integer x with $x^2 = -1$.

or no pre-image for $y = -1$

or no pre-image for $-$ve integers.

or $\underset{\mathbb{Z}}{\text{Codomain}} \neq \underset{\text{perfect squares.}}{\text{Range}}$

eg 3: $f : \mathbb{Z} \to \mathbb{Z}$ given by

$f(x) = x + 1$, is f onto ?

Ans: f is onto, as for every $y \in \mathbb{Z}$

there is $x \in \mathbb{Z}$ with $f(x) = y$,

for any $y \in \mathbb{Z}$, it's pre-image under f is $x = y-1$ so that

$$f(x) = f(y-1) = y-1+1 = y.$$

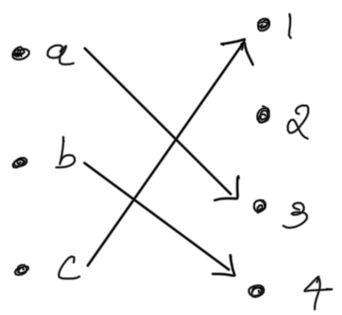
Definition: The function f is one-to-one correspondance, or a bijection, if it is both one-to-one and onto.

eg: ① $f: \{a, b, c, d\} \to \{1, 2, 3, 4\}$ given by $f(a) = 4$, $f(b) = 2$, $f(c) = 1$ and $f(d) = 3$. Is f a bijection?

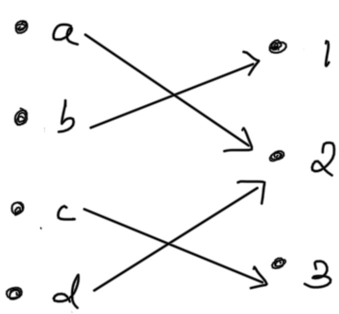
Ans: Clearly f is one-to-one, as no two values in the domain are assigned the same function value.

f is onto: because all four elements of codomain are images of elements in the domain.

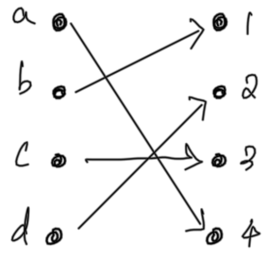
Hence f is a bijection.



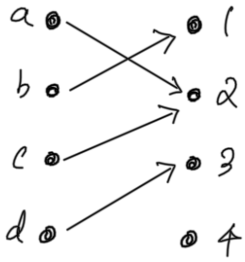
one to one not onto



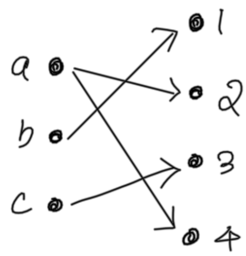
onto but not one-to-one



one-to-one and onto function.



Neither one-to one nor onto



Not a function.

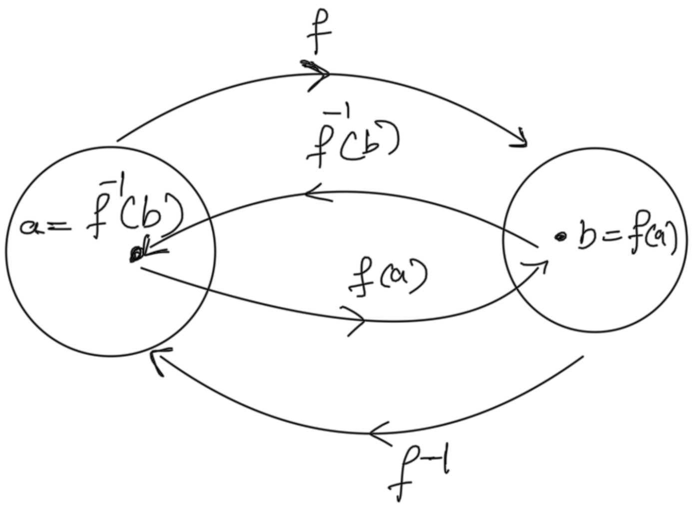
Remark: Let A be any non-empty set. $f : A \rightarrow A$ given by $f(x) = x$ is called identity function and it is denoted by $f = I_A$ note that I_A is both one-to-one and onto, Thus I_A is a bijection.

