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1. $\begin{cases} u_t = c^2 u_{xx} + ru \\ u|_\Gamma = g(t,x). \\ u(0,x) = \varphi(x). \end{cases}$ 即证 $\begin{cases} u_t = c^2 u_{xx} + ru \\ u(t,0) = \mu_1(t) \quad u(t,l) = \mu_2(t) \\ u(0,x) = \varphi(x). \end{cases}$

设 u_1, u_2 为符合上述条件的不同解. 则 u_1-u_2 满足齐次边界条件和初始条件.

由极值定理 $(u_1-u_2)_{max} \le 0 = ((u_1-u_2)|_{\Gamma_T})_{max}$. 故 $u_1=u_2$. 这样唯一性得证.

同理. 若 $|u_1-u_2||_{\Gamma_t} \le \varepsilon$. 则 $|u_1-u_2||_{\Gamma_T} \le \varepsilon$. 故解是稳定的.

2. $u_{xx}+u_{yy}=0$. 不妨取有界闭区域为矩形 $R=[a,b]\times[c,d]$. 若不然 对区域作划分即可.

记 M 为 u在 R_T 内最大值. m 为 ∂R 上最大值.

若 $M>m$. 则 $\exists (x_0,y_0) \in R^0$, s.t. $u(x_0,y_0)=M$.

令 $v(x,y) = u(x,y) + \frac{M-m}{4l^2}(x-x_0)^2 \quad l=d-c$

$v|_{\partial R} < m + \frac{M-m}{4} = \frac{1}{4}M + \frac{3}{4}m < M$. 而 $v(x_0,y_0) = M > v|_{\partial R}$

故 v 也不在 ∂R上取最大值. 故 $\exists P(x',y') \in R^0$. s.t. v在P处取最大值.

则在P处有 $\frac{\partial^2 v}{\partial x^2} \le 0$, $\frac{\partial^2 v}{\partial y^2} \le 0$. i.e. $u_{xx} + \frac{1}{2l^2}(M-m) = 0$. $u_{yy}=0$.

但由于 $u_{xx}+u_{yy}=0$. 故 $u_{xx}=u_{yy}=0$.

这就导致了 $M=m$. 与假设矛盾. 故u至少在∂R上取到最大值

故u在有界闭区域上的最大值不超过其在边界上最大值

3. $\begin{cases} u_t - c^2 u_{xx} = f(t,x) \\ u(t,0) = \mu_1(t) \quad (u_x + \sigma u)|_{x=l} = \mu_2(t) \\ u(0,x) = \varphi(x) \end{cases}$

令 $v(t,x) = e^{-\lambda t} u$. 则有

$\begin{cases} v_t - c^2 v_{xx} + \lambda v = e^{-\lambda t} f(t,x) \\ v(t,0) = e^{-\lambda t}\mu_1(t) \quad (v_x + \sigma v)|_{x=l} = e^{-\lambda t}\mu_2(t) \\ v(0,x) = \varphi(x). \end{cases}$

若v在D_t内取最大值. 则有 $v_t \ge 0$, $v_{xx} \le 0$, 且 $v>0$. 不可能.

则有: $v \le \max\limits_{0\le x\le l}\{\varphi(x)\}$ 或 $v \le \max\limits_{0\le t\le t_1}\{e^{-\lambda t}\mu_1(t)\}$.

或 $v \le \max\limits_{0\le t\le t_1}\left\{\frac{e^{-\lambda t}\mu_2(t)}{\sigma}\right\}$ 或. $v \le \frac{1}{\lambda}\max\{e^{-\lambda t} f\}$

故 $u \le e^{\lambda t_1} \max\{0, \max\{\varphi(x)\}, \max\{e^{-\lambda t}\mu_1(t)\}, \frac{e^{-\lambda t}\mu_2(t)}{\sigma}\}, \frac{1}{\lambda}\max\{e^{-\lambda t} f\}\}$.

一. 证明：$u(t,x) \geq e^{\lambda t_1} \min\{0, \min\limits_{0\leq x\leq l}\{\varphi(x)\}, \max\limits_{0\leq t\leq t_1}\{e^{-\lambda t_1} \mu_1(t), \frac{e^{-\lambda t}\mu_2(t)}{\sigma}\}\}$

$$\begin{cases} u_t - c^2 u_{xx} = 0 \\ u(t,0) = \mu_1(t) \\ (u_x + \sigma u)|_{x=l} = \mu_2 t \\ u(0,x) = \varphi(x). \end{cases}$$

令 $v(t,x) = e^{-\lambda t} u$. 则有 $\begin{cases} v_t - c^2 v_{xx} + \lambda v = 0 \\ v(t,0) = e^{-\lambda t}\mu_1(t) \\ (v_x + \sigma v)|_{x=l} = e^{-\lambda t}\mu_2(t) \\ v(0,x) = \varphi(x). \end{cases}$

若 v 在 $0<x<l$，$0<t_0\leq t_1$ 处 达到负最小. 则有

$v_t \leq 0$, $v_{xx} \geq 0$ $v<0$. 矛盾. 故 v 的负最小值只能在边界上取.

① 若在 $t=0$ 时取得. 有：$v \geq \min\limits_{0\leq x\leq l}\{\varphi(x)\}$.

② 若在 $x=0$ 时，有 $v \geq \min\limits_{0\leq t\leq t_1}\{e^{-\lambda t}\mu_1(t)\}$

③ 若在 $x=l$ 时. 由于 $v_x \leq 0$，有 $v \geq \frac{e^{-\lambda t_0}}{\sigma}\mu_2(t_0) \geq \min\limits_{0<t\leq t_0}\{\frac{e^{-\lambda t}\mu_2(t)}{\sigma}\}$

考虑到最小值可能为正，则有 $v_{min} \geq 0$.

从而有 $v \geq \min\{0, \min\{\varphi(x)\}, \min\limits_{0\leq t\leq t_1}\{e^{-\lambda t}\mu_1(t), \frac{e^{-\lambda t}\mu_2(t)}{\sigma}\}\}$

故有 $u(t,x) \geq e^{\lambda t_1}\min\{0, \min\limits_{0\leq x\leq l}\{\varphi(x)\}, \min\limits_{0\leq t\leq t_1}\{e^{-\lambda t}\mu_1(t), \frac{e^{-\lambda t}\mu_2(t)}{\sigma}\}\}$