COSC 290 Discrete Structures

Lecture 8: Predicate Logic

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1. Predicate Logic

2. Nesting Quantifiers

3. Theorems

Predicate Logic

An atomic proposition p is a Boolean variable.

A predicate $P(x)$ is a Boolean function. A predicate can take one or more arguments.

Examples:

- *isPrime*(x) returns true if x is a prime number and false otherwise.
- *isDivisibleBy*(x, y) returns true if x is evenly divisible by y.

Propositions that include predicates

We can form a proposition by supplying arguments to each predicate.

$$\varphi := isPrime(8) \vee isDivisibleBy(8, 2)$$

The truth of this proposition requires *interpreting* the predicates: *isPrime*(8) is false whereas *isDivisibleBy*(8, 2) is true according to definitions of these predicates.

Universal Quantification

Let $P := \{\, p_1, p_2, \ldots, \}$ be the (infinite) set of all persons.

$$\forall p \in P : At(p, Colgate) \implies BrushesTeeth(p)$$

means "Every person at Colgate brushes their teeth."

The above is *roughly* equivalent to

$$
\begin{aligned}
&(At(p_1, Colgate) \implies BrushesTeeth(p_1)) \\
\wedge\, &(At(p_2, Colgate) \implies BrushesTeeth(p_2)) \\
\wedge\, &(At(p_3, Colgate) \implies BrushesTeeth(p_3)) \\
\wedge\, &\ldots
\end{aligned}
$$

Common mistake with universal quantification

Typically, $\implies$ is the main connective with $\forall$.

Common mistake: using $\wedge$ as the main connective with $\forall$:

$$\forall p \in P : At(p, Colgate) \wedge BrushesTeeth(p)$$

means "Every person is at Colgate and everyone brushes their teeth."

Existential Quantification

Let $P \{\, p_1, p_2, \ldots, \}$ be the (infinite) set of all persons.

$$\exists p \in P : At(p, Bucknell) \wedge BrushesTeeth(p)$$

means "Some person at Bucknell brushes their teeth."

The above is *roughly* equivalent to

$$
\begin{aligned}
&(At(p_1, Bucknell) \wedge BrushesTeeth(p_1)) \\
\vee\, &(At(p_2, Bucknell) \wedge BrushesTeeth(p_2)) \\
\vee\, &(At(p_3, Bucknell) \wedge BrushesTeeth(p_3)) \\
\vee\, &\ldots
\end{aligned}
$$

Typically, $\wedge$ is the main connective with $\exists$.

Common mistake: using $\implies$ as the main connective with $\exists$:

$$\exists p \in P : At(p, Bucknell) \implies BrushesTeeth(p)$$

is true provided that there is some person who is *not* at Bucknell!

In programming, we can define functions that call other functions.

In predicate logic, we can define predicate in terms of other predicates.

Examples:

- $follows(x, y)$ means that x follows the tweets of y
- $TrumpFollower(x) := follows(x, @realDonaldTrump)$
- $popularTweeter(y) := \forall x \in P : follows(x, y)$

Let $faster(x, y)$ be true if x runs faster than y or x and y run the same speed and false otherwise.

Which of the following is the correct definition for $fastest(x)$?

A) $\exists y \in P : faster(x, y)$

B) $\neg (\exists y \in P : faster(y, x))$

C) $\forall y \in P : faster(x, y)$

D) $\neg (\forall y \in P : faster(y, x))$

E) None of the above / More than one of the above

Let $faster(x, y)$ be true if x runs faster than y or x and y run the same speed and false otherwise.

Let $lax(x)$ be true if x plays lacrosse.

Which of the following is the correct definition for $fastestLacrossePlayer(x)$?

A) $\forall y \in P : lax(y) \wedge faster(x, y)$

B) $\forall y \in P : lax(y) \implies faster(x, y)$

C) $lax(x) \wedge \forall y \in P : lax(y) \wedge faster(x, y)$

D) $lax(x) \wedge \forall y \in P : lax(y) \implies faster(x, y)$

E) None of the above / More than one of the above

Nesting Quantifiers

Consider universes of professors P, students S, and courses C.