Definition: Let f be a one-to-one correspondence from A to B. The inverse function of f is the function from B to A denoted by — f^{-1} and that assigns to an element $b \in B$ the unique element a in A such that $f(a) = b$. Thus we can write $f^{-1}(b) = a$.

Remark: The one-to-one correspondence or bijections are also called invertible functions.

Remark: Note that in many places of mathematics for an element or object $a \neq 0$, the element $1/a$ denoted by a^{-1} and call inverse of element a. But for the function f the inverse f^{-1} is not equivalent to this concept, that is $f^{-1}(x) \neq 1/f(x)$. Note that $1/f(x)$ is defined only for $f(x) \neq 0$, but even if $f(x) = 0$, there is $f^{-1}(x)$ possible.

diagrammatic representation of inverse function:



Ex!: If a function is not one-to-one correspondance (not a bijection) then we cannot define the inverse of the function. Why?

Ans: f not bijection means
f is either not one-one or
f is not onto.

If f is not one-one, then there is atleast one element b in co-domain of f which is assigned to more than one element of domain, then when we trying to make inverse function, then the element b will have more than one element assigned in codomain of f^{-1} this makes f^{-1} not a function

If f is not onto, then there is atleast one element b in co-domain of f which does not have pre-image in domain of f, so when we trying to make/define inverse function, the element b of domain of f^{-1} will not have any image, this makes f^{-1} not a function.

Example: Let $f : \{a, b, c\} \longrightarrow \{1, 2, 3\}$ be a function given by $f(a) = 2$, $f(b) = 3$ and $f(c) = 1$. Is f is invertible? if it is invertible, what is it's inverse.

Sln: we can see that

f is 1-1 & onto

$\Rightarrow f$ is bijection

$\therefore f$ is invertible.

$\therefore f$ has an inverse.

and it is given by

$f^{-1} : \{1, 2, 3\} \longrightarrow \{a, b, c\}$ with

$f^{-1}(1) = c$, $f^{-1}(2) = a$, & $f^{-1}(3) = b$.

eg: $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function given by $f(x) = x + 1$.

Is f invertible? if so what is it's inverse?

Ans: f is 1-1, as:

If $f(x) = f(y)$

$\Rightarrow x + 1 = y + 1$

$\Rightarrow \quad x = y$

Thus f is one-one.

f is onto, as :

for any $y \in \mathbb{Z}$, $\exists$ $x = y-1$

such that $f(x) = y$.

Thus f is onto.

$\Rightarrow$ f is a bijection.

Thus inverse exist.

To find inverse

we let $\quad y = f(x)$

$$= x+1$$

$$\Rightarrow \quad y-1 = x.$$

Thus we write $f^{-1}(y) = x = y-1$

that is $f^{-1}(y) = y-1$

eg: $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

is f is invertible ?

Sl^n : as $f(2) = f(-2) = 4$

but $2 \neq -2$

$\Rightarrow$ f is not one-one

Thus f is not invertible.

eg: Let $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ given by $f(x) = x^2$. Then f is invertible.

as f is 1-1:

If $f(x) = f(y)$

$\Rightarrow x^2 = y^2$

$\Rightarrow (x^2 - y^2) = 0$

$\Rightarrow (x-y)(x+y) = 0$

$\Rightarrow (x-y) = 0$ or $x+y = 0$

$\Rightarrow x = y$ or $x = -y$

$\hookrightarrow$ as $x, y \in \mathbb{R}^+ \cup \{0\}$ in this case $x = y = 0$.

Thus $x = y$.

$\Rightarrow f$ is one-one.

f is onto:

as for any $y \in \mathbb{R}^+ \cup \{0\}$ we have $x = \sqrt{y} \in \mathbb{R}^+ \cup \{0\}$ with

$$f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y$$

$\Rightarrow f$ is onto.

Thus f is invertible.

To find inverse of f:

Let $y = f(x)$

$\Rightarrow y = x^2$

$\Rightarrow \sqrt{y} = x$ as $x, y \in \mathbb{R}^+ \cup \{0\}$

Thus $f^{-1}(y) = x = \sqrt{y}$.

i.e. $f^{-1}(x) = \sqrt{x}$.

