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Q14:

令 $\zeta_n = \xi_n\eta_n$. 则 $E\zeta_n = E\xi_n E\eta_n = 0$. $Var\zeta_n = E(\xi_n\eta_n - E\zeta_n)^2 = E(\xi_n\eta_n)^2 = E\xi_n^2 = Var\xi_n = 1$

故 $\{\zeta_n\}$ 是一列独立同分布的随机变量且 $E\zeta_n = 0$, $Var\zeta_n = 1$.

故由中心极限定理 $\frac{\sum_{i=1}^{n}\zeta_n - nE\zeta_n}{\sqrt{nVar\xi_n}} \xrightarrow{d} N(0,1)$. i.e. $S_n \xrightarrow{d} N(0,1)$.

故 S_n 的分布函数收敛于 $\Phi(x)$.

Q19:

(1): $E\xi_k = 0$, $Var\xi_k = \ln k$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{Var\xi_k}{n^2} = \lim_{n\to\infty}\frac{\ln n!}{n^2} \overset{Stirling}{=\!=\!=} \lim_{n\to\infty}\frac{\ln\left(\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\right)}{n^2} = \lim_{n\to\infty}\left[\frac{\ln n}{n} - \frac{\ln e}{n} + \frac{\ln 2\pi n}{2n^2}\right] = 0.$$

故由切比雪夫大数定律. $\{\xi_k\}$ 服从大数定律.

(2): $E\xi_k = 0$. $Var\xi_k = \frac{1}{2^{2k+1}}\cdot(2^k)^2\times 2 = 1$

故 $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{Var\xi_k}{n^2} = \lim_{n\to\infty}\frac{1}{n} = 0$. 由切比雪夫大数定律知 $\{\xi_k\}$ 服从大数定律.

(3) $E|\xi_k| = \sum_{i=1}^{\infty}\frac{1}{2^i}\cdot\frac{2^i}{i^2} = \sum_{i=1}^{\infty}\frac{1}{i^2} = \frac{\pi^2}{6} < \infty$.

由辛钦大数定律, $\{\xi_k\}$ 服从大数定律.

(4): $E|\xi_k| = E\xi_k = \sum_{i=2}^{\infty}\frac{c}{i^2\ln^2 i}\cdot i = \sum_{i=2}^{\infty}\frac{c}{i\ln^2 i}$

令 $f(x) = \frac{1}{x\ln^2 x}$. 考虑 $f(x)$ 在区间 $[2,+\infty)$ 上的积分

$$I = \int_2^{+\infty} f(x)dx = \int_2^{+\infty}\frac{1}{x\ln^2 x}dx = \int_2^{+\infty}\frac{d\ln x}{\ln^2 x} = \int_{\ln 2}^{+\infty}\frac{1}{t^2}dt = -\frac{1}{t}\Big|_{\ln 2}^{+\infty} = \ln^{-1}2 < \infty.$$

故级数 $\sum_{i=2}^{\infty}\frac{c}{i\ln^2 i} < \infty$.

由辛钦大数定律. $\{\xi_k\}$ 服从大数定律.

Q23:

$E\ln\xi_n = E\frac{1}{n}\ln\prod_{k=1}^{n}\xi_k = \int_0^1 \ln x\,dx = x(\ln x - 1)\Big|_0^1 = -1$.

故 $\frac{1}{n}\sum_{k=1}^{n}\ln\xi_k \xrightarrow{P} -1$. 从而 $\left(\prod_{k=1}^{n}\xi_k\right)^{\frac{1}{n}} \xrightarrow{P} \frac{1}{e}$.

即 $\eta_n \xrightarrow{P} \frac{1}{e}$.

Q25:

Q28:

$E\xi_k = 0$, $\mathrm{Var}\,\xi_k = 1$. $\dfrac{\sum_{k=1}^{n} \xi_k^2}{n} \xrightarrow{P} 1$ $\xi_{n+1} \xrightarrow{d} N(0,1)$.

故 $\eta_n = \dfrac{n}{\sum_{k=1}^{n} \xi_k^2}\xi_{n+1} \xrightarrow{d} N(0,1)$

Q29:

$\xi_n \sim N(0,1)$. 有 $\dfrac{\xi_1^2+\xi_2^2+\cdots+\xi_n^2}{n} \xrightarrow{P} 1$ 因为 $E\xi_i^2 = \mathrm{Var}\,\xi_i = 1$

$\xi_1/\sqrt{n} \xrightarrow{d} N(0,\frac{1}{n})$ 故 $\dfrac{\xi_1+\cdots+\xi_n}{\sqrt{n}} \xrightarrow{d} N(0,1)$.

故 $\eta_n \to N(0,1)$.