Lecture Notes on

The Foundations: Logic and Proofs

By

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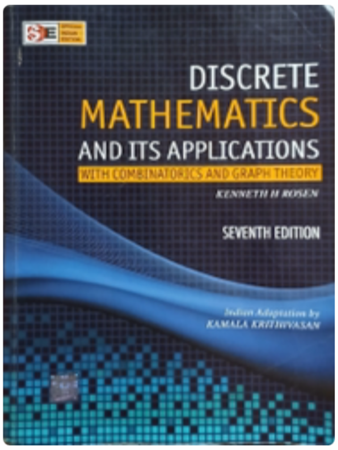

BY KENNETH H ROSEN, SEVENTH EDITION (Indian Adaption by KAMALA KRITHIVASAN)

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1 The Foundations: Logic and Proofs

1.1 PROPOSITIONAL LOGIC

☆ The aim of study of logic is to give precise meaning to mathematical statements.

↓

☆ from mathematical statements we can construct correct mathematical arguments.

↓

☆ The study of logic helps to understanding mathematical reasoning (↔ Thinking in a logical, sensible way).

1 Assignment: Logic has numerous applications in computer science. List out the applications.

Def: A proposition is a declarative sentence (that is a sentence that declares a fact) that is either true or false, but not both.

Example: 1. Washington, D.C. is the capital of united states of America

2. Toronto is the capital of canada.

3. $1+1 = 2$

4. $2+2 = 3$

Here the proposition 1 and 3 are true, whereas 2 and 4 are false.

↓ as

Ottawa is the capital of canada and $2+2 = 4$

Example: 1. what time is it?

2. Read this carefully.

3. $x+1 = 2$

4. $x+y = z$

These sentances are not propositions as

1 is a question (not declarative sentance)

2 ... (not declarative sentance)

2 is an order

3, 4 cannot say true or false. (They are depending on values of variables).

Notation: ① we use letters to represent a proposition, they are called propositional variables (or statement variables). (usually we use p, q, r, s, ... for these variables).

② If a proposition is true we say that the truth value of the proposition is true, and it is denoted by T.

③ If a proposition is false we say that the truth value of the proposition is false, and it is denoted by F.

Definition: Let p be a proposition. The negation of p, denoted by $\neg p$ (or $\bar{p}$) is the statement "It is not the case that p"

→ $\neg p$ read as not p

→ Truth value of $\neg p$ is opposite of truth value of p.

Example: find the negation of the propositions and express it in the simple english.

1. "Today is Friday"
2. "Atleast 10 inches of rain fell today in miami"

<u>Ans:</u> ① It is not the case that Today is Friday

OR

Today is not Friday / It is not Friday today

② It is not the case that Atleast 10 inches of rain fell today in miami

OR

Less than 10 inches of rain fell today in miami

<u>Truth table:</u> A truth table is a breakdown of all possible truth values returned by a logical expression.

Truth table for negation of a proposition p.

p	$\neg p$
T	F
F	T

Definition : (connectives of propositions)

Connectives are logical operators that are used to form new propositions from two or more existing propositions. and the new proposition is called compound proposition.

Types of connectives.

① Conjunction ($\wedge$)

Let p and q be propositions. The conjunction of p and q is denoted by $p \wedge q$, is the proposition "p and q".

→ The truth value of $\wedge$: The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise

Example : find the conjunction of propositions

p : Today is friday

q : It is raining today.

Ans: $p \wedge q$: Today is friday and it is raining today.

This proposition is true on rainy fridays and it is false on any day that is not friday and on fridays when it does not rain.

Truth table:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Remark: Note that the logic word "but" sometimes is used instead of "and" in a conjunction

Example : The sun is shining, but it is raining today.

② disjunction ($\vee$)

Let p and q be propositions. The disjunction of p and q is denoted by $p \vee q$, is the proposition "p or q".

→ The truth value of $\vee$: The disjunction $p \vee q$ is false when both p and q are false and is true otherwise

Example: find the disjunction of propositions

p: Today is friday

q: It is raining today.

Ans: $p \vee q$: Today is friday or it is raining today.

This proposition is true on any day that is either a friday or a rainy day (including rainy fridays). It is only false on days that are not fridays when it is also not rain.

Truth table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(3) exclusive or ($\oplus$)

Let p and q be propositions. The exclusive or of p and q denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

Truth table:

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Remark : we have seen that connective will help us to combine propositions; another <u>way to combine</u> propositions is through conditional statements.