Composition of functions

Let $f: A \longrightarrow B$ & $g: B \longrightarrow C$ be 2 functions, then the composition of f and g is denoted by fog, and it is a function from A to C defined by

$fog(x) = f(g(x))$

eg: Let f & g be functions on $\mathbb{Z}$. given by $f(x) = 2x+3$ and $g(x) = 3x+2$ find fog and gof.

Ans: $fog(x) = f(g(x))$
$= f(3x+2)$
$= 2(3x+2)+3$
$= 6x+7$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x+3) \\ &= 3(2x+3)+2 \\ &= \underline{\underline{6x+11}} \end{aligned}$$

note that $f \circ g(x) \neq g \circ f(x)$

Example: Let $f : A \to B$ be invertible. Find $f \circ f^{-1}$ and $f^{-1} \circ f$.

Ans: $f : A \to B$ and $f^{-1} : B \to A$.

with $f(a) = b$, i.e. $f^{-1}(b) = a$.

Thus $f^{-1} \circ f : A \to A$ given by

$$\begin{aligned} f^{-1} \circ f(a) &= f^{-1}(f(a)) \\ &= f^{-1}(b) \\ &= a. \end{aligned}$$

that is $f^{-1} \circ f = I_A$.

Now, we can see that

$f \circ f^{-1} : B \to B$ and given by

$$\begin{aligned} f \circ f^{-1}(b) &= f(f^{-1}(b)) \\ &= f(a) \\ &= b. \end{aligned}$$

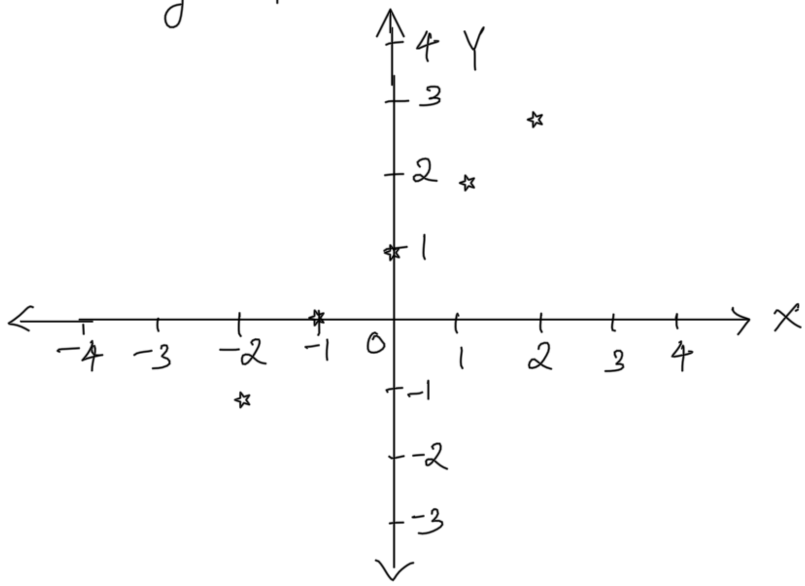
that is $\underline{\underline{f \circ f^{-1} = I_B}}$.

Graph of functions

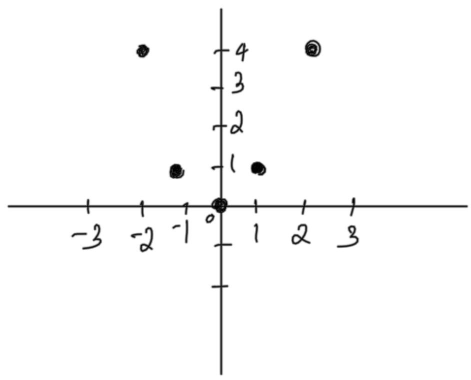
Let $f : A \longrightarrow B$. Then graph of the function is the set

$$\{ (x, y) \mid x \in A \ \& \ y = f(x) \in B \}$$

eg: Display graph of the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x+1$



eg 2: Display graph of the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$



Some important Functions.

