7. $\begin{cases} u_{tt} - c^2(u_{xx} + u_{yy}) = 0 \\ u(0,x) = x^2(x+y), \quad u_t(0,x) = 0. \end{cases}$ 即 $\varphi(x) = x^2(x+y)$, $\psi(x) = 0$.

由二维波动方程的 poisson 公式.

$$u(t,x,y) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint_{r<ct} \frac{\varphi(\xi,\eta)}{\sqrt{c^2t^2-r^2}}\,d\xi\,d\eta\right]$$

其中 $r = \sqrt{(\xi-x)^2 + (\eta-y)^2}$.

令 $\xi = x + r\cos\theta$, $\eta = y + r\sin\theta$. $\dfrac{\partial(\xi,\eta)}{\partial(r,\theta)} = r$.

$$故\ I = \iint_{r<ct} \frac{\varphi(\xi,\eta)}{\sqrt{c^2t^2-r^2}}\,d\xi\,d\eta = \int_0^{ct}\int_0^{2\pi} \frac{\varphi(x+r\cos\theta,\, y+r\sin\theta)}{\sqrt{c^2t^2-r^2}} \cdot r\,dr\,d\theta.$$

$$\varphi(x+r\cos\theta, y+r\sin\theta) = (x+r\cos\theta)^2[x+y + r(\cos\theta+\sin\theta)]$$
$$= x^2(x+y) + x^2 r(\cos\theta+\sin\theta) + 2x(x+y)r\cos\theta +$$
$$2xr^2\cos\theta(\cos\theta+\sin\theta) + (x+y)r^2\cos^2\theta + r^3\cos^2\theta(\cos\theta+\sin\theta).$$

$$\int_0^{2\pi}\varphi\,d\theta = 2\pi x^2(x+y) + 2\pi x r^2 + \pi(x+y)r^2 + = 2\pi x^2(x+y) + \pi(3x+y)r^2$$

$$故\ I = \int_0^{ct}\frac{2\pi x^2(x+y)r}{\sqrt{c^2t^2-r^2}}\,dr + \int_0^{ct}\frac{\pi(3x+y)\,r^3}{\sqrt{c^2t^2-r^2}}\,dr.$$

$$其中: \int_0^{ct}\frac{r}{\sqrt{c^2t^2-r^2}}\,dr = -\sqrt{c^2t^2-r^2}\,\Big|_{r=0}^{r=ct} = ct$$

$$\int_0^{ct}\frac{r^3}{\sqrt{c^2t^2-r^2}}\,dr = -r\sqrt{c^2t^2-r}\,\Big|_{r=0}^{r=c^2t^2} + \int_0^{t^2c^2}\sqrt{c^2t^2-r}\,dr$$
$$= -\frac{2}{3}(c^2t^2-r)^{\frac{3}{2}}\Big|_{r=0}^{r=c^2t^2} = \frac{2}{3}c^3t^3$$

故 $u(t,x) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\left(2\pi x^2(x+y)ct + \pi(3x+y)\frac{2}{3}c^3t^3\right)$

$= x^2(x+y) + (3x+y)c^2t^2$

即 ~~$u(x,y$~~ $u(t,x,y) = x^2(x+y) + (3x+y)c^2t^2$.

3. $u = u(t,r)$. $r = \sqrt{x^2+y^2}$. 且 $u_{tt} - c^2(u_{xx}+u_{yy}) = 0$.

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

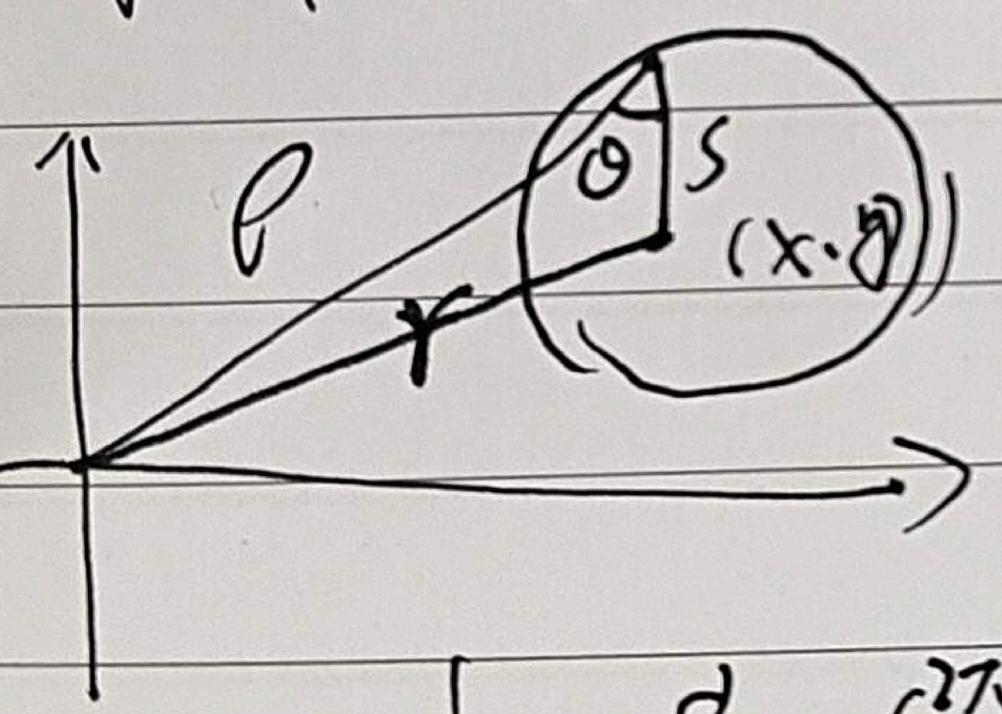
$u(t,x,y) = \frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\iint_{S_{ct}(x,y)} \frac{\varphi(\xi,\eta)}{\sqrt{c^2t^2-r^2}}\,d\xi d\eta + \frac{1}{2\pi c}\iint_{S} \frac{\psi(\xi,\eta)}{\sqrt{c^2t^2-r^2}}\,d\xi d\eta\right.$

由于 $u = u(t,r)$. 故 $\varphi(\xi,\eta) = \varphi(\rho)$. $\psi(\xi,\eta) = \varphi(\rho)$

$r = \sqrt{x^2+y^2}$. $\rho = \sqrt{\xi^2+\eta^2}$. $s = \sqrt{(\xi-x)^2+(\eta-y)^2}$

~~记ρ.s夹角为θ.~~ 记 $\langle r, s\rangle = \theta$

则有: ~~$\rho^2 = \rho^2+s^2-2\rho s$~~ $\rho^2 = s^2+r^2-2sr\cos\theta$



$d\xi d\eta = s\,ds\,d\theta$.

故 $u = \frac{1}{2\pi c}\left[\frac{\partial}{\partial t}\int_0^{2\pi}\int_0^{ct} \frac{\varphi(\sqrt{s^2+r^2-2sr\cos\theta})\,s\,ds\,d\theta}{\sqrt{c^2t^2-s^2}}\right.$

$+ \frac{1}{2\pi c}\int_0^{2\pi}\int_0^{ct} \frac{\psi(\sqrt{s^2+r^2-2sr\cos\theta})\,s\,ds\,d\theta}{\sqrt{c^2t^2-s^2}}$.