COSC 290 Discrete Structures

Lecture 3: Functions

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Colgate University

Plan for today

1. Sequences
2. CS Connections: traceroute
3. Functions
4. CS Connections: hash tables

Sequences

Sets vs. Sequences

A *set* is an unordered collection. Notation: curly braces. Example: VarsitySports $:= \{\text{basketball}, \text{soccer}, \ldots\}$.

A *sequence* is an ordered collection. Notation: angle brackets. Example: Location $:= \langle \text{latitude}, \text{longitude} \rangle$.

Cartesian product

The Cartesian product takes two sets and generates a set of ordered pairs (sequences of length two).

$$A \times B := \{ \langle a, b \rangle \mid a \in A \text{ and } b \in B \}$$

Ex: let $A := \{ \text{Colgate}, \text{'Gate} \}$ and $B := \{ \text{raiders}, \text{University}, \text{hockey} \}$.

$A \times B = ?$

More than one Cartesian product

The Cartesian product of sets A and B is defined as

$$A \times B := \{ \langle a, b \rangle \mid a \in A \text{ and } b \in B \}$$

Continuing example, if $A := \{ \text{Colgate}, \text{'Gate} \}$ and $B := \{ \text{raiders}, \text{University}, \text{hockey} \}$ and $C := \{ ! \}$.

What about

$$(A \times B) \times C = ? \qquad A \times (B \times C) = ?$$

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What about

$$(A \times B) \times C = ? \qquad A \times (B \times C) = ?$$

If you look closely at the definition... $(A \times B) \times C$ produces a set of elements of the form $\langle \langle a, b \rangle, c \rangle$.

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What about

$$(A \times B) \times C = ? \qquad A \times (B \times C) = ?$$

If you look closely at the definition... $(A \times B) \times C$ produces a set of elements of the form $\langle \langle a, b \rangle, c \rangle$.

This is awkward. Instead, we define it as

$$A \times B \times C := \{ \langle a, b, c \rangle \mid a \in A \text{ and } b \in B \text{ and } c \in C \}$$

n-ary Cartesian product

For sets A_1, A_2, ..., A_n, the n-ary Cartesian product is defined as

$$A_1 \times A_2 \times \cdots \times A_n := \{ \langle a_1, a_2, \ldots, a_n \rangle \mid a_1 \in A_1 \text{ and } a_2 \in A_2 \ldots \text{ and } a_n \in A_2 \}$$

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If $A_1, \ldots, A_n$ are all the same set A, we can use this shorthand:

$$A^n := \underbrace{A \times A \times \cdots \times A}_{n \text{ times}}$$

Back to Example

Let $A := \{ \text{Colgate}, \text{'Gate} \}$ and $B := \{ \text{raiders}, \text{University}, \text{hockey} \}$ and $C := \{ ! \}$.

$$A \times B \times C =$$

Back to Example

Let $A := \{ \text{Colgate}, \text{'Gate} \}$ and $B := \{ \text{raiders}, \text{University}, \text{hockey} \}$ and $C := \{ ! \}$.

$$\begin{aligned} A \times B \times C = \{ & \langle \text{Colgate}, \text{raiders}, ! \rangle \\ & \langle \text{Colgate}, \text{University}, ! \rangle \\ & \langle \text{Colgate}, \text{hockey}, ! \rangle \\ & \langle \text{'Gate}, \text{raiders}, ! \rangle \\ & \langle \text{'Gate}, \text{University}, ! \rangle \\ & \langle \text{'Gate}, \text{hockey}, ! \rangle \} \end{aligned}$$

Back to Example

Let $A \coloneqq \{\text{Colgate}, \text{'Gate}\}$ and $B \coloneqq \{\text{raiders}, \text{University}, \text{hockey}\}$ and $C \coloneqq \{!\}$.

$$
\begin{aligned}
A \times B \times C = \{ & \langle \text{Colgate}, \text{raiders}, ! \rangle \\
& \langle \text{Colgate}, \text{University}, ! \rangle \\
& \langle \text{Colgate}, \text{hockey}, ! \rangle \\
& \langle \text{'Gate}, \text{raiders}, ! \rangle \\
& \langle \text{'Gate}, \text{University}, ! \rangle \\
& \langle \text{'Gate}, \text{hockey}, ! \rangle \}
\end{aligned}
$$

$$A^2 =$$

Back to Example

Let $A \coloneqq \{\text{Colgate}, \text{'Gate}\}$ and $B \coloneqq \{\text{raiders}, \text{University}, \text{hockey}\}$ and $C \coloneqq \{!\}$.

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& \langle \text{Colgate}, \text{University}, ! \rangle \\
& \langle \text{Colgate}, \text{hockey}, ! \rangle \\
& \langle \text{'Gate}, \text{raiders}, ! \rangle \\
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\end{aligned}
$$

$$
\begin{aligned}
A^2 = \{ & \langle \text{Colgate}, \text{Colgate} \rangle \\
& \langle \text{Colgate}, \text{'Gate} \rangle \\
& \langle \text{'Gate}, \text{Colgate} \rangle \\
& \langle \text{'Gate}, \text{'Gate} \rangle \}
\end{aligned}
$$

Sequences and data types

In programming, we have many different *data types* that can all be thought of as special cases of sequences.