Definition (conditional statements)

Let p and q be propositions. The conditional statement $\underline{p \rightarrow q}$ is the proposition <u>"if p, then q"</u>

The truth value : $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

note ①: in $p \rightarrow q$ p is called <u>hypothesis</u> (antecedent or premise)

q is called the <u>conclusion</u> (or consequence)

note ② : $p \rightarrow q$ is also called implication.

note ③ : $p \rightarrow q$ can also read in any of the following way

- ☆ "if p, then q"
- ☆ "p implies q"
- ☆ "if p, q"
- ☆ "p only if q"
- ☆ "q if p"
- ☆ "p is sufficient for q"
- ☆ "q when p"
- ☆ "q whenever p"
- ☆ "q follows from p"
- ☆ "q is necessory for p"
- ☆ "q unless $\neg p$"
- ☆ "a sufficient condition for q is p"
- ☆ "a necessory condition for p is q"

<u>Example</u>: If I am elected, then I will lower taxes.

consider one more example

<u>Example</u>: "If you get 100% on the final exam, then you will get an A grade"

(Here if you mange to get 100% on the final,

then you would expect to receive an A. If you do not get 100%, you may or may not receive an A depending on other factors.

However if you do get 100%, but the professor does not give you an A, you will feel cheated !)

Truth table:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Exercise 1: Let p : Maria learns discrete mathematics.

q : maria will find good job

Express $p \to q$ as english statement by using all the notions in note 3

Remark: If – then construction used in many programming languages is different from that used in logic.

[to understand this

in programming

input i enter a number;
if $i \leq 10$;
print "your number is < 10";

has a precise meaning & o/p in programming.

out put you get only if $i \leq 10$.

but in logic we even cannot consider $i \leq 10$ as a proposition, as i here considered as a variable (not possible to consider as a logical statement)

<u>Definition</u>

we call $q \rightarrow p$ is the <u>converse</u> of $p \rightarrow q$

we call $\neg q \rightarrow \neg p$ is the <u>contrapositive</u> of $p \rightarrow q$

we call $\neg p \rightarrow \neg q$ is the <u>inverse</u> of $p \rightarrow q$

Result:

Show that truth values of $\neg q \rightarrow \neg p$ and $p \rightarrow q$ are same.

proof

$\neg q \rightarrow \neg p$ is false only when $\neg q$ is true and $\neg p$ is false.

that is,

q is false and p is true.

in that case $p \rightarrow q$ is false.

that means if $\neg q \rightarrow \neg p$ true then $p \rightarrow q$ also true.

Result: Show that truth values of $q \rightarrow p$ (converse) and $\neg p \rightarrow \neg q$ (inverse) are not same as that of $p \rightarrow q$.

proof. Ex!

Example: write contrapositive, converse and inverse of the statement

"The home team wins whenever it is raining" (Hint: "q whenever p" $\equiv p \rightarrow q$)

Biconditional statement (another way to combine propositions)

Let p and q be propositions. The biconditional statement $p \longleftrightarrow q$ is the proposition "p if and only if q".

☆ Truth value of $\longleftrightarrow$: $p \longleftrightarrow q$ is true when p and q have the same truth values, and is false otherwise.

☆ Truth table:

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: p: you can take the flight

q: you buy a ticket

$p \longleftrightarrow q$: you can take the flight if and only if you buy a ticket

Remark

biconditional statements can also be written as

- ☆ "p is necessary and sufficient for q"
- ☆ "If p, then q and conversly"
- ☆ "p iff q"

Remark 2

If two logical expressions have same truth table, then truth value of these statements are equal.

Ex: Show that truth value of $p \longleftrightarrow q$ and $(p \rightarrow q) \wedge (q \rightarrow p)$ are same.

(Hint: construct truth table)

Truth tables of Compound propositions.

by using different connectives studied so far (i.e. $\neg, \wedge, \vee, \oplus, \rightarrow, \longleftrightarrow$) we can create complecated compound propositions.

by noting the following order of operation one can construct truth table of compound proposition.

Order.

operation	order.
$(\quad)$	0
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Example: Construct the truth table of the compound proposition

$$(p \vee \neg q) \longrightarrow (p \wedge q)$$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Translating english sentances.

Translating sentances into compound statements (or logical expressions involving logical connectives) will helps to remove the ambiguity in the sentance.

Example: ① Translate the following english sentance to a logical expression.

" you can access the internet from the campus only if you are a computer science major or you are not a freshman"

Ans: we can write this as a simple proposition p, but that is not much useful so, we let

a : you can access internet from campus

c : you are a computer science major

f : you are a freshman

the statement is of the form

$$p \text{ only if } q \equiv p \to q$$

where $p = a$ and $q = c \vee \neg f$

$\therefore$ logical compound expression is

$$a \rightarrow (c \vee \neg f)$$

Example 2: Translate the following english sentence into compound logical expression.

"you cannot ride roller coaster if you are under 4 feet tall unless you are older than 16 years old"

Ans: Let t : you can ride roller coaster.

r : you are under 4 feet tall

s : you are older than 16 years.

the sentence is of the form q if p

that is $\underline{\text{n}}$ $p \rightarrow q$

where $q = \neg t$ and $p = (r \wedge \neg s)$

$\therefore$ The logical expression is

$$(r \wedge \neg s) \rightarrow \neg t$$

Propositional equivalences

In a mathematical argument, we can replace a statement with another statement having same truth value.

