Dynamical System

April 21, 2024

Contents

1 Stability

Definition 1.1.

- ***Stability:*** *the zero solution is called* stable *if*

$$\lim_{x_0 \to 0} \sup_{t > t_0} x(t, t_0, x_0) = 0$$

- ***Attractivity*** *the zero solution is called* attractive *if*

$$\forall x_0 \textit{in defined region}, \lim_{t \to \infty} x(t, t_0, x_0) = 0$$

- ***Asymptotical Stability:*** *the zero solution is called* asymptotically stable *if it is both stable and attractive.*

- ***Unstability:*** *A fixed point x^* is said to be unstable if it is not stable.*

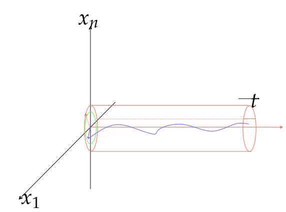

Figure 1: Stable

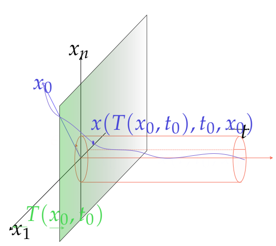

Figure 2: Attractive

1.1 Autonomous Systems

Linear Systems The one parameter group is given by the matrix exponential

$$x(t) = e^{(t-t_0)A} x_0,$$

where A is the matrix of the linear system $\partial_t x = Ax$.

The zero solution of the system is stable $\iff$

$$\lim_{x_0 \to \infty} (\sup t > t_0 e^{(t-t_0)A}) x_0 = 0$$

For simplicity we deal with the linear system on $\mathbb{C}^n$. (In fact the real case could be reduced by complexification).

Lemma 1.1. *For any $P \in GL_n(\mathbb{C})$, the zero solution of the new dynamical system for $y = Px \in \mathbb{C}^n$ given the vector field*

$$\partial_t y = (PAP^{-1})y$$

is stable(resp. attractive) if and only if the zero solution of the original system is stable(resp. attractive).

Formal Proof. *The one parameter group of the new system is given by*

$$y(t) = e^{(t-t_0)PAP^{-1}}y_0 = Pe^{(t-t_0)A}P^{-1}y_0,$$

hence

$$\begin{aligned}
&\text{the zero solution in the new system is stable} \\
\iff & P\lim_{y_0\to 0}(\sup_{t>t_0} e^{(t-t_0)A})P^{-1}y_0 = P\lim_{Px_0\to 0}(\sup_{t>t_0} e^{(t-t_0)A})x_0 = 0 \\
\iff & (P \text{ linear bijection hence homeomorphism}) \lim_{x_0\to 0}(\sup_{t>t_0} e^{(t-t_0)A})x_0 = 0 \\
\iff & \text{the zero solution in the original system is stable}
\end{aligned}$$

Similarly

$$\begin{aligned}
&\text{the zero solution in the new system is attractive} \\
\iff & \forall y_0 \text{we have} P\lim_{t\to\infty}(e^{(t-t_0)A})P^{-1}y_0 = P\lim_{t\to\infty}(e^{(t-t_0)A})x_0 = 0 \\
\iff & \forall x_0 \text{we have} \lim_{t\to\infty}(e^{(t-t_0)A})x_0 = 0 \\
\iff & \text{the zero solution in the original system is attractive}
\end{aligned}$$

Lemma 1.2. *Given diagonal blocks matrix $A := A_1, ..., A_k$ with the subsystems $x^i \in \mathbb{C}^{n_i}$ with vector fields defined by*

$$\partial_t x^i = A_i x,$$

then the zero solution of the system is stable(resp. attractive) if and only if the zero solution of each *subsystem is stable(resp. attractive).*

Formal Proof.

$$\begin{aligned} & \textit{the zero solution in the original system is stable} \\ \iff & \lim_{x_0 \to 0} (\sup_{t > t_0} e^{(t-t_0)A}) x_0 = 0 \end{aligned}$$

where

$$e^{(t-t_0)A}) x_0 = \begin{pmatrix} e^{(t-t_0)A_1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{(t-t_0)A_k}) \end{pmatrix} \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^k \end{pmatrix} = \begin{pmatrix} e^{(t-t_0)A_1}) x_0^1 \\ \vdots \\ e^{(t-t_0)A_k}) x_0^k \end{pmatrix}$$

Hence

$$\begin{aligned} & \lim_{x_0 \to 0} (\sup_{t > t_0} e^{(t-t_0)A}) x_0 = 0 \\ \iff & \lim_{x_0 \to 0} (\sup_{t > t_0} e^{(t-t_0)A_j}) x_0^j = 0 \forall 1 \le j \le k \\ \iff & \textit{the zero solutions of all subsystems are stable.} \end{aligned}$$

