

COSC 290 Discrete Structures

Lecture 8: Predicate Logic

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Plan for today

1. Predicate Logic
2. Nesting Quantifiers
3. Theorems

1

Predicate Logic

Predicate

An atomic proposition p is a Boolean variable.

A **predicate** $P(x)$ is a Boolean function. A predicate can take one or more arguments.

Examples:

- $isPrime(x)$ returns true if x is a prime number and false otherwise.
- $isDivisibleBy(x, y)$ returns true if x is evenly divisible by y .

2

Propositions that include predicates

We can form a proposition by supplying arguments to each predicate.

$$\varphi := \text{isPrime}(8) \vee \text{isDivisibleBy}(8, 2)$$

The truth of this proposition requires *interpreting* the predicates: *isPrime*(8) is false whereas *isDivisibleBy*(8, 2) is true according to definitions of these predicates.

3

Universal Quantification

Let $P := \{p_1, p_2, \dots\}$ be the (infinite) set of all persons.

$$\forall p \in P : \text{At}(p, \text{Colgate}) \implies \text{BrushesTeeth}(p)$$

means “Every person at Colgate brushes their teeth.”

The above is *roughly* equivalent to

$$(\text{At}(p_1, \text{Colgate}) \implies \text{BrushesTeeth}(p_1))$$

$$\wedge (\text{At}(p_2, \text{Colgate}) \implies \text{BrushesTeeth}(p_2))$$

$$\wedge (\text{At}(p_3, \text{Colgate}) \implies \text{BrushesTeeth}(p_3))$$

$$\wedge \dots$$

4

Common mistake with universal quantification

Typically, \implies is the main connective with \forall .

Common mistake: using \wedge as the main connective with \forall :

$$\forall p \in P : \text{At}(p, \text{Colgate}) \wedge \text{BrushesTeeth}(p)$$

means “Every person is at Colgate and everyone brushes their teeth.”

5

Existential Quantification

Let $P := \{p_1, p_2, \dots\}$ be the (infinite) set of all persons.

$$\exists p \in P : \text{At}(p, \text{Bucknell}) \wedge \text{BrushesTeeth}(p)$$

means “Some person at Bucknell brushes their teeth.”

The above is *roughly* equivalent to

$$(\text{At}(p_1, \text{Bucknell}) \wedge \text{BrushesTeeth}(p_1))$$

$$\vee (\text{At}(p_2, \text{Bucknell}) \wedge \text{BrushesTeeth}(p_2))$$

$$\vee (\text{At}(p_3, \text{Bucknell}) \wedge \text{BrushesTeeth}(p_3))$$

$$\vee \dots$$

6

Common mistake with existential quantification

Typically, \wedge is the main connective with \exists .

Common mistake: using \implies as the main connective with \exists :

$$\exists p \in P : \text{At}(p, \text{Bucknell}) \implies \text{BrushesTeeth}(p)$$

is true provided that there is some person who is *not* at Bucknell!

Constructing Predicates

In programming, we can define functions that call other functions.

In predicate logic, we can define predicate in terms of other predicates.

Examples:

- $\text{follows}(x, y)$ means that x follows the tweets of y
- $\text{TrumpFollower}(x) := \text{follows}(x, \text{@realDonaldTrump})$
- $\text{popularTweeter}(y) := \forall x \in P : \text{follows}(x, y)$

Poll: fastest person

Let $\text{faster}(x, y)$ be true if x runs faster than y and false otherwise.

Which of the following is the correct definition for $\text{fastest}(x)$?

- A) $\exists y \in P : \text{faster}(x, y)$
- B) $\neg (\exists y \in P : \text{faster}(y, x))$
- C) $\forall y \in P : \text{faster}(x, y)$
- D) $\neg (\forall y \in P : \text{faster}(y, x))$
- E) None of the above / More than one of the above

Poll: fastest lacrosse player

Let $\text{faster}(x, y)$ be true if x runs faster than y and false otherwise.

Let $\text{lax}(x)$ be true if x plays lacrosse.

Which of the following is the correct definition for $\text{fastestLacrossePlayer}(x)$?