Definition:

① A compound proposition that is always true, no matter what is the truth values of propositions that occur in it, is called a tautology.

② A compound proposition that is always false is called contradiction

③ A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

$\Rightarrow$ $\neg p$ is a contingency

$p \vee \neg p$ is a tautology

$p \wedge \neg p$ is a contradiction.

Definition : Two compound prepositions are called <u>logically equivalent</u> if both of them have <u>same truth values</u> in all possible cases. (i.e. final columns of truth tables of p and q are equal)

OR

The compound prepositions p and q are logically equivalent if $\underline{p \longleftrightarrow q}$ is a <u>tautology.</u> we use $p \equiv q$ for p and q are logically equivalent.

note : $\equiv$ is not logical connective but $p \equiv q$ means $p \longleftrightarrow q$ is a tautology.

Demorgans law

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

Proof ①:

to prove $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are equivalent it is enough to show the truth values of these compound expressions are same in all possible combinations. the following truth table shows this fact

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

② to prove $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are equivalent it is enough to show the truth values of these compound expressions are same in all possible combinations. the following truth table shows this fact

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Similarly we can prove many other equivalences.

Equivalence	name
$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
$p \vee T \equiv T$ $p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	idempotent laws
$\neg(\neg p) \equiv p$	double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	commutative laws

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg (p \wedge q) \equiv \neg p \vee \neg q$ $\neg (p \vee q) \equiv \neg p \wedge \neg q$	Demorgans laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absurption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws.

Logical equivalence
involving conditional statements

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg (q \to \neg p)$$

$$\neg (p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical equivalence involving biconditional statements

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Example: Show that $\neg(p \rightarrow q)$ and $p \wedge \neg q$ are logically equivalent without using truth table.

proof : we have the logical equivalence for conditional statement as

$p \rightarrow q \equiv \neg p \vee q$

that is

$\neg(p \rightarrow q) \equiv \neg(\neg p \vee q)$

$\equiv \neg(\neg p) \wedge \neg q$ by Demorgan law

$\equiv p \wedge \neg q$ by double negation

Hence we proved required logical equivalence.

Example : Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences

Ans: Starting with $\neg(p \vee (\neg p \wedge q))$

$$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q) \quad \text{by Demorgan law - 2}$$

$$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] \quad \text{by Demorgan - 1}$$

$$\equiv \neg p \wedge (p \vee \neg q) \quad \text{by double } \neg \text{ law}$$

$$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \quad \text{by distributive law - 2}$$

$$\equiv F \vee (\neg p \wedge \neg q) \quad \text{by } \neg \text{ law}$$

$$\equiv (\neg p \wedge \neg q) \vee F \quad \text{by Commutative}$$

$$\equiv \neg p \wedge \neg q \quad \text{by Identity law}$$

This proves required logical equivalence.

Example : Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Ans. to show this is a tautology it is enough to show this is logically equivalent to T (we can also do it by truth table)

$$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$$

by logi. equi. of. condi. stmt.
$(p \rightarrow q \equiv \neg p \vee q)$

$$\equiv (\neg p \vee \neg q) \vee (p \vee q)$$

by Demorgan 1

$$\equiv (\neg p \vee p) \vee (\neg q \vee q)$$

by commutativity and associativity of $\vee$

$$\equiv T \vee T$$

$$\equiv T$$

Hence the proof.

Predicate and Quantifiers.

Any function/ relation $P(x_1, x_2, \ldots, x_n)$ in n-variables $x_1, \ldots, x_n$ is called a propositional function. where P is called predicate.

The collection of all possible values that P can take is called domain. i.e. collection of all values of the form $(x_1, x_2, x_3, \ldots x_n)$.

Example: $P(x)$: "x is greater than 3"

here x is variable and

"is greater than 3" is predicate

in general $P(x)$ is not a proposition, as variable x involved in it. but if we specify the value of 'x' as a particular number, then it is a proposition.

eg: $P(4)$ and $P(2)$ are propositions.

what is the truth value of $P(4)$ and $P(2)$?

Quantification: The method of creating proposition from a propositional function by assigning values to variables.

Universal Quantifier.

The universal quantification of $p(x)$ is the statement

"$p(x)$ for all values of x in the domain"

we denote $\forall x\, p(x)$. (read "for every x $p(x)$")

$\forall$ is called universal quantifier.

Existential Quantifier

The existential quantification of $p(x)$ is the statement

"There exist an element x in the domain such that $p(x)$"

we use the notation $\exists x\, p(x)$. [read as. there exist x $p(x)$]

$\exists$ is called existential quantifier.

Truth values of quantifiers

statement	when true	when false
$\forall x\, p(x)$	$p(x)$ is true for every x	There is an x for which $p(x)$ is false
$\exists x\, p(x)$	There is an x for which $p(x)$ is true	$p(x)$ is false for every x.

