2. 证明：$\int_R |f(x)|\,dx \triangleq M < \infty$

$$\mathcal{F}[f](\lambda) = \int_{-\infty}^{+\infty} f(x) e^{-i\lambda x}\,dx.$$

$$|\mathcal{F}[f](\lambda+\Delta\lambda) - \mathcal{F}[f](\lambda)| = \left|\int_{-\infty}^{+\infty} f(x) e^{-i(\lambda+\Delta\lambda)x}\,dx - \int_{-\infty}^{+\infty} f(x) e^{-i\lambda x}\,dx\right|$$

$$= \left|\int_R f(x) e^{-i\lambda x}(e^{-i\Delta\lambda x} - 1)\,dx\right| \le \int_R |f(x)|\,dx \int_R |e^{-i\lambda x}||e^{-i\Delta\lambda x} - 1|\,dx$$

$$\le M \int_R e^{-|\lambda x|}(1 - e^{-|\Delta\lambda x|})\,dx = 2M\int_0^{+\infty}(e^{-\lambda x} - e^{-(\lambda+\Delta\lambda)x})\,dx$$

$$= 2M\left(-\frac{1}{\lambda}e^{-\lambda x}\Big|_0^{+\infty} + \frac{1}{\lambda+\Delta\lambda}e^{-(\lambda+\Delta\lambda)x}\Big|_0^{+\infty}\right) = 2M\left(\frac{1}{\lambda} - \frac{1}{\lambda+\Delta\lambda}\right)$$

$$= \frac{2M}{\lambda(\lambda+\Delta\lambda)}\cdot\Delta\lambda.$$

故 $\lim\limits_{\Delta\lambda\to 0}|\mathcal{F}[f](\lambda+\Delta\lambda) - \mathcal{F}[f](\lambda)| = \lim\limits_{\Delta\lambda\to 0}\frac{2M}{\lambda(\lambda+\Delta\lambda)}\cdot\Delta\lambda = \frac{2M}{\lambda^2}\lim\limits_{\Delta\lambda\to 0}\Delta\lambda = 0.$

故 $\mathcal{F}[f]$ 是连续函数

6. $v(t,x,y;\tau,\xi,\eta) = \frac{1}{4\pi c^2(t-\tau)}\exp\left\{-\frac{(x-\xi)^2+(y-\eta)^2}{4c^2(t-\tau)}\right\}.$

令 $f(t,\tau) = 1/(4c^2(t-\tau))$ $\quad g(x,y;\xi,\eta) = (x-\xi)^2 + (y-\eta)^2$

则 $v = \frac{1}{\pi} f\cdot\exp\{-gf\}.$

$v_t = \frac{1}{\pi} f_t\exp\{-gf\} - \frac{1}{\pi} gf f_t\exp\{-gf\}.$

其中 $f_t = -\frac{1}{4c^2(t-\tau)^2} = -4c^2 f^2$

故 $v_t = -\frac{4}{\pi}c^2 f^2\exp\{-gf\} + \frac{4}{\pi}c^2 gf^3\exp\{-gf\}.$

$v_x = -\frac{1}{\pi} g_x f^2\exp\{-gf\}.$ $\quad v_{xx} = -\frac{1}{\pi} g_{xx} f^2\exp\{-gf\} + \frac{1}{\pi} g_x^2 f^3\exp\{-gf\}.$

$v_y = -\frac{1}{\pi} g_y f^2\exp\{-gf\}.$ $\quad v_{yy} = -\frac{1}{\pi} g_{yy} f^2\exp\{-gf\} + \frac{1}{\pi} g_y^2 f^3\exp\{-gf\}.$

则 $v_{xx} + v_{yy} = -\frac{1}{\pi} f^2\exp\{-gf\}(g_{xx} + g_{yy}) + \frac{1}{\pi} f^3\exp\{-gf\}(g_x^2 + g_y^2)$

其中 $g_x = 2(x-\xi)$ $\quad g_{xx} = 2$ $\quad g_y = 2(y-\eta)$ $\quad g_{yy} = 2$

从而 $g_{xx} + g_{yy} = 4$ $\quad g_x^2 + g_y^2 = 4(x-\xi)^2 + 4(y-\eta)^2 = 4g$

故 $v_{xx} + v_{yy} = -\frac{4}{\pi} f^2\exp\{-gf\} + \frac{4}{\pi} gf^3\exp\{-gf\}.$

故 $v_t = c^2(v_{xx} + v_{yy}) = -\frac{4}{\pi}c^2 f^2\exp\{-gf\} + \frac{4}{\pi}c^2 gf^3\exp\{-gf\}.$

同上对 f. g 的定义. 显然有 $v_{xx} + v_{yy} = v_{\xi\xi} + v_{\eta\eta}$

而 $f_\tau = \frac{1}{4\pi c^2(t-\tau)^2} = -f_t.$ $\quad$ 故 $v_\tau = -v_t.$

从而 $v_\tau + c^2(v_{\xi\xi} + v_{\eta\eta}) = 0$

解 $v_t + c^2(v_{\xi\xi}+v_{\eta\eta})=0$.

