一.(1): $C_j = \sum_{i=1}^{j} P_i$. 时. $\bar{C} = \frac{1}{n}\sum_{j=1}^{n} C_j = \frac{1}{n}\sum_{j=1}^{n}\sum_{i=1}^{j} P_i = \frac{1}{n}\sum_{i=1}^{n}(n-i+1)P_i$.

故 $S = \sum_{j=1}^{n}(C_j-\bar{C})^2 = \sum_{j=1}^{n}\left(\sum_{i=1}^{j} P_i - \frac{1}{n}\sum_{i=1}^{n}(n-i+1)P_i\right)$

(2): 反设 $\exists$ 最优解: 最早加工的都不是加工时间最大的工件.

不会.

(3): 由(2)知 最优解加工顺序为 $\sigma = (1.2.3.4)$

故 $C_1 = P_1$ $C_2 = P_1+P_2$. $C_3 = P_1+P_2+P_3$ $C_4 = P_1+P_2+P_3+P_4$.

从而 $\bar{C} = \frac{1}{4}(4P_1+3P_2+2P_3+P_4)$.

$$S = \frac{1}{4}\Big[\frac{1}{16}(3P_2+2P_3+P_4)^2 + \frac{1}{16}(-P_2+2P_3+P_4)^2 + \frac{1}{16}(-P_2-2P_3+P_4)^2 + \frac{1}{16}(P_2+2P_3+3P_4)^2\Big]$$

$$\Rightarrow S = \frac{3}{16}(P_1^2+P_2^2+P_3^2) + \frac{1}{4}P_2P_3 + \frac{1}{8}P_2P_4 + \frac{1}{4}P_3P_4.$$

二.

(1): $S(u.v, u.v) = \{(x,y) \in [0,1) \times [0,1) \mid d(x, O_{uu}) + d(y, O_{vv}) < C_{uv}\}$

$O_{uu} = O_{vv} = 1$. 故 $S(u.v, u.v) = \{(x,y) \in [0,1)^2 \mid d(x,1) + d(y,1) < C_{uv}\}$

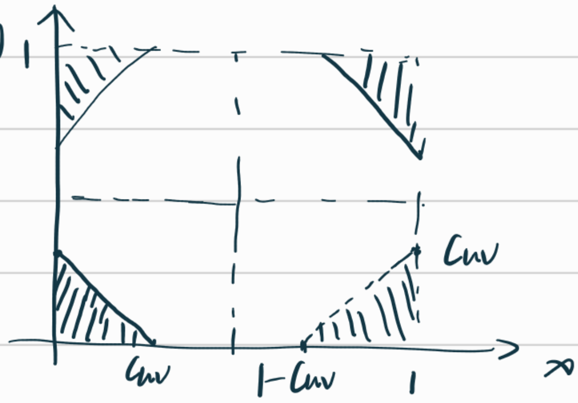
$d(x,1) = \min\{1-x, x\}$.

① $x, y < \frac{1}{2}$ 时. $x+y < C_{uv}$.

② $x \geq \frac{1}{2}, y < \frac{1}{2}$. $1-x+y < C_{uv}$.

③ $x < \frac{1}{2}, y \geq \frac{1}{2}$ $x+1-y < C_{uv}$.

④ $x, y \geq \frac{1}{2}$. $1-x+1-y < C_{uv}$.



且 $C_{uv} \leq \frac{1}{2}$. 因为 $\forall T, |T| \geq 2C_{ij}$. $\forall i, j$

故 $1 = |T^*| \geq 2C_{uv} \Rightarrow C_{uv} \leq \frac{1}{2}$

如图所示: 故 $|S| = 4 \times \frac{1}{2} \times C_{uv} \cdot C_{uv} = 2C_{uv}^2$

$|S(u.v. u.v)| = 2C_{uv}^2$

(2):

