COSC 290 Discrete Structures

Lecture 6: Logical Equivalence

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Plan for today

1. Expressing knowledge in propositional logic
2. Logical Equivalence
3. Equivalence of logical operators
4. Establishing logical equivalence

Expressing knowledge in propositional logic

Poll: Wumpus world sentences

Let atomic propositions be denoted as follows:

- p_{ij} denotes "there is a **p**it in [i, j]."
- w_{ij} denote "there is a **w**umpus in [i, j]."
- ...

Write this sentence in propositional logic: "There may be a pit in [1,2] or [2,2] and the wumpus is either in [2,1] or [3,1]."

A) $(p_{1,2} \oplus p_{2,2}) \wedge (w_{2,1} \oplus w_{3,1})$
B) $(p_{1,2} \vee p_{2,2}) \wedge (w_{2,1} \oplus w_{3,1})$
C) $(p_{1,2} \vee p_{2,2}) \wedge (w_{2,1} \vee w_{3,1})$
D) $(p_{1,2} \wedge p_{2,2}) \vee (w_{2,1} \wedge w_{3,1})$
E) More than one of the above / None of the above

Poll: Wumpus world sentences

Let atomic propositions be denoted as follows:

- p_{ij} denotes "there is a **p**it in [i, j]."
- b_{ij} denote "there is a **b**reeze in [i, j]."
- ...

Suppose we know that "pits cause breezes in adjacent cells." Which of the following sentences could we add to our *KB*?

A) $p_{2,1} \implies (b_{1,1} \wedge b_{2,2} \wedge b_{3,1})$
B) $b_{1,1} \implies (p_{1,2} \vee p_{2,1})$
C) $b_{1,1} \iff (p_{1,2} \vee p_{2,1})$
D) All of the above
E) None of the above unless we also know that $b_{1,1}$ is true or $p_{2,1}$ is true.

Bonus question: would your answer change if you also knew that pits were the *sole* cause of breezes?

Logical Equivalence

Truthiness of a sentence

Consider the sentence $p \wedge (p \implies q) \implies q$.

How do we evaluate whether this sentence is true?

1. Represent expression as a tree (relying on operator precedence to correctly parse sentence).
2. Assign true/false values to leaves of the tree (atomic propositions p and q)
3. Propagate true/false values up tree using properties of logical operators.

Shown on board

(Can this expression be simplified?)

Equivalence

Two sentences ψ and φ are logically equivalent, written $\psi \equiv \varphi$, if they have identical truth tables.

Example

Let $\psi \coloneqq p \implies q$.

Let $\varphi \coloneqq \neg p \vee q$.

ψ is logically equivalent to φ, i.e., $\psi \equiv \varphi$.

p	q	$p \implies q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Exercise

Working in small groups (2-4), try to simplify the following sentence.

$$\psi \coloneqq p \wedge (p \implies q) \implies q$$

In other words, find another sentence φ that is simpler and is equivalent to ψ. Hint: start by constructing a truth table for this sentence.

Tautology

A proposition ψ is a tautology if it is true under every assignment of its variables. In other words, ψ is a tautology if $\psi \equiv$ *true*.

The example

$$\psi \coloneqq p \wedge (p \implies q) \implies q$$

is the tautology known as *modus ponens*.

Important equivalence relationships

$(\alpha \wedge \beta)$	$\equiv$	$(\beta \wedge \alpha)$	commutativity of $\wedge$
$(\alpha \vee \beta)$	$\equiv$	$(\beta \vee \alpha)$	commutativity of $\vee$
$((\alpha \wedge \beta) \wedge \gamma)$	$\equiv$	$(\alpha \wedge (\beta \wedge \gamma))$	associativity of $\wedge$
$((\alpha \vee \beta) \vee \gamma)$	$\equiv$	$(\alpha \vee (\beta \vee \gamma))$	associativity of $\vee$
$\neg(\neg\alpha)$	$\equiv$	α	double-negation elimination
$(\alpha \implies \beta)$	$\equiv$	$(\neg\beta \implies \neg\alpha)$	contraposition
$(\alpha \implies \beta)$	$\equiv$	$(\neg\alpha \vee \beta)$	implication elimination
$(\alpha \iff \beta)$	$\equiv$	$((\alpha \implies \beta) \wedge (\beta \implies \alpha))$	biconditional elimination
$\neg(\alpha \wedge \beta)$	$\equiv$	$(\neg\alpha \vee \neg\beta)$	De Morgan's law #1
$\neg(\alpha \vee \beta)$	$\equiv$	$(\neg\alpha \wedge \neg\beta)$	De Morgan's law #2
$(\alpha \wedge (\beta \vee \gamma))$	$\equiv$	$((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$	distributivity of $\wedge$ over $\vee$
$(\alpha \vee (\beta \wedge \gamma))$	$\equiv$	$((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$	distributivity of $\vee$ over $\wedge$

Equivalence of logical operators

Poll: can we replace $\iff$?