① Floor function (Greatest integer function)

A function that assigns to any real number x the largest integer that is less than or equal to x is called floor function (Greatest integer function). The value of this function is denoted by $\lfloor x \rfloor$ ($[x]$)

eg: $\lfloor 2.1 \rfloor = 2$

$\lfloor \pi \rfloor = 3$

$\lfloor 1/2 \rfloor = 0$

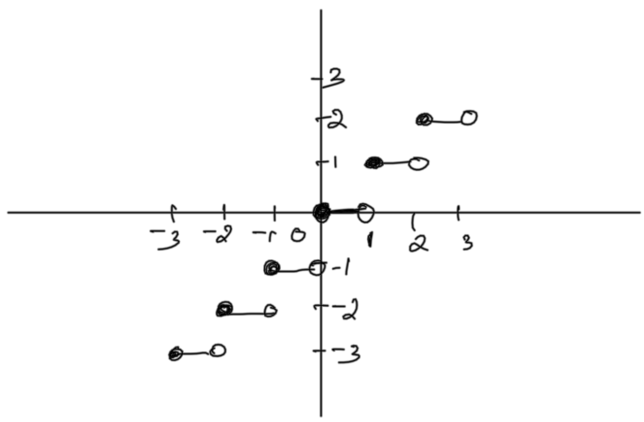
Note: note that $\lfloor\ \rfloor : \mathbb{R} \to \mathbb{R}$

where domain of $\lfloor\ \rfloor = \mathbb{R}$.

Codomain of $\lfloor\ \rfloor = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$

Range of $\lfloor\ \rfloor = \mathbb{Z}$.

Graph of $\lfloor\ \rfloor$



② Ceiling function

A function that assigns to any real number x the smallest integer that is greater than or equal to x is called ceiling function.

the value of this function is denoted by $\lceil x \rceil$

eg: $\lceil 2.1 \rceil = 3$

$\lceil \pi \rceil = 4$

$\lceil 1/2 \rceil = 1$

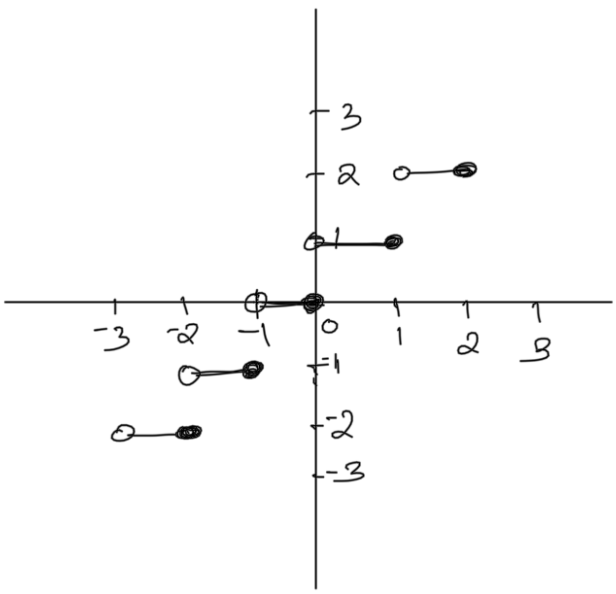
Note: note that $: \mathbb{R} \to \mathbb{R}$

where domain of $\lceil\ \rceil = \mathbb{R}$.

Codomain of $\lceil\ \rceil = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$

Range of $\lceil\ \rceil = \mathbb{Z}$.

Graph of $\lceil\ \rceil$



Properties of floor and ceiling function:

1. 1a) $\lfloor x \rfloor = n \longleftrightarrow n \le x < n+1$

   1b) $\lceil x \rceil = n \longleftrightarrow n-1 < x \le n$

   1c) $\lfloor x \rfloor = n \longleftrightarrow x-1 < n \le x$

   1d) $\lceil x \rceil = n \longleftrightarrow x \le n < x+1$

2. $x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$

3. $\lfloor -x \rfloor = -\lceil x \rceil$

   $\lceil -x \rceil = -\lfloor x \rfloor$

4. $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

   $\lceil x+n \rceil = \lceil x \rceil + n$

Proof:

1. 1a, 1b, 1c and 1d directly follows from definition of $\lfloor\ \rfloor$ and $\lceil\ \rceil$.