A course every student takes:

$$favCourse(c) := \forall s \in S : takes(s, c)$$

Professor who teaches a course that every student takes:

$$
\begin{aligned}
profOfFav(p) &:= \exists c \in C : favCourse(c) \wedge teaches(p, c) \\
&\equiv \exists c \in C : (\forall s \in S : takes(s, c)) \wedge teaches(p, c) \\
&\equiv \exists c \in C : \forall s \in S : (takes(s, c) \wedge teaches(p, c))
\end{aligned}
$$

Order of quantification

The order the of the quantifiers matters.

$$Q_1(p) := \exists c \in C : \forall s \in S : (takes(s, c) \wedge teaches(p, c))$$

$Q_1(p)$: "Prof. who teaches a course every student takes"

vs.

$$Q_2(p) := \forall s \in S : \exists c \in C : (takes(s, c) \wedge teaches(p, c))$$

$Q_2(p)$: "Prof. who teaches every student"
(but not necessarily in the same course).

Poll: nested quantifiers, part 1

Predicate $likes(p_1, p_2)$ means "p_1 likes p_2," shown by an arrow from p_1 to p_2.

Which figure represents the following proposition:

$$\forall p_1 \in P : \exists p_2 \in P : likes(p_1, p_2)$$

(a)   (b)   (c)   (d)

More than one/
None of the above

Predicate $likes(p_1, p_2)$ means "p_1 likes p_2," shown by an arrow from p_1 to p_2.

Which figure represents the following proposition:

$$\exists p_2 \in P : \forall p_1 \in P : likes(p_1, p_2)$$

(a)   (b)   (c)   More than one/ None of the above

(d)

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Theorems

Theorem

A fully quantified expression of predicate logic is a theorem if and only if it is true for every possible meaning of its predicates.

Analogous to a tautology from propositional logic.

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Two important theorems

$$\neg\,[\forall x\ P(x)] \iff [\exists x\ \neg P(x)]$$

$$\neg\,[\exists x\ P(x)] \iff [\forall x\ \neg P(x)]$$

See book for others!

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Proving a theorem

Given a universally quantified statement ($\forall x : P(x)$)

- To *prove* it: use induction, direct proof, by cases, proof by contrapositive, etc. (More on this soon! Ch. 4 & 5)
- To *disprove* it: find a single counter example, some x_{bad} where $\neg P(x_{bad})$ is true.

Given an existentially quantified statement ($\exists x : P(x)$)

- To *prove* it: find a single example x_{good} where $P(x_{good})$ is true.
- To *disprove* it: prove its *negation* is true. If claim is $\exists x : P(x)$, then negation $\neg (\exists x : P(x)) \equiv \forall x : \neg P(x)$. Use techniques listed above for universally quantified statements.

Poll: disproving a proposition

Given the proposition

$$\forall p_1 \in P : \exists p_2 \in P : likes(p_1, p_2)$$

How could you disprove it?

A) By counterexample: show there is a person who loves everyone
B) By counterexample: show there is a person who loves no one
C) By counterexample: show there is a person who nobody loves
D) By counterexample: show there is a person who everyone loves
E) Other/more/none

Poll: disproving a proposition

Given the proposition

$$\exists p_2 \in P : \forall p_1 \in P : likes(p_1, p_2)$$

What is its negation?

A) $\forall p_2 \in P : \forall p_1 \in P : \neg likes(p_1, p_2)$
B) $\forall p_2 \in P : \exists p_1 \in P : \neg likes(p_1, p_2)$
C) $\exists p_2 \in P : \forall p_1 \in P : \neg likes(p_1, p_2)$
D) $\forall p_2 \in P : \exists p_1 \in P : \neg likes(p_2, p_1)$
E) Other/more/none