Similarly

$$\begin{aligned} & \textit{the zero solution in the original system is attractive} \\ \iff & \lim_{t \to \infty} e^{(t-t_0)A}) x_0 = 0 \forall x_0 \\ \iff & \lim_{t \to \infty} e^{(t-t_0)A_j}) x_0^j = 0 \forall x_0, 1 \le j \le k \\ \iff & \textit{the zero solutions in all subsystems are attractive} \end{aligned}$$

Lemma 1.3. *For a "Jordan" system defined by*

$$\partial_t x = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} x =: J_m(\lambda) x,$$

the zero solution is stable if and only if $\Re(\lambda) < 0$ or ($\Re(\lambda) = 0$ and $m = 1$).
the zero solution is attractive if and only if $\Re(\lambda) < 0$.

Formal Proof. *For the euclidean topology over $M_m(\mathbb{R})$ we use the* *standard norm*

$$\|A\| := \sqrt{\sum_{ij} a_{ij}^2}$$

We use the fact that in phase space the limit is the origin $\iff$ *the norm is tending to 0, without explicitly ramble about it.*
A quick computation:

$$e^{(t-t_0)J_m(\lambda)} = e^{(t-t_0)\lambda} \begin{pmatrix} 1 & \frac{t-t_0}{1!} & \cdots & \frac{(t-t_0)^{m-1}}{(m-1)!} \\ 0 & 1 & \cdots & \frac{(t-t_0)^{m-2}}{(m-2)!} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{t-t_0}{1!} \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

hence

$$\begin{aligned} n(t) &:= \|e^{(t-t_0)J_m(\lambda)}\| \\ &= e^{(t-t_0)\Re\lambda}\sqrt{m\cdot 1^2 + (m-1)(\frac{t-t_0}{1!})^2 + \cdots + 1\cdot(\frac{(t-t_0)^{m-1}}{(m-1)!})^2} \\ &=: e^{(t-t_0)\Re\lambda}\sqrt{P}(t) \end{aligned}$$

It suffices to consider the following cases

Case I: *For* $\Re(\lambda) < 0$ *we need to show the zero solution is stable and attractive. Firstly,*

$$\|e^{(t-t_0)J_m(\lambda)}x_0\| \leq \|e^{(t-t_0)J_m(\lambda)}\| \cdot \|x_0\|,$$

where $n(t) = \|e^{(t-t_0)J_m(\lambda)}\| \to 0$ *as* $t \to \infty$*, hence the zero solution is attractive.*

Secondly, we could do "one point compactification" to get the space

$$[t_0, \infty) \cup \infty,$$

and extend n *by mapping* $\infty \mapsto 0$*. And since* $n(t) \to 0$*, the extended mapping is a continuous real valued function over a compact space, hence admits a maximum, say* M*. Therefore*

$$\|e^{(t-t_0)J_m(\lambda)}x_0\| \leq \|e^{(t-t_0)J_m(\lambda)}\| \cdot \|x_0\| \leq M \cdot \|x_0\|,$$

hence the zero solution is stable.

Case II: *For* ($\Re(\lambda) = 0$ *and* $m = 1$) *we need to show the zero solution is stable. But* $n(t) \equiv 1$*, hence*

$$\|e^{(t-t_0)J_m(\lambda)}x_0\| = \|x_0\|,$$

thus the zero solution is stable.

Case III: *For $\Re(\lambda) > 0$ we need to show the zero solution is neither stable nor attractive. Given $x_0 \neq 0$, without loss of generality, let $(x_0)_1 = 1$(since we could focus our attention at a small ball centered at origin, which is homeomorphic to $\mathbb{E}^m$)*

therefore

$$\begin{aligned}\|e^{(t-t_0)J_m(\lambda)}x_0\| &\geq n(e^{(t-t_0)J_m(\lambda)})_1 \\ &= e^{(t-t_0)\Re\lambda} \to \infty,\end{aligned}$$

it is unbounded, hence the limit as $t \to \infty$ could not exist, hence the zero solution is NOT attractive.

Now consider the sequence of initial positions $(x_0)_n := \{\frac{1}{t'-t_0}\}e_1$ where t' takes $t_0 + n$. Of course the sequence is tending to the origin. But as $t_0 + n$ grows,

$$\|e^{(t-t_0)J_m(\lambda)}x_0\| = \frac{e^{n\Re\lambda}}{n} \to \infty,$$

hence the zero solution is NOT stable.