2. This also follow from definitions.

3. $\lfloor -x \rfloor = -\lceil x \rceil$.

   note that $-\lfloor -x \rfloor$ is the negation of $\lfloor -x \rfloor$

i.e. $-\lfloor -x \rfloor = \neg \lfloor -x \rfloor =$ It is not the case that largest integer less than or equal to $-x$.

$= $ smallest integer $\geq x$

$= \lceil x \rceil$

$\therefore \quad \lfloor -x \rfloor = -\lceil x \rceil$.

Similarly $\lceil -x \rceil = -\lfloor x \rfloor$

as $-\lceil -x \rceil = 7 \lceil -x \rceil$

$= $ greatest integer $\leq x$

$= \lfloor x \rfloor$

$\therefore \quad \lceil -x \rceil = -\lfloor x \rfloor$

4 $\quad \lfloor x+n \rfloor = \lfloor x \rfloor + n$.

Let $\lfloor x \rfloor = m$

then by 1a we have

$m \leq x < m+1$

add n entirely to this inequality,

we get $m+n \leq x+n < m+n+1$

again by 1a,

the later inequality equivalent to

$\lfloor x+n \rfloor = m+n$

$= \lfloor x \rfloor + n$

Hence the proof.

Next To prove $\lceil x+n \rceil = \lceil x \rceil + n$

Let $\lceil x \rceil = m$

then by 1b,

$$m-1 < x \le m$$

add n entirely to this inequality — to get $m+n-1 < x+n \le m+n$

again by 1b), we have this under inequality is equivalent to

$$\lceil x+n \rceil = m+n$$
$$= \lceil x \rceil + n$$

Q. prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Ans. To prove this, let $x = n + \varepsilon$ with $n \in \mathbb{Z}$ and $0 \le \varepsilon < 1$

now we prove the result by 2 cases.

Case-1 : when $0 \le \varepsilon < 1/2$.

here $2x = 2n + 2\varepsilon$

with $0 \le 2\varepsilon < 1$

Then $\lfloor 2x \rfloor = \lfloor 2n + 2\varepsilon \rfloor = 2n$. — (1)

now $\lfloor x + 1/2 \rfloor = \lfloor n + \varepsilon + 1/2 \rfloor$

note that $1/2 \le \varepsilon + 1/2 < 1$

Then $\lfloor n + \varepsilon + 1/2 \rfloor = n$

i.e. $\lfloor x + 1/2 \rfloor = n$. — (2)

also $\lfloor x \rfloor = \lfloor n + \varepsilon \rfloor$

$= n$ as $0 \le \varepsilon < 1/2$. — (3)

from (1), (2) and (3) it is clear that

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Thus result proved in this case.

Case-2: when $1/2 \le \varepsilon < 1$

in this case

$$2x = 2n + 2\varepsilon$$
$$= 2n + 1 + 2\varepsilon - 1$$

where $0 \le 2\varepsilon - 1 < 1$

Thus $\lfloor 2x \rfloor = \lfloor 2n+1 + 2\varepsilon - 1 \rfloor$

$= 2n+1$ — 4

now $x + 1/2 = n + \varepsilon + 1/2$

$= n + 1 + \varepsilon - 1/2$

with $0 \le \varepsilon - 1/2 < 1/2$.

Thus $\lfloor x + 1/2 \rfloor = \lfloor n+1 + \varepsilon - 1/2 \rfloor$

$= n+1$ — 5

also $\lfloor x \rfloor = \lfloor n + \varepsilon \rfloor$

$= n$ as $1/2 \le \varepsilon < 1$ — 6

Thus from 4, 5 and 6 we have

$\lfloor 2x \rfloor = 2n+1 = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Thus in this case also we have

$\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Hence the proof.

Q. prove or disprove that

$\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.

Ans note that we can easily get a counter example for which the given statement is false

as $\lceil 1/2 \rceil = 1$ and $\lceil 1 \rceil = 1$

we see $\lceil 1/2 + 1/2 \rceil = 1 \neq \lceil 1/2 \rceil + \lceil 1/2 \rceil = 2$.

③ factorial function

$f : \mathbb{N} \to \mathbb{Z}^+$ is a function given by

$f(n) = 1 \times 2 \times 3 \times \cdots \times n$

this function is denoted by $n!$.

☆ ☆ ☆ ☆ ☆