1. 求解 $\begin{cases} u_t = c^2 u_{xx} & (t>0,\ 0<x<1) \\ u(t,0)=u_x(t,1)=0. & (t>0) \\ u(0,x)=f(x). & 0<x<1. \end{cases}$

令 $u(t,x)=X(x)T(t)$. 则有 $\frac{T'}{Tc^2}=\frac{X''}{X}=-\lambda$

① 若 $\lambda\le 0$. 则由 $X''+\lambda X=0$. $\Rightarrow X=Ae^{-\sqrt{-\lambda}x}+Be^{\sqrt{-\lambda}x}$

由 $u(t,0)=0 \Rightarrow A+B=0$

由 $u_x(t,1)=0 \Rightarrow -\sqrt{-\lambda}(Ae^{-\sqrt{-\lambda}}-Be^{\sqrt{-\lambda}})=0$ $\Rightarrow A\sqrt{-\lambda}=0$.

则若 $A=0$. 则 $u(t,x)=0$, 若 $\lambda=0$. 则 $u(t,x)=0$.

此时当且仅当 $f(x)\equiv 0$, 否则无解.

② $\lambda>0$ 时. $X(x)=A\sin\sqrt{\lambda}x+B\cos\sqrt{\lambda}x$, $T(t)=C_0e^{-\lambda c^2t}$

则有: $\begin{cases} B=0 \\ A\sqrt{\lambda}\cos\sqrt{\lambda}-B\sqrt{\lambda}\sin\sqrt{\lambda}=0 \end{cases}$ $\Rightarrow \lambda=\left(\frac{\pi}{2}+k\pi\right)^2$ $k=0,1,2\cdots$

故 $u_k(t,x)=A_ke^{-(\frac{\pi}{2}+k\pi)c^2t}\sin(\frac{\pi}{2}+k\pi)x$ $u(t,x)=\sum_{k=0}^{\infty}u_k(t,x)$.

由 $u(0,x)=f(x) \Rightarrow f(x)=\sum_{k=0}^{\infty}A_k\sin(\frac{\pi}{2}+k\pi)x$ $T=2$

$A_k=\int_0^2 f(x)\sin(\frac{\pi}{2}+k\pi)x\,dx$ $k\ge 1$. $A_0=\frac{1}{2}\int_0^2 f(x)\,dx$

故 $u(t,x)=\sum_{k=1}^{\infty}\left(\int_0^2 f(\xi)\sin(\frac{\pi}{2}+k\pi)\xi\,d\xi\right)\sin(\frac{\pi}{2}+k\pi)x.+\frac{1}{2}\int_0^2 f(x)\,dx$.

3. $x=0$ —— l —— $x=l$

即有 $\begin{cases} u_t=c^2u_{xx} \\ u_x(t,0)=u_x(t,l)=0 \\ u(0,x)=f(x) \end{cases}$ 令 $u(t,x)=X(x)T(t)\Rightarrow \begin{cases} X''+\lambda X=0 \\ T'+\lambda c^2T=0 \\ X'(0)=X'(l)=0. \end{cases}$

① $\lambda<0$ 时: $X(x)=c_1e^{-\sqrt{-\lambda}x}+c_2e^{\sqrt{-\lambda}x}$. 易知 此时无非零解.

② $\lambda=0$ 时. $X(x)=c_1x+c_2$. 由于 $X'(x)|_{x=0}=0$. 故 $c_1=0$. 故 $u(t,x)=c_2'$

由 $u(0,x)=f(x)$. 故当且仅当 $f(x)\equiv C\in\mathbb{R}$ 时. $u(t,x)=C$ 符合.

③ $\lambda>0$ 时. $X(x)=A\sin\sqrt{\lambda}x+B\cos\sqrt{\lambda}x$ 由 $X'(0)=X'(l)=0$

$\Rightarrow \begin{cases} A\sqrt{\lambda}=0 \\ A\sqrt{\lambda}\cos\sqrt{\lambda}l-B\sqrt{\lambda}\sin\sqrt{\lambda}l=0 \end{cases}$ $\Rightarrow \begin{cases} A=0 \\ \lambda=k^2\pi^2/l^2 \end{cases}$ $k=0,1,2\cdots$

故 $X_k(x)=B_k'\cos\frac{k\pi}{l}x$ $T_k(t)=Ce^{-\frac{k^2\pi^2c^2}{l^2}t}$

$\Rightarrow u(t,x)=\sum_{k=0}^{\infty}A_ke^{-(\frac{k\pi c}{l})^2t}\cos\frac{k\pi}{l}x$

由 $u(0,x)=f(x) \Rightarrow f(x)=\sum_{k=0}^{\infty}A_k\cos\frac{k\pi}{l}x$ i.e. $A_k=\frac{2}{l}\int_0^l f(\xi)\cos\frac{k\pi}{l}\xi\,d\xi$

故 $u(t,x)=\frac{1}{l}\int_0^l f(\xi)d\xi+\sum_{k=1}^{\infty}\left[\frac{2}{l}\int_0^l f(\xi)\cos\frac{k\pi}{l}\xi d\xi\right]\cos\frac{k\pi}{l}x$

当 $f(x)\equiv u_0\in R$ 时. $A_0=\frac{1}{l}\int_0^l u_0\,d\xi=u_0$. $A_k=\frac{2}{l}\int_0^l u_0\cos\frac{k\pi}{l}\xi d\xi=\frac{2u_0}{l}\cdot\frac{l}{k\pi}\sin\frac{k\pi}{l}\xi\Big|_0^l=0$.

故 $u(t,x)=u_0$.