R. & M. No. 3562, Part 3

R. & M. No. 3562, Part 3

MINISTRY OF TECHNOLOGY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

A Scheme of Notation and Nomenclature for Aircraft Dynamics and Associated Aerodynamics

by H. R. HOPKIN
Aero. Dept., R.A.E., Farnborough

Part 3—Aircraft Dynamics

LONDON: HER MAJESTY'S STATIONERY OFFICE
1970
PRICE £1 13s 0d [£1·65] NET

PREFACE

For many years the notation and nomenclature used in the UK for aircraft dynamics has consisted of a basic scheme introduced by Bryant and Gates (R. & M. 1801, 1937) together with various additions and amendments due to Neumark, Mitchell, and others. Modifications were not co-ordinated and resulted in a complex mixture having at least two serious drawbacks. First, further rational extensions would be extremely difficult to make and probably confusing. Secondly, some parts of the scheme appeared to possess a pattern which in fact they did not possess, and this hidden ambiguity sometimes led to mistakes.

The present Report is the third in a series of five separate parts of R. & M. 3562 in which a unified system of notation and nomenclature is described. The system will present few difficulties to those familiar with the scheme of Bryant and Gates and its variants, and has the great advantage that it has a built-in potential for extension. At the same time, reliable patterns are incorporated and furthermore a great deal of freedom is available to an author who wishes in special cases to simplify the notation without ambiguity—for example, by omitting suffixes.

The new system is the outcome of many years of discussion at the Royal Aircraft Establishment, in co-operation with the National Physical Laboratory. It has been adopted by the Royal Aeronautical Society for its Data Items on Dynamics. Moreover, agreements reached by the International Organisation for Standardisation on terms and symbols used in flight dynamics have so far been completely consistent with the principles of the system.

The Aeronautical Research Council hopes that publication in the R. & M. Series will encourage the acceptance of the new notation and nomenclature and its use in the general field of aircraft dynamics by workers in research establishments and universities and in industry.

A Scheme of Notation and Nomenclature for Aircraft Dynamics and Associated Aerodynamics

by H. R. HOPKIN

Aero. Dept., R.A.E., Farnborough

Part 3—Aircraft Dynamics

Reports and Memoranda No. 3562 *Part* 3*

June, 1966

Summary.

A scheme of notation and nomenclature applicable to the dynamics and associated aerodynamics of both aeroplanes and missiles is proposed. The proposals are intended to supersede the attempts made, notably by Bryant and Gates, to revise and extend the existing standard reference in this field, namely R. & M. 1801.

Part 1 contains an extensive introduction describing the main objectives and summarising a considerable amount of historical background. It also lists the symbols, references, and most of the tables for the whole report, and provides an index. All illustrations are appended to Part 1, and copies included in the remaining parts where required.

Parts 2, 3, and 4 deal with basic notation and nomenclature, aircraft dynamics, and associated aerodynamic data respectively, and they can be read almost independently of Part 1. A great deal of reference information is included in the main text and in the twelve appendices which comprise Part 5.

Parts 2 to 5 do not contain separate lists of symbols, references or indexes.

LIST OF CONTENTS

*Replaces R.A.E. Tech. Report 66 200, Part 3 (A.R.C. 28 971), and includes the corrections and amendments listed in R.A.E. 66 200A.

---

10. *Equations of Motion.*

It is not practicable to set out equations of motion that are in a convenient form for all investigations in aircraft dynamics. A large display of the application of the notation and nomenclature is provided by considering the equations relevant for a rigid object of constant mass in still air, and the following sections in this part of the report are entirely restricted to this case. However, the procedures to be adopted in more general cases are indicated in Appendices C and M. The equations of motion are presented here in their full component form, but a matrix form is given in Appendix M.

10.1. *General Equations of Motion for a Rigid Object of Constant Mass.*

It will be assumed that there are no forces or moments present due to contact with the earth, but if this were not true the symbols X, Y, Z, $\mathscr{L}$, $\mathscr{M}$, $\mathscr{N}$ would merely be replaced by $X+X^{C}$, etc. according to the notation explained in Section 3. X, etc. represent the forces and moments other than gravitational and earth contact effects.