Case IV: *For $\Re(\lambda) = 0$ we need to show the zero solution is NOT attractive. WLG, we consider the initial position $x_0 = e_1$(since we could focus our attention at a small ball centered at origin, which is homeomorphic to $\mathbb{E}^m$)*

$$\|e^{(t-t_0)J_m(\lambda)}x_0\| = \|e_1\| = 1,$$

which is independent of t, hence the zero solution is NOT attractive.

Case V: *For ($\Re(\lambda) = 0$ and $m > 1$)we need to show the zero solution is NOT stable.*

WLG, we consider the path $(x_0)_\delta = \delta e_2$. hence

$$\|e^{(t-t_0)J_m(\lambda)}x_0\| = \delta(t - t_0 + 1),$$

now consider the sequence $(x_0)_n = \frac{1}{(n+t_0)}e_2$ hence the norm is just

$$\frac{t - t_0 + 1}{n + t_0} \to 1,$$

hence as the sequence of pts in phase space approaching the origin, the norm is no where near touching the 0, i.e., the zero solution could not possibly be stable.

Theroem 1.4. *1. the zero solution of the linear system $\partial_t x = Ax$ is assymtotically stable*
$\iff$ All the subsystems with vector fields $J_{n_i}(\lambda_i)$ is both attractive and stable
$\iff$ All the eigenvalues of A have negative real parts.

2. the zero solution of the linear system $\partial_t x = Ax$ is stable
$\iff$ All the subsystems are stable
$\iff$ All the eigenvalues of A have non-positive real parts, and for $\lambda_i = 0$ the jordan block is of size $n_i = 1$.

In particular, if there exists $\lambda_i > 0$ or ($\lambda_i = 0$ and $n_i > 1$), then the zero solution is unstable.

Formal Proof. *it follows directly from the lemmas above.*

Liapunov2: V Function Intuition If using an Euclidean metric(inner product), we find

$$r(f(x), x) \le 0' \iff '\nabla r^2(x) \cdot f(x) \le 0$$

i.e., the vector field is pointing inwards, then given x_0, we could have a pretty decent predection of x having smaller magnitude as time evolves.

In fact, we could consider a broader class of positive(≥ 0) definite($= 0 \iff x = 0$) functions(Liapunov V functions) in a neighborhood of $x = 0$ rather than just the global norm r^2, which are not necessarily quadratic, but still retain the sense that $f(x)$ is "pointing inwards"($\nabla V(x) \cdot f(x) \le 0$).

Theroem 1.5. *If such a function $V(x)$ exists, then the system is stable.(Later we will see the existence of V is not only sufficient, but also necessary)*
Moreover, if the set of points where $\nabla V(x)$ is orthogonal to $f(x)$ does not contain any other phase curves of the system(in particular, if $\nabla V(x) \cdot f(x)$ is positive definite), then the system is asymptotically stable.

Note that all problems come from the textbook [1] Problem set 3, Page 122-124.

[1] Zhien Ma, Yicang Zhou, and Chengzhi Li. *Qualitative and Stability Methods for Ordinary Differential Equations*. Science Press, 2001

Problem 14(1) Let

$$V(x) = \frac{x_1^2 + x_2^2}{2},$$

then

$$\nabla V(x) \cdot \begin{pmatrix} -x_1 + x_1 x^2 \\ -2x_1^2 x_2 - x_2 \end{pmatrix} = -x_1^2 - x_2^2 - (x_1 x_2)^2 \le 0.$$

Using the same intuition of the direction of $f(x)$ we would have 2 sufficient conditions of the unstability of the system.

Theroem 1.6. *If there exists a function $U(x)$ in a small neighborhood of $x = 0$ such that*

1. *$U(0) = 0$ and at some pt. $\overline{x_0}$ in the small nbh. we have $U(\overline{x_0}) > 0$,*

2. *$\nabla U(x) \cdot f(x) > 0$ for all x in the nbh. except $x = 0$(i.e., positive definite; in other words, the vector field is pointing outwards),*

   then the system is unstable.

Problem 2(4) Let

$$U(x) = \frac{x_1^2 + x_2^2}{2},$$

which is postive definite, then

$$\nabla U(x) \cdot \begin{pmatrix} x_1 - x_2 + x_1 x_2 + x_1^3 \\ x_1 + x_2 - x_1^3 - x_2^2 \end{pmatrix} = r^2 + O(r^3) = r^2(1 + O(r))$$

where $r = \sqrt{x_1^2 + x_2^2}$. Hence there exists $\delta > 0$ s.t. for all $(x_1, x_2) \in B(O, \delta)/O$ we have

$$\nabla U(x) \cdot f(x_1 x_2) > 0.$$

therefore the zero solution is unstable.