Example

What is truth value of $\forall x\, p(x)$, where

$p(x): x+1 > x, \quad x \in \mathbb{R}$

Ans. $\forall x\, P(x)$ is true.

What is truth value of $\exists x\, P(x)$, where $P(x)$: "$x^2 \leq 10$", $x \in \mathbb{N}$, $x \geq 4$

Ans : $\exists x\, P(x)$ false

Logical Equivalence involving quantifiers.

Statements involving predicate and quantifiers are logically equivalent if and only if they have same truth value.

(no matter which predicate used and what domain used)

Q. Show that $\forall x\,(P(x) \land Q(x))$ and $\forall x\, P(x) \land \forall x\, Q(x)$ are logically equivalent.

Ans.

To prove this it is enough to show

$$\forall x\,(P(x) \land Q(x)) \longleftrightarrow \forall x\, P(x) \land \forall x\, Q(x)$$

is a tautology.

OR

$\forall x\,(P(x) \land Q(x)) \longrightarrow \forall x\, P(x) \land \forall x\, Q(x)$ is true and

$\forall x\, P(x) \land \forall x\, Q(x) \longrightarrow \forall x\,(P(x) \land Q(x))$ is true.

Suppose $\forall x\,(P(x) \land Q(x))$ is true.

$\Rightarrow$ $P(a) \wedge Q(a)$ is true $\forall\, a \in D$.

$\Rightarrow$ $P(a)$ is true $\forall\, a \in D$ &
$Q(a)$ is true $\forall\, a \in D$.

$\Rightarrow$ $\forall x\, P(x)$ is true and
$\forall x\, Q(x)$ is true.

$\Rightarrow$ $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true.

i.e. $\forall x\, (P(x) \wedge Q(x)) \rightarrow \forall x\, P(x) \wedge \forall x\, Q(x)$ is true.

On other hand suppose.

$\Rightarrow$ $\forall x\, P(x) \wedge \forall x\, Q(x)$ is true.

$\Rightarrow$ $\forall x\, P(x)$ is true and
$\forall x\, Q(x)$ is true.

$\Rightarrow$ $P(a)$ is true $\forall\, a \in D$ &
$Q(a)$ is true $\forall\, a \in D$.

$\Rightarrow$ $P(a) \wedge Q(a)$ is true $\forall\, a \in D$.

$\Rightarrow$ $\forall x\, (P(x) \wedge Q(x))$ is true.

i.e. $\forall x\, P(x) \wedge \forall x\, Q(x) \rightarrow \forall x\, (P(x) \wedge Q(x))$ is true.

$\therefore$ $\forall x\, (P(x) \wedge Q(x)) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

Remark:

1. $\exists x\, (P(x) \wedge Q(x)) \not\equiv \exists x\, P(x) \wedge \exists x\, Q(x)$

2. $\forall x\, (P(x) \vee Q(x)) \not\equiv \forall x\, P(x) \vee \forall x\, Q(x)$

proof : Ex 1

Negating Quantified Expressions.

we have seen that quantified expressions are propositions.

i.e. $\forall x\, P(x)$ and $\exists x\, Q(x)$ are propositions.

thus the negation of these prepositions are

$\neg \forall x\, P(x)$: "It is not the case that $\forall x\, P(x)$"

$\neg \exists x\, Q(x)$: "It is not the case that $\exists x\, Q(x)$"

Example: consider the statement

"Every student in your class has taken a course in calculus"

$\simeq \forall x\, P(x)$ where

$P(x)$: "x has taken a course in calculus"

Domain = { all students in your class }.

Here $\neg \forall x\, P(x)$ = It is not the case that every student in your class has taken a course in calculus.

This is equivalent to

there is a student in your class who has not taken a course in calculus.

i.e. $\simeq$ $\exists x\, \neg P(x)$.

Thus $\neg \forall x\, P(x) = \exists x\, \neg P(x)$

Similarly

$$\neg \exists x \, Q(x) = \forall x \, \neg Q(x)$$

Truth values of negated quantifiers.

Negation	equivalent stmnt	when true	when false
$\neg \exists x \, P(x)$	$\forall x \, \neg P(x)$	for every x, $P(x)$ is false	There is an x for which $P(x)$ is true
$\neg \forall x \, P(x)$	$\exists x \, \neg P(x)$	There is an x for which $P(x)$ is false	$P(x)$ is true for every x

Q. show that $\neg \forall x \, (P(x) \rightarrow Q(x))$ and $\exists x \, (P(x) \wedge \neg Q(x))$ are logically equivalent.

Ans. by the negation of quantifiers, we have

$$\neg \forall x \, (P(x) \rightarrow Q(x)) \equiv \exists x \, \neg ((P(x) \rightarrow Q(x)))$$

$$\equiv \exists x \, (P(x) \wedge \neg Q(x))$$

($\because$ we know that $p \rightarrow q \equiv p \wedge \neg q$)

Hence the proof.

