1. 令 $V=(h-x)u$.

则可得 $V_{tt}=(h-x)u_{tt}$,

$$V_{xx}=(h-x)u_{xx}-2u_x.$$

对原式 $\frac{\partial}{\partial x}[(1-\frac{x}{h})^2u_x]=\frac{2(h-x)}{h^2}u_x+\frac{(h-x)^2}{h^2}u_{xx}$ 有

$$\frac{\partial}{\partial x}[(1-\frac{x}{h})^2u_x]=\frac{h-x}{h^2}V_{xx}.$$

同理有 $\frac{1}{a^2}(1-\frac{x}{h})^2\frac{\partial^2u}{\partial t^2}=\frac{1}{a^2}\frac{h-x}{h^2}\cdot V_{tt}$

则方程等价于 $\cancel{a}\ \frac{h-x}{h^2}(V_{tt}-a^2V_{xx})=0$. 且 $x\neq h$.

即 $V_{tt}-a^2V_{xx}=0$. 已知此方程的解为

$$V(t,x)=F(x-at)+G(x+at). \quad F,G\in C^2.$$

i.e. $(h-x)u(t,x)=F(x-at)+G(x+at)$.

故 $u(t,x)=\dfrac{F(x-at)+G(x+at)}{h-x}$.

则有 $\begin{cases} V_{tt}-a^2V_{xx}=0 \\ V(0,x)=(h-x)\varphi(x). \\ V_t(0,x)=(h-x)\psi(x). \end{cases}$ 由达朗贝尔公式知.

$$V(t,x)=\frac{1}{2}[(h-x-ct)\varphi(x+ct)+(h-x+ct)\varphi(x-ct)]+\frac{1}{2a}\int_{x-at}^{x+at}(h-s)\psi(s)\,ds.$$

故 $u(t,x)=\frac{1}{2}[(1-\frac{ct}{h-x})\varphi(x+ct)+(1+\frac{ct}{h-x})\varphi(x-ct)+\frac{1}{2a(h-x)}\int_{x-at}^{x+at}(h-s)\psi(s)\,ds.$

7. 由 $u_{tt}-u_{xx}=0$. $\Rightarrow u(t,x)=F(x-t)+G(x+t)$.

$u|_{t=x}=\varphi(x) \Leftrightarrow u(x,x)=F(0)+G(2x)=\varphi(x)$

$u|_{t=f(x)}=\psi(x) \Leftrightarrow u(f(x),x)=F(x-f(x))+G(x+f(x))=\psi(x)$.

$\Rightarrow G(x)=\varphi(\frac{1}{2}x)-F(0)$.

$\cancel{F(x)=}$ 令 $x-f(x)=s$. 由 $\forall x$, $f'(x)\neq 1$. 知 $s_x\neq 0$. 故 $\exists$ $\cancel{s(x)}$ $\delta(s)$.

~~其~~. ~~x~~ 是由 $x-f(x)=s$ 确定的逆函数 $x=\delta(s)$.

则有: $F(s)=\psi(\delta(s))-G(\delta(s)+f(\delta(s)))$

$=\psi(\delta(s))-\varphi(\frac{1}{2}\delta(s)+\frac{1}{2}f(\delta(s)))+F(0)$

$=\psi(\delta(s))-\varphi(\delta(s)-\frac{1}{2}s)+F(0)$.

故 $u(t,x)=F(x-t)+G(x+t)$

$\Rightarrow u(t,x)=\psi(\delta(x-t))-\varphi(\delta(x-t)-\frac{1}{2}(x-t))+\varphi(\frac{1}{2}(x+t))$

8. 令 $u=v+w$. 其中. v,w 是 Cauchy 问题.

$$\begin{cases} v_{tt}-v_{xx}=\sin x \\ v(0,x)=0,\ v_t(0,x)=0 \end{cases} \qquad \begin{cases} w_{tt}-w_{xx}=0 \\ w(0,x)=0,\ w_t(0,x)=\sin x. \end{cases}$$

由达朗贝尔公式: $w(t,x)=\frac{1}{2}\int_{x-t}^{x+t}\sin s\,ds=\frac{1}{2}\cos(x-t)-\frac{1}{2}\cos(x+t)$.

$v(t,x)=\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\tau\sin\xi\,d\xi\,d\tau=\frac{1}{2}\int_0^t\left(\tau\cancel{d\tau}\int_{x-(t-\tau)}^{x+(t-\tau)}\sin\xi\,d\xi\right)d\tau$

$=\frac{1}{2}\int_0^t\tau[\cos(x-t+\tau)-\cos(x+t-\tau)]\,d\tau$

$\cancel{=\frac{1}{2}(2\sin x-\sin(x-t)-\sin(x+t))}$

$\cancel{=\sin x-\sin x\cos t=\sin x(1-\cos t)}$

$=t\sin x-\sin x\sin t=(t-\sin t)\sin x$.

故 $u(t,x)=v(t,x)+w(t,x)=t\sin x$.