The equations of motion of a rigid object of constant mass (m) in still air and referred to any system of body axes with origin at the c.g. are as follows:

$$\left.\begin{aligned} m(\dot{u}+qw-rv) &= X+mg_x \\ m(\dot{v}+ru-pw) &= Y+mg_y \\ m(\dot{w}+pv-qu) &= Z+mg_z \end{aligned}\right\} \tag{10.1}$$

$$\left.\begin{aligned} I_x\dot{p}-I_{yz}(q^2-r^2)-I_{zx}(\dot{r}+pq)-I_{xy}(\dot{q}-rp)-(I_y-I_z)\,qr &= \mathscr{L} \\ I_y\dot{q}-I_{zx}(r^2-p^2)-I_{xy}(\dot{p}+qr)-I_{yz}(\dot{r}-pq)-(I_z-I_x)\,rp &= \mathscr{M} \\ I_z\dot{r}-I_{xy}(p^2-q^2)-I_{yz}(\dot{q}+rp)-I_{zx}(\dot{p}-qr)-(I_x-I_y)\,pq &= \mathscr{N} \end{aligned}\right\} \tag{10.2}$$

The moments of inertia (I_x, I_y, I_z) and the products of inertia (I_{yz}, I_{zx}, I_{xy}) of a rigid object are of course constant. The symbols u, v, w; p, q, r denote the components of linear velocity of the c.g. and the angular velocity of the body respectively. The components of the weight are denoted by mg_x, mg_y, mg_z, and these may be expressed in various forms in terms of attitude angles, as shown in equations (5.14). A 'dot' over a variable denotes differentiation with respect to time.

It is usually convenient to reduce equations (10.1) and (10.2) to a concise form by dividing the force equations through by the mass, and each moment equation by the appropriate moment of inertia. We obtain

$$\left.\begin{aligned} \dot{u}+qw-rv-g_x &= \frac{X}{m}, \\ \dot{v}+ru-pw-g_y &= \frac{Y}{m}, \\ \dot{w}+pv-qu-g_z &= \frac{Z}{m}, \end{aligned}\right\} \tag{10.3}$$

$$\left.\begin{aligned} \dot{p}+d_x(q^2-r^2)+e_x(\dot{r}+pq)+f_x(\dot{q}-rp)+b_xqr &= \frac{\mathscr{L}}{I_x}, \\ \dot{q}+e_y(r^2-p^2)+f_y(\dot{p}+qr)+d_y(\dot{r}-pq)+b_yrp &= \frac{\mathscr{M}}{I_y}, \\ \dot{r}+f_z(p^2-q^2)+d_z(\dot{q}+rp)+e_z(\dot{p}-qr)+b_zpq &= \frac{\mathscr{N}}{I_z}, \end{aligned}\right\} \tag{10.4}$$

where d_x, d_y, d_z; e_x, e_y, e_z; f_x, f_y, f_z; b_x, b_y, b_z are the constant-of-inertia ratios defined in Section 9.

10.2. *Equations of Motion Expanded in Terms of Force and Moment Derivatives.*

10.2.1. *General form of the equations.* It is often convenient to expand the forces X, Y, Z and the moments $\mathscr{L}, \mathscr{M}, \mathscr{N}$ on the right-hand sides of equations (10.1) and (10.2) in terms of the variables defining the motion, including the motivator deflections. There are several ways of doing this, since alternative sets of variables exist and also alternative expressions for the forces and moments. The notation required for coping with the various expansions is discussed in Part 4, and in this Section just one form of expansion is given for illustration.

It should be noted that the equations of motion are very often put in a normalised form to suit the dynamic analysis, and that the force and moment derivatives are almost invariably normalised to suit the aerodynamicist. It is thought better, however, to postpone the presentation of normalised systems until Section 11, and to develop the equations first in their basic form.