Theroem 1.7. *If there exists a function $U(x)$ in a small neighborhood of $x = 0$ such that*

1. *$U(0) = 0$ and for any small nbh. of $x = 0$ there exists a pt. $\overline{x_0}$ s.t. $U(\overline{x_0}) > 0$,*

2. $$\nabla U(x) \cdot f(x) = \lambda U(x) + W(x),$$

   where $\lambda > 0, W(x) \geq 0$(essentially the same as $\partial_t(U(x(t))) \geq \lambda U(x)$),

then the system is unstable.

Formal Proof. *(Proof by contradiction) Assume that the system is stable, then for any small $\epsilon > 0$ we could find $\delta > 0$ s.t. whenever*

$$x_0^T x_0 < \delta,$$

for any time t after t_0 we could control

$$x(t, t_0, x_0)^T x(t, t_0, x_0) < \epsilon.$$

For such a small nbh. of $B(\delta, 0)$ we could find $\overline{x_0} \in B(\delta, 0)$ s.t.

$$U(\overline{x_0}) > 0$$

Now since $\partial_t(U(x(t, t_0, \overline{x_0}))) \geq \lambda U(x(t, t_0, \overline{x_0}))$, by Gronwall's Inequality,

$$\partial_t U(x(t, t_0, \overline{x_0})) \geq \lambda U(x(t, t_0, \overline{x_0})) \geq \lambda U(\overline{x_0}) e^{\lambda(t-t_0)} > 0.$$

Since $U(x)$ is continuous and $U(0) = 0$ there exists $\epsilon > \eta > 0$ s.t. for all time $t > t_0$

$$x(t, t_0, \overline{x_0})^T x(t, t_0, \overline{x_0}) \geq \eta,$$

therefore,

$$\begin{aligned} U(x(t, t_0, \overline{x_0})) &= U(\overline{x_0}) + \int_{t_0}^{t} \partial_t U(x(\tau, t_0, \overline{x_0})) d\tau \\ &\geq U(\overline{x_0}) + \int_{t_0}^{t} \lambda U(x(\tau, t_0, \overline{x_0})) + W(x(\tau, t_0, \overline{x_0})) d\tau \end{aligned}$$

but the function inside the integral is actually $\geq \lambda U(\overline{x_0})$, hence

$$U(x(t, t_0, \overline{x_0})) \geq U(\overline{x_0}) + \lambda U(\overline{x_0})(t - t_0) \to \infty,$$

as time evolves $t \to \infty$.
However, the integral curve $x(t, t_0, \overline{x_0})$ is bounded inside $B(0, \epsilon)$, and since $U(x)$ is continuous, the values of $U(x(t, t_0, \overline{x_0}))$ is also bounded, leading to the contradiction, completing the proof.

Problem 2(7) Let

$$U(x) = \frac{x_1^2 - x_2^2}{2},$$

then

$$\nabla U(x) \cdot \begin{pmatrix} x_1 + x_2 + x_1 x_2^2 \\ 2x_1 + x_2 - x_1^2 x_2 \end{pmatrix} = x_1^2 - x_2^2 + 2(x_1 x_2)^2 = 2U + W,$$

where $W \geq 0$, therefore the zero solution of the system is stable.

Liapunov1: Linear Appoximation

Theroem 1.8. *Let f be a non-autonomous vector field on the phase space satisfying:*

1. *there exists $k \geq 0$ s.t.*

$$\|f(t, x_1) - f(t, x_2)\| \leq k \|x_1 - x_2\|,$$

2. f *is of order greater than* $\|x\|$. *more precisely,*

$$\lim_{x\to 0}\sup_{t>t_0}\frac{\|f(t,x)\|}{\|x\|}.$$

Suppose A *has no pure imaginary(0 included) eigenvalue, then*

$$\begin{aligned}&\text{the zero solution of the vector field } Ax+f \text{ is asymptotically stable}\\ \Longleftrightarrow\ &\text{the zero solution of the vector field } Ax \text{ is asymptotically stable}\\ \Longleftrightarrow\ &\text{All eigenvalues of } A \text{ have negative real part.}\end{aligned}$$

Likewise,

$$\begin{aligned}&\text{the zero solution of the vector field } Ax+f \text{ is unstable}\\ \Longleftrightarrow\ &\text{the zero solution of the vector field } Ax \text{ is unstable}\\ \Longleftrightarrow\ &\text{At least one eigenvalue of } A \text{ has positive real part.}\end{aligned}$$

