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P105. Q1. Q3. Q5. Q7

Q1: $u = f(r)$,

$u_{x_i} = f'(r) r_{x_i} = \frac{x_i}{r} f'(r)$. $\frac{\partial u^2}{\partial x_i^2} = \frac{1}{r} f'(r) - \frac{x_i^2}{r^3} f'(r) + \frac{x_i^2}{r^2} f''(r)$

由 $\Delta u = 0 \Rightarrow \frac{n}{r} f'(r) - \frac{\Sigma x_i^2}{r^3} f'(r) + \frac{\Sigma x_i^2}{r^2} f''(r) = \frac{n-1}{r} f'(r) + f''(r) = 0$.

① 当 $n=2$ 时. $-\frac{1}{r} = \frac{f''(r)}{f'(r)} \Rightarrow \ln|f'(r)| = \ln c_1 r^{-1} \Rightarrow |f'(r)| = \frac{c_1}{r}$

$\Rightarrow f(r) = \pm c_1 \ln r + c_2$. 即 $f(r) = c_3 + c_4 \ln r^{-1}$ （$\mp c_1 = c_4$. $c_2 = c_3$）.

② 当 $n > 2$ 时.

$-\frac{n-1}{r} = \frac{f''(r)}{f'(r)} \Rightarrow \ln|f'(r)| = \ln c_2 r^{1-n}$

$\Rightarrow f(r) = c_1 + c_2 r^{2-n}$

故有 $u = f(r) = \begin{cases} c_1 + c_2 r^{2-n} & n \neq 2 \\ c_1 + c_2 \ln r^{-1} & n = 2 \end{cases}$

3. $\Delta u = u_{xx} + u_{yy} + u_{zz}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$

$r_x = \cos\theta \quad r_{xx} = \frac{y^2}{r^3} \quad r_y = \sin\theta \quad r_{yy} = \frac{x^2}{r^3}$

$\theta_x = \frac{-1}{\sqrt{1-\frac{x^2}{r^2}}}\left(\frac{1}{r} - \frac{x^2}{r^3}\right) = -\frac{y}{r^2} \quad \theta_{xx} = \frac{2xy}{r^4}$

$\theta_y = \frac{1}{\sqrt{1-\frac{x^2}{r^2}}}\left(\frac{1}{r} - \frac{y^2}{r^3}\right) = \frac{x}{r^2} \quad \theta_{yy} = -\frac{2xy}{r^4}$

$u_x = u_r r_x + u_\theta \theta_x \Rightarrow u_{xx} = u_{rr}(r_x)^2 + 2u_{r\theta} r_x \theta_x + u_{\theta\theta}(\theta_x)^2 + u_r r_{xx}$

同理 $u_{yy} = u_{rr}(r_y^2) + 2u_{r\theta} r_y \theta_y + u_{\theta\theta}(\theta_y)^2 + u_r r_{yy}$

$\Rightarrow \quad u_{xx} + u_{yy} = (r_x^2 + r_y^2) u_{rr} + 2u_{r\theta}(r_x\theta_x + r_y\theta_y) + u_{\theta\theta}(\theta_y^2 + \theta_x^2)$

$= u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$

故 $\Delta u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\partial^2 u}{\partial z^2}$

Q5:

$r_x = \frac{x}{r} \quad r_{xx} = \frac{y^2}{r^3} \quad \theta_x = -\frac{y}{r^2} \quad \theta_{xx} = \frac{2xy}{r^4}$

$\theta_y = \frac{x}{r^2} \quad \theta_{yy} = -\frac{2xy}{r^4}$

(1) $\ln r$ 与 θ.

$\Delta \ln r = \Delta\left(\ln(x^2+y^2)^{\frac{1}{2}}\right) = \frac{1}{2}\left[\left(\frac{2x}{x^2+y^2}\right)_x + \left(\frac{2y}{x^2+y^2}\right)_y\right] = \frac{1}{2}\left(\frac{-2x^2+2y^2}{(x^2+y^2)^4} + \frac{2x^2-2y^2}{(x^2+y^2)^4}\right) = 0$

$\Delta\theta = \Delta(x\tan\theta) = 0$.

(2): $r^n\cos n\theta$ 与 $r^n\sin n\theta$.

$\frac{\partial^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta}\right) = \frac{\partial^2 r}{\partial x^2}\frac{\partial}{\partial r} + \left(\frac{\partial r}{\partial x}\right)^2\frac{\partial^2}{\partial r^2} + \frac{\partial^2\theta}{\partial x^2}\frac{\partial}{\partial\theta} + \left(\frac{\partial\theta}{\partial x}\right)^2\frac{\partial}{\partial\theta} = \frac{y^2}{r^3}\frac{\partial}{\partial r} + \frac{x^2}{r^2}\frac{\partial^2}{\partial r^2} + \frac{2xy}{r^4}\frac{\partial}{\partial\theta} + \frac{y^2}{r^4}\frac{\partial^2}{\partial\theta^2}$

同理 $\frac{\partial^2}{\partial y^2} = \frac{x^2}{r^3}\frac{\partial}{\partial r} + \frac{y^2}{r^2}\frac{\partial^2}{\partial r^2} - \frac{2xy}{r^4}\frac{\partial}{\partial\theta} + \frac{x^2}{r^4}\frac{\partial^2}{\partial\theta^2}$

同理 $\frac{\partial^2}{\partial y^2}$ …

$\Rightarrow \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\theta}$.

故 $\Delta(r^n\cos n\theta)=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(r^n\cos n\theta)=\frac{1}{r}nr^{n-1}\cos n\theta+n(n-1)r^{n-2}\cos n\theta-n^2r^{n-2}\cos n\theta$

$=r^{n-2}\cos n\theta\,(n+(n-1)n-n^2)=0$.

同理 $\Delta(r^n\sin n\theta)=0$.

故 $r^n\cos n\theta$ 与 $r^n\sin n\theta$ 满足Laplace方程.

$\sin\theta+\theta\cos\theta$

$\cos\theta+\cos\theta-\theta\sin\theta$

(3): $r\ln r\cos\theta-r\theta\sin\theta$ 与 $r\ln r\sin\theta+r\theta\cos\theta$.

$\ln r+1$

同(2)中作法有:

$\Delta(r\ln r\cos\theta-r\theta\sin\theta)=\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(r\ln r\cos\theta-r\theta\sin\theta)$

$=\frac{1}{r}\ln r\cos\theta+\frac{1}{r}\cos\theta-\frac{1}{r}\theta\sin\theta+\frac{1}{r}\cos\theta-\frac{\ln r}{r}\cos\theta-\frac{2}{r}\cos\theta+\frac{\theta\sin\theta}{r}$

$=0$.

同理 $\Delta(r\ln r\sin\theta+r\theta\cos\theta)=0$. 故满足Laplace方程

Q7: 令 $\Gamma=\{(x,y): x^2+y^2=1\}$ $g\equiv1$.

令 $u_1=\begin{cases}1 & |r|\geq0.5\\ 2 & |r|<0.5\end{cases}$ $u_2=\begin{cases}1 & |r|\geq0.5\\ 0.5 & |r|<0.5\end{cases}$

则 u_1, u_2 在 Γ 外部满足Laplace方程，在 $\Omega\cup\Gamma$ 上连续. 且 $u_i|_\Gamma=g\equiv1$. $i=1,2$.

故解不唯一.