We may express the force Z as a Taylor series in terms of increments* such as u', w', q', h', η', ... from datum values u_e, w_e, q_e, h_e, η_e, and in terms of partial derivatives such as $Z_u = \partial Z/\partial u$, which are evaluated at the datum values u_e, w_e, ... The full Taylor expansion is written

$$Z = Z_e + \left(T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots\right) Z, \tag{10.5}$$

where T stands for the differential operator

$$T \equiv h'\frac{\partial}{\partial h} + \left(u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v} + w'\frac{\partial}{\partial w}\right) + \left(p'\frac{\partial}{\partial p} + q'\frac{\partial}{\partial q} + r'\frac{\partial}{\partial r}\right) + \left(\dot{u}'\frac{\partial}{\partial \dot{u}} + \dot{v}'\frac{\partial}{\partial \dot{v}} + \dot{w}'\frac{\partial}{\partial \dot{w}}\right) + \dots .$$

Retaining only a few typical variables we thus write

$$Z = Z_e + Z_u u' + Z_w w' + Z_{\dot{w}} \dot{w}' + Z_q q' + Z_h h' + Z_\eta \eta' + \tfrac{1}{2} Z_{uu} u'^2 + \dots ,$$

and likewise

$$\mathcal{M} = \mathcal{M}_e + M_u u' + M_w w' + \dots ,$$

the script letters not being required for the derivatives $\partial \mathcal{M}/\partial u$, etc. The reference values Z_e, $\mathcal{M}_e$ are the values of Z, $\mathcal{M}$ when the expansion variables u, w, ... have their datum values.

These partial derivatives Z_u, ... are called *force* or *moment derivatives*, and it is convenient to define related *concise derivatives*, such as

$$x_u = -\frac{X_u}{m}, \qquad z_w = -\frac{Z_w}{m},$$

$$m_q = -\frac{M_q}{I_y}, \qquad n_\zeta = -\frac{N_\zeta}{I_z},$$

and likewise other *concise quantities* such as $z_e = -Z_e/m$, so that on substituting in the equations of motion we obtain the compact forms

$$\dot{u} + qw - rv - g_x + x_e + x_u u' + x_w w' + \dots\dots = 0, \text{etc.}$$

---

*Sometimes variables representing deviations from the datum condition will be used which are functions of the increments and more convenient than the increments themselves. This is especially so for representing perturbations in the attitude or position of the aircraft. Terms of this character are not included here since in free stream the aerodynamic forces and moments do not depend on attitude or position (other than altitude). A fuller discussion of the Taylor expansions will be found in Section 17, and the introduction of additional terms when required, for example on account of ground effect, is straightforward.

The definitions of all concise quantities incorporate negative signs in order to make the majority of these quantities positive. When a concise quantity turns out to be negative it may be convenient to introduce a dressing to denote sign reversal, and the notation suggested is, for example,

$$\bar{n}_v = -n_v = \frac{N_v}{I_z},$$

so that $\bar{n}_v$ and N_v have the same sign. Such a 'reversed' concise quantity* could be referred to as 'n-v-neg'.

As an example of the equations of motion we give below those obtained when the x and z-axes lie in a plane of mass symmetry so that I_{yz} and I_{xy}, and hence $d_x, d_y, d_z, f_x, f_y, f_z$, are zero.

$$\left.\begin{aligned}
&\dot{u}+qw-rv-g_x+x_e+x_u u'+x_w w'+x_q q'+x_{\dot{w}}\dot{w}'+x_h h'+x_\eta \eta'+\ldots = 0\\
&\dot{v}+ru-pw-g_y+y_e+y_v v'+y_p p'+y_r r'+y_\xi \xi'+y_\zeta \zeta'+\ldots = 0\\
&\dot{w}+pv-qu-g_z+z_e+z_u u'+z_w w'+z_q q'+z_{\dot{w}}\dot{w}'+z_h h'+z_\eta \eta'+\ldots = 0\\
&\dot{p}+e_x(\dot{r}+pq)+b_x qr+l_e+l_v v'+l_p p'+l_r r'+l_\xi \xi'+l_\zeta \zeta'+\ldots = 0\\
&\dot{q}+e_y(r^2-p^2)+b_y rp+m_e+m_u u'+m_w w'+m_q q'+m_{\dot{w}}\dot{w}'+\ldots = 0\\
&\dot{r}+e_z(\dot{p}-qr)+b_z pq+n_e+n_v v'+n_p p'+n_r r'+n_\xi \xi'+n_\zeta \zeta'+\ldots = 0
\end{aligned}\right\} \quad (10.6)$$

