No.

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一.

(1): $C^{\sigma}(I) = \max\{\sum_{j=1}^{k-1} p_j, p_k\}$.

$$C^*(I) \le \sum_{j=1}^{k-1} p_j + \frac{p_k}{s_k}\left(S_{max} - \sum_{j=1}^{k-1} s_j\right)$$

$$< \sum_{j=1}^{k-1} p_j + \frac{p_k}{s_k}\cdot s_k = \sum_{j=1}^{k-1} p_j + p_k.$$

故 $\frac{C^*(I)}{C^{\sigma}(I)} \le \frac{\sum_{j=1}^{k-1} p_j + p_k}{\max\{\sum_{j=1}^{k-1} p_j, p_k\}} \le 2$.

(2). 设 S 为背包容量.

若 $S > S_{max} + s_k$. 则 $l \ge k+1$. 故 $S \le S_{max} + s_k$.

若 $S = S_{max} + s_k$. 则 $C^*(I) = C^{\sigma}(I)$.

若 $S < S_{max} + s_k$. 则同(1)中讨论.

$C^*(I) / C^{\sigma}(I) \le 2$.

故仍有 $\frac{C^*(I)}{C^{\sigma}(I)} \le 2$.

(3). $\sum_{j=1}^{i-1} p_j = \sum_{j=1}^{k} p_j + \sum_{j=k+1}^{i-1} p_j \ge S_{max} + \sum_{j=k+1}^{i-1} p_j \ge p_i$. $(i \ge k+2)$

$i = k+1$时. $\sum_{j=1}^{i-1} p_j = \sum_{j=1}^{k} p_j = S_{max} > p_i = p_{k+1}$.

故 $\forall i > k$. 恒有 $\sum_{j=1}^{i-1} p_j > p_i$.

(4). $l > k$ 则算法等价于按序 $\sigma = \{1, 2, \dots, n\}$ 放入背包

$\cancel{s_l < \frac{1}{2}S,\ \text{否则}\ l \le \min\{i \mid \sum_{j=1}^{i} s_i > S_{max}\}.\ \text{则}\ l \le k.}$

$$C^*(I) \le \sum_{j=1}^{l-1} p_j + \frac{p_l}{s_l}\left(S - \sum_{j=1}^{l-1} p_j\right)$$

$$\le \sum_{j=1}^{l-1} p_j + \frac{p_l}{s_l}\cdot s_l = \sum_{j=1}^{l-1} p_j + p_l.$$

$C^{\sigma}(I)=\sum_{j=1}^{k-1} p_j$.

现有三种情况① $\begin{cases} p_l > \sum_{j=1}^{k-1} p_j \\ s_l < \sum_{j=1}^{k-1} s_j \end{cases}$. 不可能. 因为 $\frac{p_l}{s_l} < \frac{p_{l-1}}{s_{l-1}} \cdots < \frac{p_1}{s_1}$, 矛盾.

② $\begin{cases} p_l > \sum_{j=1}^{k-1} p_j \\ s_l > \sum_{j=1}^{k-1} s_j . \end{cases}$ ~~此时~~ 注意到 $s_l > \sum_{j=1}^{k-1} s_j$. 则 $s_{max} \in \{s_l, s_{l+1}, \ldots, s_n\}$.

因有 $k = \min\{i \mid \sum_{j=1}^{i} s_j > s_{max}\} > l$. 矛盾!

③ $\begin{cases} p_l \le \sum_{j=1}^{k-1} p_j \\ * \end{cases}$. 故由排除法. $p_l \le \sum_{j=1}^{k-1} p_j$.

从而 $\frac{C^*(I)}{C^{\sigma}(I)} \le \frac{\sum_{j=1}^{k-1} p_j + p_l}{\sum_{j=1}^{k-1} p_j} \le \frac{2\sum_{j=1}^{k-1} p_j}{\sum_{j=1}^{k-1} p_j} = 2$.

故当 $l > k$ 时. 仍有 $C^*/C^{\sigma} \le 2$.

2.

(1): 令 $S_1=\{s_n\}$. $S_2=\{s_1, s_2\cdots s_{n-1}\}$. 则 $\sigma(S)=\frac{l(S_1)}{l(S_2)}$.

若否. 则至少有 $i\in\{1,2\ldots n-1\}$ s.t. $\{s_n, s_i\}\subset S_1'$, $S_2'=S\setminus S_1'$

从而 $\frac{l(S_1')}{l(S_2')}>\frac{l(S_1)}{l(S_2)}$ 故此算法为最优. 显然上述步骤在多项式时间内可完成.

(2): 不妨设 $k=1$, $j\geq 2$.

