COSC 290 Discrete Structures

Lecture 8: Predicate Logic

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Plan for today

1. Predicate Logic
2. Nesting Quantifiers
3. Theorems

Predicate Logic

Predicate

An atomic proposition p is a Boolean variable.

A predicate $P(x)$ is a Boolean function. A predicate can take one or more arguments.

Examples:

- $isPrime(x)$ returns true if x is a prime number and false otherwise.
- $isDivisibleBy(x, y)$ returns true if x is evenly divisible by y.

Propositions that include predicates

We can form a proposition by supplying arguments to each predicate.

$$\varphi \coloneqq \mathit{isPrime}(8) \vee \mathit{isDivisibleBy}(8, 2)$$

The truth of this proposition requires *interpreting* the predicates: $\mathit{isPrime}(8)$ is false whereas $\mathit{isDivisibleBy}(8, 2)$ is true according to definitions of these predicates.

Universal Quantification

Let $P \coloneqq \{ p_1, p_2, \ldots, \}$ be the (infinite) set of all persons.

$$\forall p \in P : \mathit{At}(p, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p)$$

means "Every person at Colgate brushes their teeth."

The above is *roughly* equivalent to

$$\begin{aligned} &(\mathit{At}(p_1, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p_1)) \\ \wedge\, &(\mathit{At}(p_2, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p_2)) \\ \wedge\, &(\mathit{At}(p_3, \mathit{Colgate}) \implies \mathit{BrushesTeeth}(p_3)) \\ \wedge\, &\ldots \end{aligned}$$

Common mistake with universal quantification

Typically, $\implies$ is the main connective with $\forall$.

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall p \in P : \mathit{At}(p, \mathit{Colgate}) \wedge \mathit{BrushesTeeth}(p)$$

means "Every person is at Colgate and everyone brushes their teeth."

Existential Quantification

Let $P \{ p_1, p_2, \ldots, \}$ be the (infinite) set of all persons.

$$\exists p \in P : \mathit{At}(p, \mathit{Bucknell}) \wedge \mathit{BrushesTeeth}(p)$$

means "Some person at Bucknell brushes their teeth."

The above is *roughly* equivalent to

$$\begin{aligned} &(\mathit{At}(p_1, \mathit{Bucknell}) \wedge \mathit{BrushesTeeth}(p_1)) \\ \vee\, &(\mathit{At}(p_2, \mathit{Bucknell}) \wedge \mathit{BrushesTeeth}(p_2)) \\ \vee\, &(\mathit{At}(p_3, \mathit{Bucknell}) \wedge \mathit{BrushesTeeth}(p_3)) \\ \vee\, &\ldots \end{aligned}$$

Common mistake with existential quantification

Typically, $\wedge$ is the main connective with $\exists$.

Common mistake: using $\implies$ as the main connective with $\exists$:

$$\exists p \in P : At(p, Bucknell) \implies BrushesTeeth(p)$$

is true provided that there is some person who is *not* at Bucknell!

Credit: slides adapted from Russell & Norvig, AI: A Modern Approach

Constructing Predicates

In programming, we can define functions that call other functions.

In predicate logic, we can define predicate in terms of other predicates.

Examples:

- $follows(x, y)$ means that x follows the tweets of y
- $TrumpFollower(x) := follows(x, \text{@realDonaldTrump})$
- $popularTweeter(y) := \forall x \in P : follows(x, y)$

Poll: fastest person

Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Which of the following is the correct definition for $fastest(x)$?

A) $\exists y \in P : faster(x, y)$
B) $\neg(\exists y \in P : faster(y, x))$
C) $\forall y \in P : faster(x, y)$
D) $\neg(\forall y \in P : faster(y, x))$
E) None of the above / More than one of the above

Poll: fastest lacrosse player

Let $faster(x, y)$ be true if x runs faster than y and false otherwise.

Let $lax(x)$ be true if x plays lacrosse.

Which of the following is the correct definition for $fastestLacrossePlayer(x)$?