For principal axes the d's, e's, and f's would be zero. We have retained only a representative selection of the variables and of first-order derivatives. In general other terms, including higher-order derivatives, should be added as required, on the basis of (10.5).

This form of expansion is well suited to problems in which the motion can be treated as a disturbance from some fixed datum condition of flight: that is, the undisturbed flight is a steady condition corresponding to constant values of all variables except those that are naturally absent or redundant. The possible sorts of such undisturbed flight are

(a) level flight (straight or turning) at constant speed,

(b) non-level flight (straight or turning) at constant speed provided that the atmosphere can be treated as uniform.

In these steady states $u, v, w, p, q, r, \Phi, \Theta$ are constant, while x, y, z, h, or Ψ may be absent or redundant.

It should be noted that if the motivators are fixed steady turning flight is possible about a vertical axis only, and the datum values of the angular velocity components are related to the datum attitude as follows:

$$p_e = -\Omega_e \sin\Theta_e,$$

$$q_e = \Omega_e \sin\Phi_e \cos\Theta_e,$$

$$r_e = \Omega_e \cos\Phi_e \cos\Theta_e,$$

where $\Omega_e = \bar{\Psi}_f$ and denotes the steady resultant angular velocity.

It may be practicable to employ the form of expansion given above even when the datum values of some variables are not constant, provided that these values are defined in some functional way and always

*Confusion with mean derivatives (*see* Section K.2) is unlikely, but in emergency n-v-neg could be written as $\underline{n}_v$ or n_v^-.

provided that increments relative to datum values are sufficiently small for the expansions to be valid. This technique will introduce derivatives that are functions of time or of other variables. As mentioned previously the suffix f should replace e when datum values are not constant.

In some applications, for example stability investigations, the disturbances are assumed to be so small that the equations may be linearized, and the higher-order derivatives are taken as zero. Acceleration terms such as $(\dot{u}+qw-rv)$ are then written as $(\dot{u}'+q_e w'-r_e v'+w_e q'-v_e r')$, and so on (*see* Section 10.2.2).

The more general types of undisturbed flight conditions such as climbing or diving in a non-uniform atmosphere, or flight at varying speed, are probably best treated by expanding the equations in terms of force- and moment-coefficient derivatives as set out in Section 10.3; otherwise the accumulating deviations in altitude or speed would entail a large number of terms in the Taylor series to preserve much degree of accuracy, and it would be difficult and rather artificial to relate the corresponding high-order derivatives to the aerodynamic data by relations like those in Appendix E.

To solve equation (10.6) we must use the kinematic relationships between the angular velocity components p, q, r and the rates of change of attitude angles, examples of which are equations (5.3) and (5.12). Normally the same attitude angles will be chosen for this purpose and for expressing the gravity components g_x, g_y, g_z as for example in equations (5.14).

We must also know how the increments in equivalent motivator deflections ξ', η', ζ', ν', δ', κ' vary. They may, for example, be fixed, or changed by an automatic pilot. In the latter case ξ', etc. may be functions not only of the increments in velocity components u', v', w', p', q', r', but also of perturbations in attitude angles: on occasion they may also depend on variables defining the deviations of the aircraft from some datum flight path. When such additional variables are introduced, further kinematic relationships will be required of the form given in Section 7.