$1, 2 \cdots k-1, k, k+1, \cdots l-1, l. l+1 \cdots n$

$|T'| = |T| - C_{k,k+1} + C_{k,l} - C_{l,l+1} + C_{k+1,l+1} \geq |T|$

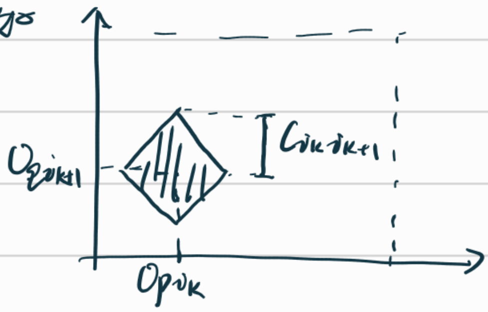
$|T'| = |T| - c_{k,k+1} - c_{l,l+1} + c_{k,l} + c_{k+1,l+1} \ge |T|$

即 $c_{k,k+1} + c_{l,l+1} \le c_{k,l} + c_{k+1,l+1}$

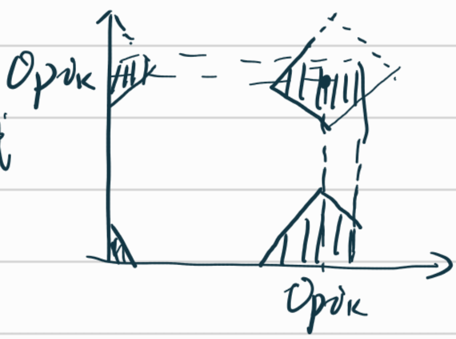
小性质. 对上述 $k.l$ 来说. $\sigma = (k, k+1, l, l+1)$ 是局部环游的最优解.

证明: $S(p, q, i_k, i_{k+1}) = \{(x,y) \in [0,1]^2 \mid d(x, Op_{i_k}) + d(y, Oq_{i_{k+1}}) < c_{i_k, i_{k+1}}\}$.

S形如



或



要证 $S(p, q, i_k, i_{k+1})$ 与 $S(p, q, i_l, i_{l+1})$ 不交.

只需证: $\frac{\sqrt{2}}{2} c_{i_k, i_{k+1}} + \frac{\sqrt{2}}{2} c_{i_l, i_{l+1}} \le \sqrt{(Op_{i_k} - Op_{i_l})^2 + (Op_{i_{k+1}} - Op_{i_{l+1}})^2}$

即证: $\frac{1}{2}(c_{i_k, i_{k+1}} + c_{i_l, i_{l+1}})^2 \le (Op_{i_k} - Op_{i_l})^2 + (Op_{i_{k+1}} - Op_{i_{l+1}})^2$

而有: $RHS = O^2_{i_k, i_l} + O^2_{i_{k+1}, i_{l+1}} \ge O^2_{i_k, i_{k+1}} + O^2_{i_l, i_{l+1}}$.

这是因为 T 为 2-opt 环游.

从而 $RHS \ge O^2_{i_k, i_{k+1}} + O^2_{i_l, i_{l+1}} \ge c^2_{i_k, i_{k+1}} + c^2_{i_l, i_{l+1}}$

$\ge \frac{1}{2}(c_{i_k, i_{k+1}} + c_{i_l, i_{l+1}})^2$.

第二个"$\ge$"是因为 $c_{i,l} \le O_{i,l} \in \{c_{i,l}, 1 - c_{i,l}\}$ 而 $c_{i,l} \le \frac{1}{2}$.

第三个"$\ge$"是由基本不等式 $a^2 + b^2 \ge \frac{1}{2}(a+b)^2$.

综上: $S(p, q, i_k, i_{k+1})$ 与 $S(p, q, i_l, i_{l+1})$ 交为空.

(3): 如(2)中图. $|S(p, q, u, v)| = 4 \times \frac{1}{2} c_{uv} \cdot c_{uv} = 2 c_{uv}^2 = f(u, v)$.

故 $\forall p_1, p_2, q_1, q_2$. $|S(p_1, q_1, u, v)| = |S(p_2, q_2, u, v)|$.

(4): T 为最优解. T^* 为 2-opt 算法解.

$|T| \le \sum c_{uv} \le n \cdot \frac{\sum c_{uv}}{n} \le n \cdot \sqrt{\frac{\sum c_{uv}^2}{n}} \le \sqrt{\frac{n}{2}} |T|^*$.

故 $|T| / |T^*| \le \sqrt{\frac{n}{2}}$.

即最坏情况比不超过 $\sqrt{\frac{n}{2}}$