Translating English into logical expressions.

Example: Express the statement as logical stmnt "Every student in this class has studied calculus"

① by considering domain is all student in your class,

(i) introduce the variable x

for every student x in this class, x has studied calculus.

(ii) introduce the predicate

Let $c(x)$ = "x has studied calculus"

(iii) Translating the statement

by joining the quantifier $\forall x$, we get the equivalent logical expression as

$$\forall x \; c(x)$$

② by considering domain as all people (in your college)

(i) introducing the variable x

for every person x, if the person x in this class then x has studied calculus.

(ii) introducing predicates.

Here we can see two predicates

$S(x)$: the person x is in this class.

$c(x)$: x has studied calculus.

(iii) converting into logical expression

$\forall x \; (S(x) \rightarrow c(x))$

Remark : The expression $\forall x \; (S(x) \wedge c(x))$ cannot be true in this case, because this expression means that "all people are students in this class and have studied calculus" !

Remark : In above example suppose the subject calculus belongs to one of the subject from a domain of subjects, then we can introduce a two variable predicate as

$Q(x, y)$ = student x has studied subject y.

Then we can replace $C(x)$ by $Q(x, \text{calculus})$ in both of above two approaches to obtain

$\forall x\, Q(x, \text{Calculus})$ or

$\forall x\, (S(x) \rightarrow Q(x, \text{calculus}))$

Q2: Express the statement as logical stmnt by using predicate and quantifier

"Some student in this class has visited USA"

Ans. ① by considering domain as all students of the class.

(i) introduce variable

There is a student x in this class having the property that x has visited USA.

(ii) introducing predicate

Let $M(x)$ = "x has visited USA"

(iii) translating using quantifier.

The stmnt equivalent to $\exists x\, M(x)$

② If domain is all people, then

(i) introducing variable

there is a person x having the properties that x is a student in this class and x has visited USA

(ii) Introducing predicates

Let $S(x)$ = x is a student in this class.

$M(x)$ = x has visited USA.

(iii) Converting to logical stmnt by using quantifiers and logical operators

Here the statement is equivalent to $\exists x\,(S(x) \wedge M(x))$

Remark: the given statement cannot be expressed as $\exists x\,(S(x) \rightarrow M(x))$, because the truth value of this statement and given statement may not be equal in some cases. for example if some one not in the class, then given statement is always false. but the statement $\exists x\,(S(x) \rightarrow M(x))$ can be true as, $F \rightarrow T$ is true in conditional statement case. Similarly $F \rightarrow F$ is also true in conditional statement.

Nested Quantifiers

Two contifiers are nested if one is with in the scope of the other.

Example: $\forall x\, \exists y\, (x + y = 0) \quad x, y \in \mathbb{R}$.

This means that "for every real number x there is a real number y such that $x + y = 0$." True

Example 2: $\forall x\, \forall y\, (x + y = y + x) \quad x, y \in \mathbb{R}$.

for every real number x and for every real number y it satisfies $x + y = y + x$. True

Example 3: Translate into English statement

$$\forall x \forall y \left((x > 0) \wedge (y < 0) \rightarrow (xy < 0) \right)$$

The statement says for every real number x and for every real number y if $x > 0$ and $y < 0$ then $xy < 0$

$\simeq$ for real numbers x and y, if x is +ve and y is −ve, then xy is −ve.

$\simeq$ the product of a +ve real no. and −ve real no is always negative real number.

The order of quantifiers:

For mathematical statements involving multiple quantifiers for a multivariable propositional function, the order of the quantifier is important

Ex: Let $Q(x,y) = $ "$x + y = 0$". what are the truth values of

1. $\exists y \forall x \; Q(x,y)$
2. $\forall x \exists y \; Q(x,y)$

Ans. The Stmnt 1 means

"There is a real number y such that for every real number x, $Q(x,y)$"

no matter what value of y is chosen, there is only one value of x for which $x+y=0$. Because there is no number y such that $x+y=0$ for all real numbers x, thus the stmnt $\exists y \forall x\, Q(x,y)$ is false.

The stmnt Q means

"for every real number x, there is a real number y such that $Q(x,y)$"

Given a real number x, there is a real number y such that $x+y=0$; namely $y=-x$. Hence the statment $\forall x \exists y\, Q(x,y)$ is true.

Truth value of quantification of two variables.

statement	when true?	when false?
$\forall x \forall y\, P(x,y)$ $\forall y \forall x\, P(x,y)$	$P(x,y)$ is true for every pair x,y	There is a pair x,y for which $P(x,y)$ is false.
$\forall x \exists y\, P(x,y)$	for every x there is a y for which $P(x,y)$ is true	There is an x such that $P(x,y)$ is false for every y.
$\exists x \forall y\, P(x,y)$	There is an x for which $P(x,y)$ is true for every y.	for every x there is a y for which $P(x,y)$ is false.
$\exists x \exists y\, P(x,y)$ $\exists y \exists x\, P(x,y)$	There is a pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y.