Are all of the logical connectives really necessary? You already know that $\implies$ is unnecessary because $p \implies q \equiv \neg p \vee q$.

What about $\iff$ (if and only if)?

A) Replace with $\neg(\neg p \vee \neg q)$
B) Replace with $(\neg p \wedge \neg q) \vee (p \wedge q)$
C) Replace with $\neg(p \wedge q) \wedge (p \vee q)$
D) Replace with something else
E) $\iff$ in necessary

Hint: write out a truth table and evaluate each expression.

Poll: can we replace $\oplus$?

Are all of the logical connectives really necessary? You already know that $\implies$ is unnecessary because $p \implies q \equiv \neg p \vee q$.

What about $\oplus$ (exclusive or)?

A) Replace with $\neg(\neg p \vee \neg q)$
B) Replace with $(\neg p \wedge \neg q) \vee (p \wedge q)$
C) Replace with $\neg(p \wedge q) \wedge (p \vee q)$
D) Replace with something else
E) $\iff$ in necessary

Poll: can we replace $\wedge$?

Are all of the logical connectives really necessary? You already know that $\implies$ is unnecessary because $p \implies q \equiv \neg p \vee q$.

What about $\wedge$ (and)?

A) Replace with $\neg(\neg p \vee \neg q)$
B) Replace with $(\neg p \wedge \neg q) \vee (p \wedge q)$
C) Replace with $\neg(p \wedge q) \wedge (p \vee q)$
D) Replace with something else
E) $\iff$ in necessary

Minimal set of logical connectives

Not necessary:

- if, $\Longrightarrow$
- iff, $\Longleftrightarrow$
- exclusive or, $\oplus$
- and, $\wedge$

We can represent all of the above using only two connectives:

- Or $\vee$ and Not $\neg$

The set of connectives $\{\vee, \neg\}$ is functionally complete, meaning that any statement we can write in propositional logic we can write with only these two connectives.

Can we get it down to just one?

NAND connective

The NAND connective, denoted $\uparrow$, is logically equivalent to the "not" of an "and."

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \uparrow q$
T	T	T	F	F
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

NAND is also functionally complete. *Given what we already know*, what work remains to show this?

Poll: use NAND to simulate NOT

$\neg p$ is equivalent to...

A) $p \uparrow p$
B) $(p \uparrow p) \uparrow p$
C) $p \uparrow (p \uparrow p)$
D) $\uparrow p$
E) None of the above / more than one of the above

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Poll: use NAND to simulate AND

$p \wedge q$ is equivalent to...

A) $p \uparrow q$
B) $(p \uparrow p) \uparrow (q \uparrow q)$
C) $(p \uparrow q) \uparrow (p \uparrow q)$
D) None of the above / more than one of the above

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Hint: the double-negation elimination rule may be helpful

$$\neg(\neg\alpha) \quad \equiv \quad \alpha \quad \text{double-negation elimination}$$

as in $p \wedge q \equiv \neg(\neg(p \wedge q))$

Poll: use NAND to simulate OR

$p \vee q$ is equivalent to...

A) $p \uparrow q$

B) $(p \uparrow p) \uparrow (q \uparrow q)$

C) $(p \uparrow q) \uparrow (p \uparrow q)$

D) None of the above / more than one of the above

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Hint: double-negation elimination and DeMorgan's law #2 might be helpful

$$\begin{array}{rcll} \neg(\neg\alpha) & \equiv & \alpha & \text{double-negation elimination} \\ \neg(\alpha \wedge \beta) & \equiv & (\neg\alpha \vee \neg\beta) & \text{De Morgan's law \#1} \\ \neg(\alpha \vee \beta) & \equiv & (\neg\alpha \wedge \neg\beta) & \text{De Morgan's law \#2} \end{array}$$

as in, $p \vee q \equiv \neg(\neg(p \vee q))$.

Establishing logical equivalence

Ways to show logical equivalence

There are basically two ways to show logical equivalence $\psi \equiv \varphi$:

1. Using a truth table.
   - Make a truth table with columns for ψ and φ.
   - Equivalent if and only if the T/F values in each row are identical between the two columns.
2. Using known logical equivalences.
   - Step-by-step approach, resembling a proof.
   - Equivalent if and only if one can start with ψ and gradually transform it into φ using only known logical equivalence properties.

Example

A proposition ψ is a tautology if it is true under every assignment of its variables. In other words, ψ is a tautology if $\psi \equiv \textit{true}$.

Can we show that

$$\psi := p \wedge (p \implies q) \implies q$$

is a tautology by transforming it into $\varphi := \textit{true}$.

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CNF and DNF

Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in disjunctive-normal form (DNF) – an "or" of a bunch of "ands".

Given a proposition ψ, it is possible to write a proposition φ such that $\psi \equiv \varphi$ and φ is in conjunctive-normal form (CNF) – an "and" of a bunch of "ors".

Problem set

Any questions from the problem set?