Sketch of Proof. *It suffices to show*

Case I:

$$\begin{aligned}&\text{All eigenvalues of } A \text{ have negative real part.}\\ \Longrightarrow\ &\text{the zero solution of the vector field } Ax+f \text{ is asymptotically stable}\end{aligned}$$

Step I: *Constructing a Liapunov function* $V(x)$

Method I: *using* $(A^T * + * A)^{-1}(-E)$ *as the(note that all eigenvlues of* A *lies in the left half plane hence could not be symmetric about the 0 on the complex plane thus the mapping is a linear isomorphism by 1.9) quadratic form for* V.

Method II: *finding a (almost) proper basis(to controll* $\nabla V\cdot A$*), and use the standard norm for* V.

Step II: *(cf. [2]) Controling* $\nabla V(x)\cdot f(x)$ *in a small enough nbh. of the origin of the phase space.*

[2] Vladimir I Arnold. *Ordinary differential equations*. Springer Science & Business Media, 1992

Case II:

$$\begin{aligned}&\text{At least one eigenvalue of } A(\text{ say } \lambda) \text{ has positive real part.}\\ \Longrightarrow\ &\text{the zero solution of the vector field } Ax+f \text{ is unstable}\end{aligned}$$

Step I: *Choose a basis s.t. the operator* A *looks like*

$$\begin{pmatrix}\lambda & *\\ O & A'\end{pmatrix}$$

Step II: *Suppose the zero solution is stable, contain the phase curve to sit inside a small nbh. of the origin, in order to control* $\|f(x)\|$

Step III: *use the estimation*

$$\partial_t x_1^2 = 2\lambda x_1^2 + 2x_1 f_1 \geq \lambda x_1^2,$$

and Gronwall inequality to blows

$$x_1^2(t) \geq x_1^2(0) \cdot e^{\lambda t}$$

up, arriving at the contradiction.

Problem 2(4) The linear part of the field has $\det A = 3; \mathrm{tr} A = -3$ hence both of the eigenvalues are negative, while the nonlinear part

$$-(x_1^2 + x_2^2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = O(\|x\|^3)$$

hence the zero solution of the system is stable.

1.2 *Non-Autonomous Systems*

3

[3] Zhien Ma, Yicang Zhou, and Chengzhi Li. *Qualitative and Stability Methods for Ordinary Differential Equations*. Science Press, 2001

1.3 *Constructions of Liapunov V function*

If the dynamical system is stable, then

If it is a planar vector field, and x_2 is the velocity of x_1

If it is a constant coeffient linear vector field, We choose such V from the set of quadratic forms $x^T Bx$, which is essentially the set of real symmetric matrices $\mathscr{S}$, then the derivative of V along the flow of the vector field Ax is just $x^T(A^T B + BA)x =: x^T Cx$.

As we know, in the topological vector space of all real matrices of size $n \times n$ $M_n(\mathbb{R})$

$$(d_{ij}) \in \mathscr{S} \iff (d_{ij}) \in \cap_{ij} \ker(d_{ij} - d_{ji}) = \begin{bmatrix} d_{12} - d_{21} \\ \vdots \\ d_{n-1,n} - d_{n,n-1} \end{bmatrix}^{-1} (0).$$

In other words, $\mathscr{S} \subset M_n(\mathbb{R})$ is a topological linear closed subspace.

Looking at this linear (hence continuous) operator $A^T * + * A$ on the space of real symmetric matrices more closely,

Lemma 1.9. *if eigenvalues of A satisfy*

$$\lambda_i + \lambda_j \neq 0,$$

then the operator $A^T * + * A$ is injective hence bijective.

Assuming $C = (c_{ij}), A = (a_{ij})$
For planar dynamical system,

$$V(x) = \frac{-1}{\Delta}\begin{vmatrix} 0 & x_1^2 & 2x_1x_2 & x_2^2 \\ c_{11} & a_{11} & a_{21} & 0 \\ 2c_{12} & a_{12} & a_{11}+a_{22} & a_{21} \\ c_{22} & 0 & a_{12} & a_{22} \end{vmatrix}$$

where $\Delta = trA \times \det A \neq 0$.

References

[1] Vladimir I Arnold. *Ordinary differential equations*. Springer Science & Business Media, 1992.

[2] Zhien Ma, Yicang Zhou, and Chengzhi Li. *Qualitative and Stability Methods for Ordinary Differential Equations*. Science Press, 2001.