H.W

Q. what are truth values of

Let $Q(x,y,z) = x+y=z$

1. $\forall x \forall y \exists z \; Q(x,y,z)$
2. $\exists z \forall x \forall y \; Q(x,y,z)$

Translating logical expression to english sentance.

1. Translate the statement

$\forall x \, (C(x) \vee \exists y \, (C(y) \wedge F(x,y)))$

into english where

$C(x)$: 'x has a computer"

$F(x,y)$: x and y are friends.

Ans. Let us take domain as students in your class. The statement is $\cong$

for every student x in your class has a laptop or there exist a student y who has a computer and he is friend of x.

in simple english it is equivalent to

"every student in your class has a computer or has a friend who has a computer."

2. Translate the statement

$\exists x \forall y \forall z \, ((F(x,y) \wedge F(x,z)) \wedge (y \neq z)) \rightarrow \neg F(y,z)$.

where $F(a,b)$ means a and b are friends- and domain for x, y is students of your college.

Ans : The statement means

There is a student x such that for all students y and all students z other than y, if x and y are friends and x and z are friends then y and z are not friends.

Simply the given expression is $\cong$

there is a student non of his friends are also friends with each other.

Translating english sentences to logical expression

Translate

"The sum of two positive integers is always positive" into logical expression.

Ans. when domain is all integers.

The given statement can be re-written as

(i) introduce variables.

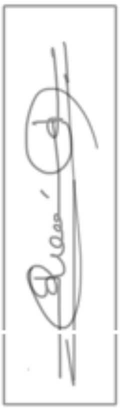
for any two integers x & y, If x & y are +ve then the sum $x+y$ is +ve.

(ii) introduce predicate

$P(x)$: $x > 0$

$Q(x)$: $y > 0$

$R(x,y)$: $x+y > 0$



(iii) use quantifiers & logical operators.

$$\forall x \forall y ((P(x) \land Q(x)) \rightarrow R(x,y))$$

☆ when domain is +ve integers the statement simply is $\forall x \forall y\, R(x,y)$.

Translate: "every real number except zero has a multiplicative inverse"

$\downarrow$

$(\quad y \in \mathbb{R}$ s.t $x \cdot y = 1)$

Ans: the statement is :-

" for every real number x, if $x \neq 0$ then there exist a real number y such that $xy = 1$ "

:- $\forall x\ (x \neq 0) \rightarrow \exists y\ (xy = 1)$

3. Express the definition of limit using quantifiers.

<u>Ans</u>

Recall that $\lim_{x \to a} f(x) = L$

for every real number $\varepsilon > 0$ there exist a real number $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

$\cong \forall \varepsilon \exists \delta \forall x \,(0 < |x-a| < \delta \rightarrow |f(x) - 2| < \varepsilon)$

domain of $\varepsilon, \delta \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$.

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4. Every one has exactly one best friend

Ans. is for every person x has exactly one best friend

Let $B(x,y)$: y is best friend of x

The stmnt 'x has exactly one best friend' $\cong$

$$\exists y \,(B(x,y) \wedge \forall z((z \neq y) \rightarrow \neg B(x,z)))$$

$\therefore$ original statement is $\cong$

$$\forall x \exists y \,(B(x,y) \wedge \forall z((z \neq y) \rightarrow \neg B(x,z)))$$

---

5. There is a women who has taken a flight on every airline in the world.

Ans $P(w, f)$: w has taken f

$Q(f, a)$: f is a flight on a

$\exists w \forall a \exists f \,(P(w,f) \wedge Q(f,a))$

domain : all women in the world

all airoplane flights.

all airlines.

OR

$\exists w \forall a \exists f \; R(w, f, a)$

$R(w, f, a)$: w has taken f on a

Negating nested quantifiers.

Q. express the negation of the stmnt $\forall x \exists y (xy = 1)$ so that no negation precedes a quantifier.

Ans. $\neg(\forall x \exists y (xy = 1)) = \exists x \neg \exists y (xy = 1)$
$= \exists x \forall y \neg (xy = 1)$
$= \exists x \forall y (xy \neq 1)$

Rules of Inference

Definition: An argument in propositional logic is a sequence of propositions. All but the final proposition in the argument are called premices, and the final proposition is called conclusion. An argument is valid if the truth of all it's premises implies that the conclusion is true.

Example:

$p \to q$: If you have access to network, then you can change your grade.

p: you have access to network.

q: you can change your grade.

here $p \to q$, p are premices and

q is conclusion.

moreover we can see that the argument is valid.

Remark : an argument with premises $p_1, \dots, p_n$ and conclusion q is valid when

$$p_1 \wedge p_2 \wedge \dots \wedge p_n \Rightarrow q \text{ is a tautology.}$$

Remark : we can always use truth table to show that an argument form is valid.
But when the argument involve too many propositional variables then we should require large number of rows in truth table (eg: if there are 10 variables one may require 2^{10} rows.) to overcome this difficulty we establish the validity of some relatively simple argument form called rules of inference. These rules of inference can be used as building blocks to construct more complicated valid argument forms.