- A string is a sequence of characters.
- A vector is meant to represent a sequence of $\mathbb{R}$, but each real number is approximated by a float
- In Java, `int[] x` is a sequence of integers in the interval $[-2^{31}, 2^{31} - 1]$
- In Python, `a_list = ['a', 4.3, True]` is a sequence of Python objects.

Pset 3

We will go over a couple of problems from the *first half* of pset 3. Which ones would you like to review?

Pset 3

Please turn in problem set 3. Pass to middle aisle.

CS Connections: traceroute

CS Connections: traceroute

When you type `www.auckland.ac.nz` into a web browser, what path through the Internet is taken to reach this website?

```
$ man traceroute
```

```
$ traceroute -m 10 -n www.auckland.ac.nz
```

Geography overlaid:
`http://csvoss.scripts.mit.edu/traceroute/`

Poll: traceroute

Let's say that traceroute returns a path from your IP address to a target IP address. The path can be at most 10 hops long.

Let S denote the set of all possible IP addresses.

Which of the following is the best representation for the set of all possible traceroute outputs?

A) S

B) $S \times S$

C) $\mathcal{P}(S)$

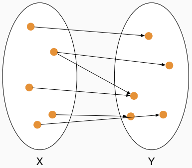
D) $S \cup S^2 \cup S^3 \cup \cdots \cup S^{10}$

E) S^{10}

Functions

What is a function?

Is this a function from X to Y?



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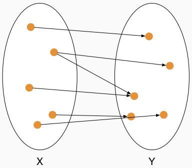
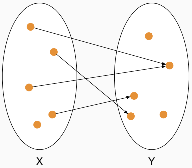


No. Some x is mapped to more than y.

What is a function?

Is this a function from X to Y?



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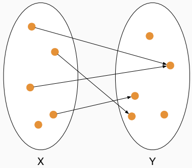
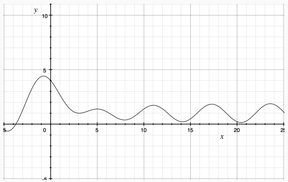
Is this a function from X to Y?



No. Some x is not mapped to any y.

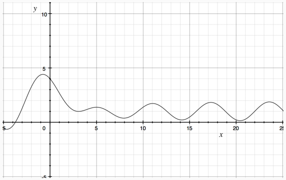
What is a function?

Is this a function from X to Y?



What is a function?

Is this a function from Y to X?



What is a function?

Let X and Y be sets.

A function f from X to Y, written $f : X \to Y$, assigns each input value $x \in X$ to a unique output value $y \in Y$.

We use $f(x)$ to denote the unique value from Y assigned to x by function f.

- For every element $x \in X$, $f(x)$ is always defined.
- Every element $x \in X$ is mapped to only one value in Y.

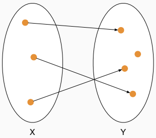
Terminology

- Domain: X
- Codomain: Y
- Image (aka range): the y values that correspond to function outputs.

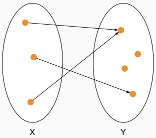
$$\{\, y \in Y : \text{there is some } x \in X \text{ where } f(x) = y \,\}$$

- Function composition: $(f \circ g)(x)$ is same as $f(g(x))$
- Onto, one-to-one, bijective functions
- Polynomials

One-to-one

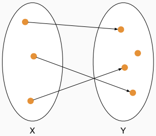


Yes.

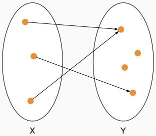


No.

One-to-one



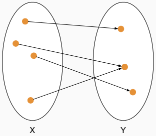
Yes.



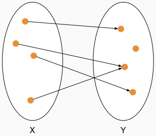
No.

One-to-one: for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.

Onto

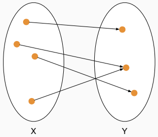


Yes.



No.

Onto



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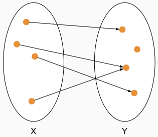

No.

Onto: for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.

Bijective

A function is bijective if it is both one-to-one and onto.

Poll: analyzing functions, part 0

Let $f : \{ 0, 1, 2, 3, 4, 5 \} \to \{ 0, 1, 2, 3, 4, 5 \}$ where

$$f(x) := 2x \bmod 6$$

This function is...

A) One-to-one
B) Onto
C) Bijective
D) None of the above

Poll: analyzing functions, part 1

Let $g : \{ 0, 1, 2, 3, \ldots, 9 \} \to \{ 0, 1, 2, 3, 4 \}$ where

$$g(x) := \lfloor x/2 \rfloor \bmod 5$$

This function is...

A) One-to-one
B) Onto
C) Bijective
D) None of the above

Poll: analyzing functions, part 2

Let $h : \{0, 1, 2, 3, 4\} \to \{0, 1, 2, 3, 4\}$ where

$$h(x) \coloneqq 2x \bmod 5$$

This function is...

A) One-to-one
B) Onto
C) Bijective
D) None of the above

CS Connections: hash tables

Poll: perfect hash function

A *perfect hash function* for a set S is a hash function $h : S \to \mathbb{Z}^{\geq 0}$ that maps distinct elements in S to a set of non-negative integers, with no collisions.

For a hash function to be perfect, it must be...

A) One-to-one
B) Onto
C) Bijective
D) None of the above