一. 知 $u(t,x,y)=\frac{1}{2\pi c}\iint\limits_{r<ct}\frac{\psi(\eta_1,\eta_2)}{\sqrt{c^2t^2-r^2}}d\eta_1 d\eta_2+\frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{r<ct}\frac{\varphi(\eta_1,\eta_2)}{\sqrt{c^2t^2-r^2}}d\eta_1 d\eta_2\right]$.

不妨设 $\varphi.\psi>0$.

由于 $\varphi.\psi$ 有紧支集. 即 $\exists\ M\subset R^2$. s.t. $\varphi(R^2\setminus M)=\{0\}$.

而 M 有界闭. 故 $\varphi.\psi$ 在 M 有最大值. 记为 $\varphi_m.\ \psi_m$.

从而有: $\exists t_0$ 充分大. s.t.

$$u(t,x,y)\le\frac{1}{2\pi c}\iint\limits_{r<ct_0}\frac{\psi_m}{\sqrt{c^2t^2-r^2}}d\eta_1 d\eta_2+\frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{r<ct_0}\frac{\varphi_m}{\sqrt{c^2t^2-r^2}}d\eta_1 d\eta_2\right]$$

$$=\frac{\psi_m}{2\pi c}\left(-\sqrt{c^2t^2-r^2}\Big|_{r=0}^{r=ct_0}\right)+\frac{\partial}{\partial t}\left[\frac{\varphi_m}{2\pi c}\left(-\sqrt{c^2t^2-r^2}\Big|_{r=0}^{r=ct_0}\right)\right]$$

$$=\frac{\psi_m}{2\pi c}\cdot\frac{c^2t_0^2}{c^2t^2+\sqrt{c^2t^2-c^2t_0^2}}+\frac{\varphi_m}{2\pi c}\left[c-\frac{c^2t}{\sqrt{c^2t^2-c^2t_0^2}}\right].$$

故 $\lim\limits_{t\to\infty}u(t,x,y)\le\lim\limits_{t\to\infty}\left(\frac{\psi_m}{2\pi c}\cdot 0+\frac{\varphi_m}{2\pi c}[c-c]\right)=0$.

而 $u\ge 0$. 故. $u=0$. $(t\to\infty)$.

即: $\lim\limits_{t\to\infty}u(t,x,y)=0$. 故 $\forall(x_0,y_0)\in R^2$

$$\lim_{t\to\infty}u(t,x_0,y_0)=0.$$

二. $E(t)=\int_0^l(u_t^2+c^2u_x^2)dx$. 弦长 l.

$$\frac{dE(t)}{dt}=\int_0^l(2u_tu_{tt}+2c^2u_xu_{xt})dx=\int_0^l 2u_tu_{tt}+2c^2u_xu_t\Big|_0^l$$

$$-\int_0^l 2c^2u_{xx}u_t\,dx=2c^2u_xu_t\Big|_0^l+\int_0^l 2u_t(u_{tt}-c^2u_{xx})dx$$

$$=2c^2u_xu_t\Big|_0^l+\int_0^l -2\gamma u_t^2\,dx.$$

而 $u(t,0)=u(t,l)=u_t(t,0)=u_t(t,l)=0$.

故 $\frac{d(E(t))}{dt}=-\int_0^l 2\gamma u_t^2\,dx<0$. 故能量是衰减的!

设 u_1, u_2 为 $\begin{cases} u_{tt} - c^2 u_{xx} = -\gamma u_t + f \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \\ u(t,0) = \mu_1(t),\ u(t,l) = \mu_2(t) \end{cases}$ 解.

则 $u = u_1 - u_2$ 为 $\begin{cases} u_{tt} - c^2 u_{xx} = -\gamma u_t \\ u(0,x) = 0 \quad u_t(0,x) = 0 \\ u(t,0) = u(t,l) = 0 \end{cases}$ 的解.

则其能量 $E(t)$ 衰减. 又 $t=0$ 时. $E(0) = \int_0^l (u_t^2 + a^2 u_x^2)_{t=0}\, dx = 0$.

故 $E(t) \equiv 0$. 从而 $u_1 = u_2$. 故解是唯一的.