Rules of inference	Tautology	Name
p $p \to q$ $\therefore q$	$p \wedge (p \to q) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$\neg q \wedge (p \to q) \to \neg p$	modus tollens

$p \to q$ $q \to r$ $\therefore p \to r$	$(p \to q) \land (q \to r) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$(p \lor q) \land \neg p \to q$	Disjunctive syllogism
p $\therefore p \land q$	$p \to (p \land q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$(p) \land (q) \to (p \land q)$	conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$(p \lor q) \land (\neg p \lor r) \to (q \lor r)$	Resolutio.

Example :

Let us consider the argument

$p \to q$: "If it sunny today, then we will go Hiking"

p : It is sunny today

q : we will go Hiking.

$p \to q$, p are premises.

q is the conclusion.

by using modus ponens, here the argument is valid

and as premises are true then conclusion is also true.
if

Remark : valid argument can lead to incorrect conclusion.

Example: consider the argument

$p \to q$: If $\sqrt{2} > \frac{3}{2}$ then $(\sqrt{2})^2 > (\frac{3}{2})^2$

p : $\sqrt{2} > \frac{3}{2}$

$\therefore q$: $2 > (\frac{3}{2})^2$ $\qquad (\sqrt{2} = 1.414)$

The given argument is a valid argument as it is constructed by using modus ponens. Here one of the premises p is false, thus the conclusion is also false.

Example : State which rule of inference is the basis of the following argument

1. " It is below freezing now. Therefore, it is either below freezing or raining "

Ans

Let p : It is below freezing now

q : It is raining now

Then argument is of the form

$$\frac{p}{\therefore p \vee q}$$

This is an argement that uses the addition rule

2. "It is below freezing now and it is raining now Therefore it is below freezing now"

Ans Let p: It is below freezing now

q: It is raining now

Then argument is of the farm

$$\frac{p \wedge q}{\therefore p}$$

The argument uses simplification rule

3 If it is rains today, then we will not have a barbecue today. If we donot have a barbecue today, then we will have a barbecue tomorrow. Therefore if it rains today, then we will have a barbecue tomorrow

Ans p: It is raining today

q: we will not have a barbecue today

r: we will have a barbecue tomorrow

Then the argument is of the form

$p \to q$

$q \to r$

$\therefore p \to r$

Hence this argument is a hypothetical syllogism

Using Rules of Inference to build arguments

when there are many premises, several rules of inference are often needed to show that an argument is valid

Example : show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday" "we will go swimming only if it is sunny" "If we do not go swimming, then we will take a canoe trip" and "If we take a canoe trip, then we will be home by sunset" lead to the conclusion "we will be home by sunset"

Ans

Let

p : It is sunny this afternoon.

q : It is colder than yesterday

r : we will go swimming

s : we will take a canoe trip

t : we will be home by sunset.

ponemises are

$\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, $s \rightarrow t$

and conclusion t

construction of argument to show that our hypothesis leads to desired conclusion. :

	Step	Reason
1.	$\neg p \wedge q$	given Hypothesis
2.	$\neg p$	simplification using 1.
3.	$r \rightarrow p$	Hypothesis
4.	$\neg r$	modus follens by using 2 and 3
5	$\neg r \rightarrow s$	Hypothesis
6	s	modus ponens using 4 and 5
7	$s \rightarrow t$	Hypothesis
8	t	modus ponens using 5 and 6.

we can use truth table to show that whenever each of the four hypothesis is true then conclusion also true. but as we are using 5 variables the truth table requires $2^5 = 32$ rows.

Example

show that the hypothesis

" If you send me an e-mail message, then I will finish writing the program"

" If you do not send me an e-mail message, then I will go to sleep early" and

" If I go to sleep early, then I will wake up feeling refreshed "

lead to the conclusion

" If I donot finish writing the program, then I will wake up feeling refreshed "

Ans. Let

p : you send me an e-mail message

q : I will finish writing the program

r : I will go to sleep early

s : I will wake up feeling refreshed

Then hypothesis are

$p \to q$, $\neg p \to r$ and $r \to s$ and conclusion is $\neg q \to s$

	Steps	Reason
1.	$p \to q$	Hypothesis
2.	$\neg q \to \neg p$	Contrapositive of 1.
3.	$\neg p \to r$	Hypothesis
4.	$\neg q \to r$	Hypothetical syllogism using 2 & 3
5.	$r \to s$	Hypothesis
6.	$\neg q \to s$	Hypothetical syllogism using 4 & 5

Example: Show that the following argument is valid.

"If today is Tuesday, I have a test in mathematics or C.S. If my C.S. professor is sick, I will not have a test in C.S. Today is Tuesday and my C.S. professor is sick. Therefore I have a test in mathematics."

