COSC 290 Discrete Structures

Lecture 5: Propositional Logic

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Plan for today

1. Propositional logic: syntax and semantics
2. Evaluating propositions

Propositional logic: syntax and semantics

Poll: what is a proposition?

Propositional logic is based around the concept of a proposition.

Why isn't

"Where does COSC 290 lab meet?"

considered a proposition?

A) Because only "yes/no" questions can be propositions.
B) Because questions can never be propositions.
C) Because the answer changes over time (last semester it was held in a different room).
D) More than one of the above / None of the above.

Proposition

A proposition is a sentence that is either true or false.

- $3 + 4 = 6$
- My middle name is Herbert or my dog's name is Rufus.
- One of these three propositions is true.

These are *not* propositions:

- Questions: is $3 + 4 = 7$?
- Imperatives: You should major in computer science.
- Opinions: CS majors have more fun.

Syntax of propositional logic

Recall: syntax defines what sentences are permissible in the language.

A sentence can consist of a single atomic proposition. Example: "The chair is red." Such propositions are represented using variables p, q, r, etc.

More complex sentences can be constructed from simpler sentences using logical connectives.

- If S is a sentence, $\neg S$ is a sentence (negation)
- If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)
- If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)
- If S_1 and S_2 are sentences, $S_1 \implies S_2$ is a sentence (implication)
- If S_1 and S_2 are sentences, $S_1 \iff S_2$ is a sentence (iff)

Semantics of propositional logic

Recall: the semantics defines the rules for determining the *truth* of a sentence.

A simple sentence consisting of a single atomic proposition p is either true or false.

For more complex sentences, the truth can be evaluated using these rules:

$\neg S$	is true iff	S	is false		
$S_1 \wedge S_2$	is true iff	S_1	is true *and*	S_2	is true
$S_1 \vee S_2$	is true iff	S_1	is true *or*	S_2	is true
$S_1 \implies S_2$	is true iff	S_1	is false *or*	S_2	is true
$S_1 \iff S_2$	is true iff	$S_1 \implies S_2$	is true *and*	$S_2 \implies S_1$	is true

In propositional logic, a model is simply an assignment of truth values to the atomic variables.

Truth tables

A truth table can be used to compactly represent semantics of a connective.

p	q	$p \wedge q$
T	T	?
T	F	?
F	T	?
F	F	?

Ex: Consider proposition: "x is prime and even." For what x is it true?

Poll: exclusive vs. inclusive or

There two "or" operators, the "inclusive or" and the "exclusive or."

Inclusive or

p	q	$p \vee q$
T	T	?
T	F	?
F	T	?
F	F	?

Exclusive or

p	q	$p \oplus q$
T	T	?
T	F	?
F	T	?
F	F	?

In which rows do their truth tables differ?

A) The T T row
B) The T F row
C) The F T row
D) The F F row
E) None of the above / More than one of the above

Exclusive or vs. inclusive or

Inclusive or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Ex: "Bob likes chicken or fish." (He might like both.)

Exclusive or

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Ex: "Alice is either in her office or exercising at the gym." (She can't be in both places at once.)

Implication

The proposition $p \implies q$ is true when the truth of p implies the truth of q. In other words, $p \implies q$ is true unless p is true and q is false.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

You are drinking alcohol legally $\implies$ you are at least 21.

In a natural language like English, this *same* implication can be expressed in *many* different forms.

- If you are drinking legally, then you are at least 21.
- You are drinking legally only if you are at least 21.
- You are at least 21 if you are drinking legally.
- Being at least 21 is necessary for you to be drinking legally.
- Knowing that you are drinking legally is sufficient information to conclude you are at least 21.

Problem set 4

DLN 3.9. Given,

- $p :=$ "`x + y` is valid Python"
- $q :=$ "`x ** y` is valid Python" see footnote
- $w :=$ "x is a list"

write the following sentence in propositional logic:

"`x + y` and `x ** y` are both valid Python only if x is not a list."

$$(p \wedge q) \implies \neg w$$

I slightly modified the definition of q to make the statement true in python.

Evaluating propositions

Evaluating propositions

Question: How do we know whether a given proposition is true?

- If the answer is, "Well, we have to read the proposition and decide if it seems true on a case-by-case basis," then logic FAIL.
- Truth should be determined based on the *form* of the proposition, the semantics of logical operators, and the truth of each input variable.

Poll: evaluating propositions

Let p, q, r be the following *atomic* propositions.

- $p =$ "In COSC 290, Alice earned an A on each lab, the take-home midterm, and the final." (assume this is true)
- $q =$ "Alice's final grade for COSC 290 was an A." (assume this is true)
- $r =$ "7 is prime." (this is true)

Which of the following *compound* propositions are true?

A) $p \implies q$
B) $q \implies p$
C) $r \implies q$
D) All of the above
E) None of the above

Implication and causality

In logic, we are looking at the *form* of the arguments.

To know if $p \implies q$, it is not necessary for p to *cause* q.

To determine truth of $p \implies q$, we need to know the truth values of p and q and then consult the truth table.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: "7 is prime implies Alice's final grade for COSC 290 was an A" is true because both statements are given as true.

Counter-intuitive nature of implication

A second counter-intuitive aspect is that $p \implies q$ is true whenever p is false.

Example:

- Let $\psi =$ "If Bob earns an A on each lab, the take-home midterm, and the final, then Alice will earn an A."
- Suppose Bob earns a C on his labs and exams.
- Then ψ is true.

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

Poll: evaluating implications

Let ψ be "If Prof. Gember-Jacobson bikes to work today, then he wears his helmet."

Which of the following facts will allow us to conclude that ψ is false?

A) Prof. Gember-Jacobson is at work but his bike helmet is visible on his front porch.

B) Prof. Gember-Jacobson is seen wearing his helmet today but not biking.

C) Prof. Gember-Jacobson is seen biking today but not wearing his helmet.

D) More than one of the above / None of the above.

Correct answer is C.

Poll: evaluating implications, part 2

Let ψ be "If Prof. Gember-Jacobson bikes to work today, then he wears his helmet."

Which of the following facts will allow us to conclude that ψ is true?

A) Prof. Gember-Jacobson is seen today wearing a helmet.

B) Prof. Gember-Jacobson did not bike to work today.

C) Prof. Gember-Jacobson is seen biking to work today, and he is wearing a helmet.

D) More than one of the above / None of the above.

Correct answer is D because both B and C allow us to conclude that ψ is true.

Truthiness of a sentence

Consider the sentence $p \wedge (p \implies q) \implies q$.

How do we evaluate whether this sentence is true?

1. Represent expression as a tree (relying on operator precedence to correctly parse sentence).
2. Assign true/false values to leaves of the tree (atomic propositions p and q)
3. Propagate true/false values up tree using properties of logical operators.

Shown on board

(Aside: can this expression be simplified?)