A) $\forall y \in P : lax(y) \wedge faster(x, y)$
B) $\forall y \in P : lax(y) \implies faster(x, y)$
C) $lax(x) \wedge \forall y \in P : lax(y) \wedge faster(x, y)$
D) $lax(x) \wedge \forall y \in P : lax(y) \implies faster(x, y)$
E) None of the above / More than one of the above

[[MH: double check these]]

Nesting Quantifiers

Unpacking a complex predicate can reveal nested quantifiers

Consider universes of professors P, students S, and courses C.

A course every student takes:

$$favCourse(c) \coloneqq \forall s \in S : takes(s, c)$$

Professor who teaches a course that every student takes:

$$\begin{aligned} profOfFav(p) &\coloneqq \exists c \in C : favCourse(c) \wedge teaches(p, c) \\ &\equiv \exists c \in C : (\forall s \in S : takes(s, c)) \wedge teaches(p, c) \\ &\equiv \exists c \in C : \forall s \in S : (takes(s, c) \wedge teaches(p, c)) \end{aligned}$$

Order of quantification

The order the of the quantifiers matters.

$$Q_1(p) \coloneqq \exists c \in C : \forall s \in S : (takes(s, c) \wedge teaches(p, c))$$

vs.

$$Q_2(p) \coloneqq \forall s \in S : \exists c \in C : (takes(s, c) \wedge teaches(p, c))$$

$Q_1(p)$: "Prof. who teaches a course every student takes"

vs.

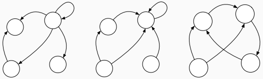
$Q_2(p)$: "Prof. who teaches every student"
(but not necessarily in the same course).

Poll: nested quantifiers, part 1

Predicate $likes(p_1, p_2)$ means "p_1 likes p_2," shown by an arrow from p_1 to p_2.

Which figure represents the following proposition:

$$\forall p_1 \in P : \exists p_2 \in P : likes(p_1, p_2)$$



More than one/ None of the above

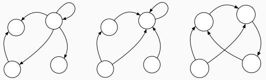
(a) **(b)** **(c)** **(d)**

Poll: nested quantifiers, part 2

Predicate $likes(p_1, p_2)$ means "p_1 likes p_2," shown by an arrow from p_1 to p_2.

Which figure represents the following proposition:

$$\exists p_2 \in P : \forall p_1 \in P : likes(p_1, p_2)$$



More than one/ None of the above

(a) **(b)** **(c)** **(d)**

Theorems

Theorem

A fully quantified expression of predicate logic is a theorem if and only if it is true for every possible meaning of its predicates.

Analogous to a tautology from propositional logic.

Two important theorems

$$\neg [\forall x\ P(x)] \iff [\exists x\ \neg P(x)]$$

$$\neg [\exists x\ P(x)] \iff [\forall x\ \neg P(x)]$$

See book for others!

Proving a theorem

Given a universally quantified statement ()$\forall x : \ldots$)

- To *prove* it: use induction, direct proof, by cases, proof by contrapositive, etc. (More on this soon! Ch. 4 & 5)
- To *disprove* it: find a single counter example.

Given an existentially quantified statement ()$\exists x : \ldots$)

- To *prove* it: find a single example.
- To *disprove* it: prove its *negation* is true. If claim is $\exists x : P(x)$, then negation $\forall x : \neg P(x)$. Use techniques listed above for universally quantified statements.

Poll: disproving a proposition

Given the proposition

$$\forall p_1 \in P : \exists p_2 \in P : likes(p_1, p_2)$$

How could you disprove it?

A) By counterexample: show there is a person who loves everyone
B) By counterexample: show there is a person who loves no one
C) By counterexample: show there is a person who nobody loves
D) By counterexample: show there is a person who everyone loves
E) Other/more/none

Poll: disproving a proposition

Given the proposition

$$\exists p_2 \in P : \forall p_1 \in P : likes(p_1, p_2)$$

What is its negation?

A) $\forall p_2 \in P : \forall p_1 \in P : \neg likes(p_1, p_2)$
B) $\forall p_2 \in P : \exists p_1 \in P : \neg likes(p_1, p_2)$
C) $\exists p_2 \in P : \forall p_1 \in P : \neg likes(p_1, p_2)$
D) $\forall p_2 \in P : \exists p_1 \in P : \neg likes(p_2, p_1)$
E) Other/more/none

[[MH: check these]]