Ans. Let

T : today is Tuesday

M : I have a test in mathematics.

C : I have a test in C.S.

S : my C.S. professor is sick.

Thus the premises are

1. $T \to (M \vee C)$
2. $S \to \neg C$
3. $T \wedge S$

$\therefore M$

from 3 we get

$$\left.\begin{array}{ll} 4. & T \\ 5. & S \end{array}\right\} \text{simplification}$$

6. $M \vee C$ modus ponens from 4, 1
7. $\neg C$ modus ponens 5, 2
8. M disjunctive syllogism 6, 7.

Example : Show that the following argument is valid

"If mohan is a lawyer then he is ambitious. If mohan is an early riser, then he does not like idlies. If mohan is ambitious, then he is an early riser. Then if mohan is a lawyer, then he does not like idlies."

Ans. Let

L : mohan is a lawyer

A : mohan is ambitious

E : mohan is early riser

I : mohan likes idlies.

Then premises and conclusion

1. $L \rightarrow A$

2. $E \rightarrow \neg I$

3. $A \rightarrow E$

$\therefore L \rightarrow \neg I$

4. $L \rightarrow E$ hypothetical syllogism from 1 and 3.

5. $L \rightarrow \neg I$ hypothetical syllogism from 4 and 2.

Hence argument is vali .

Rules of inference for quantified statements

Rules of inference	Name
$\dfrac{\forall x \; p(x)}{\therefore p(c)}$	universal instantiation
$\dfrac{p(c) \text{ for an arbitrary } c}{\therefore \forall x \; p(x)}$	universal generalization

$\exists x\, P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x\, P(x)$	Existential generalisation

Example

Show that

"every one in this discrete mathematics class has taken a course in computer science" and

" Maria is a student in this class"

imply that the conclusion

" Maria has taken a course in computer science"

Ans.

Let $D(x)$: "x is in this D.M. class"

$C(x)$: "x has taken a course in C.S."

Then the premises are

$\forall x\, (D(x) \rightarrow C(x))$ and

$D(\text{maria})$

and conclusion is $C(\text{maria})$

Steps	Reason
1. $\forall x\,(D(x) \rightarrow C(x))$	premise
2. $D(\text{maria}) \rightarrow C(\text{maria})$	universal instantiation from 1
3. $D(\text{maria})$	premise
4. $C(\text{maria})$	modus ponens from 2 and 3

Example

show that the premises

"A student in this class has not read the book" and

"every one in this class has passed the first exam"

imply the conclusion

"some one who passed first exam has not read the book"

Ans. Let

$C(x)$: "x is in this class"

$B(x)$: "x has read the book"

$P(x)$: "x passed the first exam"

The premises are

$\exists x\,(C(x) \wedge \neg B(x))$ and

$\forall x\,(C(x) \rightarrow P(x))$

The conclusion is

$$\exists x\,(P(x) \land \neg B(x)).$$

Steps	Reason
1. $\exists x\,(C(x) \land \neg B(x))$	premise
2. $C(a) \land \neg B(a)$	Existential instantiation from 1.
3. $C(a)$	simplification from 2.
4. $\forall x\,(C(x) \rightarrow P(x))$	premise
5. $C(a) \rightarrow P(a)$	universal instantiation fro 4.
6. $P(a)$	modus ponens from 3 and 5.
7. $\neg B(a)$	simplification from 2.
8. $P(a) \land \neg B(a)$	conjunction from 6 & 7
9. $\exists x\,(P(x) \land \neg B(x))$	Existential generalization from 8.

universal modus ponens: It is the combination of modus ponens Rules of inference for proposition and universal instantiation Rules of inference for quantified statements as follows

$\forall x\,(P(x) \rightarrow Q(x))$

$P(a)$ where a is a particular element in domain

$\therefore Q(a)$

Example: assume that "for all positive integers n, if n is greater than 4, then n^2 is less than 2^n" is true use universal modus ponens to show that $100^2 < 2^{100}$.

Ans- Let $P(n)$: "$n > 4$" and $Q(n)$: "$n^2 < 2^n$"

thus the statement "for all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be written as

$$\forall n \,(P(n) \rightarrow Q(n))$$

Domain: all positive integers.

$P(100)$ is true as $n = 100 > 4$

$\therefore$ by universal modus ponens that

$$\begin{array}{l} \forall n \,(P(n) \rightarrow Q(n)) \\ P(100) \\ \hline \therefore Q(100) \end{array}$$

as $Q(100) = 100^2 < 2^{100}$ is true

universal modus tollens: It is the combination of modus tollens Rules of inference for proposition and universal instantiation Rules of inference for quantified statements as follows

$$\begin{array}{l} \forall x \,(P(x) \rightarrow Q(x)) \\ \neg Q(a) \quad \text{where } a \text{ is a particular element in the domain} \\ \hline \therefore \neg P(a) \end{array}$$