- A) $\forall y \in P : \text{lax}(y) \wedge \text{faster}(x, y)$
- B) $\forall y \in P : \text{lax}(y) \implies \text{faster}(x, y)$
- C) $\text{lax}(x) \wedge \forall y \in P : \text{lax}(y) \wedge \text{faster}(x, y)$
- D) $\text{lax}(x) \wedge \forall y \in P : \text{lax}(y) \implies \text{faster}(x, y)$
- E) None of the above / More than one of the above

Nesting Quantifiers

Unpacking a complex predicate can reveal nested quantifiers

Consider universes of professors P , students S , and courses C .

A course every student takes:

$$\text{favCourse}(c) := \forall s \in S : \text{takes}(s, c)$$

Professor who teaches a course that every student takes:

$$\begin{aligned}\text{profOfFav}(p) &:= \exists c \in C : \text{favCourse}(c) \wedge \text{teaches}(p, c) \\ &\equiv \exists c \in C : (\forall s \in S : \text{takes}(s, c)) \wedge \text{teaches}(p, c) \\ &\equiv \exists c \in C : \forall s \in S : (\text{takes}(s, c) \wedge \text{teaches}(p, c))\end{aligned}$$

11

Order of quantification

The order the of the quantifiers matters.

$$Q_1(p) := \exists c \in C : \forall s \in S : (\text{takes}(s, c) \wedge \text{teaches}(p, c))$$

vs.

$$Q_2(p) := \forall s \in S : \exists c \in C : (\text{takes}(s, c) \wedge \text{teaches}(p, c))$$

$Q_1(p)$: "Prof. who teaches a course every student takes"

vs.

$Q_2(p)$: "Prof. who teaches every student"
(but not necessarily in the same course).

Poll: nested quantifiers, part 1

Predicate $\text{likes}(p_1, p_2)$ means " p_1 likes p_2 ," shown by an arrow from p_1 to p_2 .

Which figure represents the following proposition:

$$\forall p_1 \in P : \exists p_2 \in P : \text{likes}(p_1, p_2)$$



(a)



(b)



(c)

More than one/
None of the
above

(d)

12

13

Poll: nested quantifiers, part 2

Predicate $\text{likes}(p_1, p_2)$ means " p_1 likes p_2 ," shown by an arrow from p_1 to p_2 .

Which figure represents the following proposition:

$$\exists p_2 \in P : \forall p_1 \in P : \text{likes}(p_1, p_2)$$



(a)



(b)



(c)

More than one/
None of the
above

(d)

14

Theorems

Theorem

A fully quantified expression of predicate logic is a **theorem** if and only if it is true for every possible meaning of its predicates.

Analogous to a tautology from propositional logic.

15

Two important theorems

$$\neg [\forall x P(x)] \iff [\exists x \neg P(x)]$$

$$\neg [\exists x P(x)] \iff [\forall x \neg P(x)]$$

See book for others!

16

Proving a theorem

Given a universally quantified statement ($\forall x : \dots$)

- To *prove* it: use induction, direct proof, by cases, proof by contrapositive, etc. (More on this soon! Ch. 4 & 5)
- To *disprove* it: find a single counter example.

Given an existentially quantified statement ($\exists x : \dots$)

- To *prove* it: find a single example.
- To *disprove* it: prove its *negation* is true. If claim is $\exists x : P(x)$, then negation $\forall x : \neg P(x)$. Use techniques listed above for universally quantified statements.

17

Poll: disproving a proposition

Given the proposition

$$\forall p_1 \in P : \exists p_2 \in P : \text{likes}(p_1, p_2)$$

How could you disprove it?

- A) By counterexample: show there is a person who loves everyone
- B) By counterexample: show there is a person who loves no one
- C) By counterexample: show there is a person who nobody loves
- D) By counterexample: show there is a person who everyone loves
- E) Other/more/none

18

Poll: disproving a proposition

Given the proposition

$$\exists p_2 \in P : \forall p_1 \in P : \text{likes}(p_1, p_2)$$

What is its negation?

- A) $\forall p_2 \in P : \forall p_1 \in P : \neg \text{likes}(p_1, p_2)$
- B) $\forall p_2 \in P : \exists p_1 \in P : \neg \text{likes}(p_1, p_2)$
- C) $\exists p_2 \in P : \forall p_1 \in P : \neg \text{likes}(p_1, p_2)$
- D) $\forall p_2 \in P : \exists p_1 \in P : \neg \text{likes}(p_2, p_1)$
- E) Other/more/none

[[MH: check these]]

19