$l(T_1)=s_{j+1}$. $l(T_2)=l(T_3)-s_1$.

① $\frac{l(T_1)}{l(T_2)}>\frac{l(T_3)}{l(T_1)}$ 时. 即 $l^2(T_1)>l(T_2)l(T_3)=l^2(T_2)+s_1l(T_2)$.

往证 $\frac{l(T_3)}{l(T_1)}\leq\alpha$. 即证. $\alpha l(T_1)\geq l(T_2)+s_1$.

即证: $\alpha^2l^2(T_1)\geq l^2(T_2)+2s_1l(T_2)+s_1^2$.

即证: $(\alpha^2-1)l^2(T_2)+(\alpha^2s_1-2s_1)l(T_2)-s_1^2\geq 0$.

$l(T_2)=s_2+s_3+\cdots+s_j\geq\alpha s_1+\alpha^2s_1+\cdots+\alpha^{j-1}s_1=s_1\alpha\frac{1-\alpha^{j-1}}{1-\alpha}$

代入上式有:

$(\alpha^2-1)\alpha^2\frac{(1-\alpha^{j-1})^2}{(1-\alpha)^2}s_1^2+(\alpha^2-2)\alpha\frac{1-\alpha^{j-1}}{1-\alpha}s_1^2-s_1^2=LHS$.

$LHS\geq s_1^2\left[\frac{\alpha-1}{\alpha+1}\right](\alpha^4-\alpha^2+\alpha^3-2\alpha-1)>0$. $(j\geq 2)$.

故 $LHS\geq 0$. 即 $\frac{l(T_3)}{l(T_1)}\leq\alpha$.

同理: 当 $\frac{l(T_1)}{l(T_2)}<\frac{l(T_3)}{l(T_1)}$ 时. 有 $\frac{l(T_1)}{l(T_2)}\leq\alpha$.

故 $\min\{\frac{l(T_1)}{l(T_2)}, \frac{l(T_3)}{l(T_1)}\}\leq\alpha$.

(3)：现对 $S=\{s_1, s_2 \cdots s_n\}$ 作如下改造.

① 若 $s_2 < \alpha s_1$，则令 $s_2' = s_2 + s_3$.

若 $s_2' \geq \alpha s_1$，则令 $S' = \{s_1, s_2', s_4 \cdots s_{n-1}, s_n\}$.

若 $s_2' < \alpha s_1$，则令 $s_2'' = s_2 + s_3 + s_4$.

$\cdots$

对 $i \geq 1$，$s_{i+1} < \alpha s_i$，同①中做法. 从而得到一个新集 S_0

满足. $s_{i+1} \geq \alpha s_i$，$\forall i$，$s_i \in S_0$. (*)

则有如下两情形.

②：若不 $\exists j$，s.t. $s_{j+1} \leq \sum_{i=1}^{j} s_i$，则 S_0 是超增集.

由(1)中构造，可直接得到最优解. 故最坏情况界 $=1$.

③ 若 $\exists j$，s.t. $s_{j+1} \leq \sum_{i=1}^{j} s_i$，则按(2)中取法.

S_0 中下标从 k 到 $j+1$ 的元素取为 $T_1=\{s_{j+1}\}$.

$T_2=\{s_{k+1} \cdots s_j\}$. $T_3=\{s_k \cdots s_j\}$.

若 (i)：$\dfrac{l(T_1)}{l(T_2)} > \dfrac{l(T_3)}{l(T_1)}$ 则将 T_3 中元加入 S_1.
T_1 中元加入 S_2.

令 $S_0' = S_0 \setminus (T_1 \cup T_3)$.

(ii)：$\dfrac{l(T_1)}{l(T_2)} < \dfrac{l(T_3)}{l(T_1)}$ 则将 T_1 中元加入 S_1
T_2 中元加入 S_2.

令 $S_0' = S_0 \setminus (T_1 \cup T_3)$.

无论何种，新的集 S_0' 依旧符合(*)式，于是再进行上述步骤. 由于 $|S| < \infty$，有限步会终止.

最终得到 S_1, S_2，符合 $\dfrac{l(S_1)}{l(S_2)} \leq \alpha$. $S_1=\{T_1^1, T_1^2 \cdots T_1^n\}$
$S_2=\{T_2^1, T_2^2 \cdots T_2^n\}$

$\dfrac{l(T_1^i)}{l(T_2^i)} \leq \alpha$，$\forall i=1,2\cdots n$. 故 $\dfrac{l(S_1)}{l(S_2)} \leq \dfrac{l(T_1^1)+\cdots+l(T_1^n)}{l(T_2^1)+\cdots+l(T_2^n)} \leq \alpha$.