18.

(1) $P(\xi=k)=\frac{n-1}{n}\frac{n-2}{n-1}\cdots\frac{n-k}{n-k+1}\frac{1}{n-k}=\frac{1}{n}$. 故 $E\xi=\sum_{k=1}^{n}\frac{k}{n}=\frac{1+n}{2}$

$Var\xi=E\xi^2-(E\xi)^2=\sum_{k=1}^{n}\frac{1}{n}\cdot k^2-\left(\frac{1+n}{2}\right)^2=\frac{n^2-1}{12}$

(2) ξ 服从成功概率为 $p=\frac{1}{n}$ 的几何分布. 故 $Var\xi=\frac{1-p}{p^2}=n(n-1)$

22.

记 ξ 为取一次的号码.

$E\xi=\sum_{k=1}^{n}k\cdot\frac{1}{n}=\frac{1+n}{2}$

$Var\sum\xi=\sum Var\xi=k\cdot Var\xi=k\cdot(E\xi^2-(E\xi)^2)=k\cdot\left(\frac{(n+1)(2n+1)}{6}-\left(\frac{1+n}{2}\right)^2\right)=k\cdot\frac{n^2-1}{12}$

故方差为 $\frac{n^2-1}{12}k$.

24. $\xi\in[a,b]$.

$Var\xi=E(\xi-E\xi)^2$ 由方差性质. $E(\xi-E\xi)^2\le E(\xi-\frac{a+b}{2})^2\le\frac{(b-a)^2}{4}$

故 $Var\xi\le\frac{1}{4}(b-a)^2$

25.

$Var\sum_{i=1}^{n}a_i\xi_i=\sum_{i=1}^{n}Var\,a_i\xi_i=\sum_{i=1}^{n}a_i^2\,Var\xi_i=\sum_{i=1}^{n}a_i^2\sigma_i^2\geqslant n\sqrt[n]{\prod_{i=1}^{n}a_i^2\sigma_i^2}$

当且仅当 $\begin{cases}a_1^2\sigma_1^2=a_2^2\sigma_2^2=\cdots=a_n^2\sigma_n^2\\ \sum a_i=1\end{cases}$ 即 $a_i=\frac{1}{\sigma_i}\left(\sum\frac{1}{\sigma_i}\right)^{-1}$ 时取 "="

此时 $Var\sum a_i\xi_i\geqslant n\left(\sum_{i=1}^{n}\frac{1}{\sigma_i}\right)^{-2}$

故当 $a_i=\frac{1}{\sigma_i}\left(\sum\frac{1}{\sigma_i}\right)^{-1}$ 时方差最小.

29. (1) $p(x,y)=\begin{cases}2-x-y & 0<x<1,\ 0<y<1\\ 0 & \text{其它}\end{cases}$ $p_X(x)=\begin{cases}\frac{3}{2}-x & 0<x<1\\ 0 & \text{其它}\end{cases}$ $p_Y(y)=\begin{cases}\frac{3}{2}-y & 0<y<1\\ 0 & \text{其它}\end{cases}$

$E\xi=\int_R x\,p_X(x)dx=\left(\frac{3}{4}x^2-\frac{1}{3}x^3\right)\Big|_0^1=\frac{5}{12}$ 同理 $E\eta=\frac{5}{12}$

$Cov(\xi,\eta)=E\xi\eta-E\xi E\eta=\int_0^1\int_0^1 xy(2-x-y)dxdy-\frac{25}{144}=\frac{1}{6}-\frac{25}{144}=-\frac{1}{144}$.

$Cov(\xi,\xi)=E\xi^2-(E\xi)^2=\frac{1}{4}-\frac{25}{144}=\frac{11}{144}$

故协方差矩阵 $B=\begin{pmatrix}\frac{11}{144} & \frac{-1}{144}\\ \frac{-1}{144} & \frac{11}{144}\end{pmatrix}$.

(2) $p(x,y)=\begin{cases}6xy^2 & 0<x<1 & 0<y<1\end{cases}$ $p_X(x)=\begin{cases}2x & 0<x<1\end{cases}$ $p_Y(y)=\begin{cases}3y^2 & 0<y<1\end{cases}$

$$p(x,y)=\begin{cases} \cdots \\ 0 & \text{其它.} \end{cases} \quad p_X(x)=\begin{cases} \cdots \\ 0 \end{cases} \quad p_Y(y)=\begin{cases} \cdots \\ 0 & \text{其它.} \end{cases}$$

$p_X(x)p_Y(y)=p(x,y)$. 故 ξ,η 相互独立. 从而 $Cov(\xi,\eta)=0$.

$Cov(\xi,\xi)=E\xi^2-(E\xi)^2=\int_0^1 2x^3\,dx-\left(\int_0^1 2x^2\,dx\right)^2=\frac{1}{2}-\frac{4}{9}=\frac{1}{18}$

$Cov(\eta,\eta)=E\eta^2-(E\eta)^2=\int_0^1 3y^4\,dy-\left(\int_0^1 3y^3\,dy\right)^2=\frac{3}{5}-\frac{9}{16}=\frac{3}{80}$

故协方差矩阵 $B=\begin{pmatrix} \frac{1}{18} & \\ & \frac{3}{80} \end{pmatrix}$

31. 令 $\alpha=p\xi+q\eta$. $\beta=u\xi+v\eta$. $\xi,\eta\sim N(a,\sigma^2)$

$r_{\alpha\beta}=\dfrac{E(\alpha-E\alpha)(\beta-E\beta)}{\sqrt{Var\alpha}\sqrt{Var\beta}}$ $\qquad E\xi=E\eta=a$.

$Cov(\alpha,\beta)=Cov(p\xi+q\eta,u\xi+v\eta)=pu\,Cov(\xi,\xi)+qv\,Cov(\eta,\eta)+(pu+qv)Cov(\xi,\eta)$

$\quad =pu\,D\xi+qv\,D\eta=(pu+qv)\sigma^2$

$Var\,\alpha=(p^2+q^2)\sigma^2 \quad Var\beta=(u^2+v^2)\sigma^2$.

故 $r_{\alpha\beta}=\dfrac{Cov(\alpha,\beta)}{\sqrt{Var\alpha\,Var\beta}}=\dfrac{pu+qv}{\sqrt{(p^2+q^2)(u^2+v^2)}}$

33. $Var\xi_i=1$. $Cov(\xi_i,\xi_j)=\rho\sqrt{Var\xi_i\,Var\xi_j}=\rho$

$Cov(\eta,\zeta)=Cov(\xi_1+\xi_2+\cdots+\xi_n,\xi_{n+1}+\cdots+\xi_{2n})=\sum_{i=1}^{n}\sum_{j=1}^{n}Cov(\xi_i,\xi_{n+j})=\rho n^2$

$Var\eta=\sum_{i=1}^{n}Var\xi_i+2\sum_{1\le i<j\le n}E(\xi_i-E\xi_i)(\xi_j-E\xi_j)=n+2\times\frac{n-1+1}{2}(n-1)\rho=n+n(n-1)\rho$

同理 $Var\zeta=n+n(n-1)\rho$.

故 $r_{\zeta\eta}=\dfrac{\rho n^2}{n+n(n-1)\rho}=\dfrac{\rho n}{1+(n-1)\rho}$

36.

ξ,η 不相关 $\Leftrightarrow r_{\xi\eta}=0\Leftrightarrow Cov(\xi,\eta)=0\Leftrightarrow E(\xi-E(\xi))(\eta-E(\eta))=0\Leftrightarrow E\xi\eta-E\xi E\eta=0$.

设 ξ,η 分布列 $\begin{bmatrix} a & b \\ p_1 & q_1 \end{bmatrix}$ $\begin{bmatrix} c & d \\ p_2 & q_2 \end{bmatrix}$

令 $\xi'=\xi-b$. $\eta'=\eta-d$. 则 $E\xi'\eta'=p_1p_2(a-b)(c-d)=P(\xi=a,\eta=c)(a-b)(c-d)$.

$\quad E\xi'E\eta'=(a-b)P(\xi=a)\cdot(c-d)P(\eta=c)$. 由 $E\xi'\eta'=E\xi'E\eta'$

知 $P(\xi=a,\eta=c)=P(\xi=a)P(\eta=c)$. 同理可证其它式.

故 ξ 与 η 相互独立.

38. 令 $\xi_k'=\dfrac{1}{\sqrt{Var\xi_k}}\xi_k$. 则 $Var\xi_k'=1$. $r_{\xi_k\xi_l}=\dfrac{Cov(\xi_k,\xi_l)}{\sqrt{Var\xi_k\,Var\xi_l}}=\rho$.

故 $\mathrm{Var}\sum \xi'_k = \sum \mathrm{Var}\,\xi_k + 2\sum_{1\le i<j\le n} \mathrm{Cov}(\xi'_i, \xi'_j) = n + n(n-1)\rho \geqslant 0$

$\Rightarrow \rho \geqslant -1/(n-1)$.