In general, equations (10.6) would only be solved by means of a computing machine, but in particular cases the equations can be linearized and useful analytical solutions can result. Sections 10.2.2 and 10.2.3 give examples of a linearized treatment which may be applicable to all types of aircraft, including missiles, and an example especially pertinent to spinning missiles is given in Appendix F. Appendix G provides an example of the linearized equations expressed in terms of 'displacement' variables. Section 12 develops some aspects of the solution for the example of Section 10.2.2.

10.2.2. *Linearized equations for small disturbances from a steady state flight condition.* As mentioned in Section 10.2.1 it is convenient to make use of body axes with the xz-plane in a plane of symmetry of the aircraft. Let us define a steady state as one in which the linear and angular velocity components are constant—these values u_e, v_e, w_e, p_e, q_e, r_e will usually be taken as datum values. This definition implies that the datum flight condition must either be level ($\gamma_e = 0$) or be assumed to exist in a uniform atmosphere (forces and moments independent of deviation in height, h'). Motivator deflections such as ξ may have to vary in the datum state in order to maintain a prescribed constant angular velocity such as p_e, and increments would then be written as $\xi' = \xi - \xi_f$, but in this case it may well be preferable to linearize the equations for small perturbations with respect to arbitrary constant values of the variables. If the motivator deflections are constant in the datum flight condition, the latter must consist of straight flight or circling about a vertical axis at a rate Ω_e such that

$$\left.\begin{aligned} p_e &= -\Omega_e \sin\Theta_e\,, \\ q_e &= \Omega_e \sin\Phi_e \cos\Theta_e\,, \\ r_r &= \Omega_e \cos\Phi_e \cos\Theta_e\,. \end{aligned}\right\} \tag{10.7}$$

When these equations are not satisfied the terms in ξ, etc. must be eliminated by manipulation of equations (10.6) and complicated forms will result[31]. In this Report we are giving examples merely to display the notation, and we shall restrict the remainder of this Section to problems where equations (10.7) are satisfied.

With the choice of axes and datum condition specified above, the linearized forms of equations (10.6) are as follows*.

$$\left.\begin{aligned}
\dot{u}'+q_e w'+w_e q'-r_e v'-v_e r'-g_x'+\Sigma x_u u' &= 0\\
\dot{v}'+r_e u'+u_e r'-p_e w'-w_e p'-g_y'+\Sigma y_v v' &= 0\\
\dot{w}'+p_e v'+v_e p'-q_e u'-u_e q'-g_z'+\Sigma z_w w' &= 0\\
\dot{p}'+e_x(\dot{r}'+p_e q'+q_e p')+b_x(q_e r'+r_e q')+\Sigma l_p p' &= 0\\
\dot{q}'+2e_y(r_e r'-p_e p')+b_y(r_e p'+p_e r')+\Sigma m_q q' &= 0\\
\dot{r}'+e_z(\dot{p}'-q_e r'-r_e q')+b_z(p_e q'+q_e p')+\Sigma n_r r' &= 0
\end{aligned}\right\}\qquad(10.8)$$

If in addition the angular velocity of the body is zero in the datum condition, considerable simplification ensues because $p_e = q_e = r_e = 0$, and the equations become

$$\left.\begin{aligned}
\dot{u}'+w_e q'-v_e r'-g_x'+\Sigma x_u u' &= 0\,,\\
\dot{v}'+u_e r'-w_e p'-g_y'+\Sigma y_v v' &= 0\,,\\
\dot{w}'+v_e p'-u_e q'-g_z'+\Sigma z_w w' &= 0\,,\\
\dot{p}'+e_x\dot{r}'+\Sigma l_p p' &= 0\,,\\
\dot{q}'+\Sigma m_q q' &= 0\,,\\
\dot{r}'+e_z\dot{p}'+\Sigma n_r r' &= 0\,.
\end{aligned}\right\}\qquad(10.9)$$

It is often permissible to break up these equations into two groups of three: the first, third, and fifth constitute the longitudinal group, and the others the lateral group. It may then be possible to solve separately for the longitudinal variables u', w', q', θ (and perhaps h'), and likewise for the lateral variables v', p', r', ϕ, ψ; and perhaps for variables defining the position of the aircraft c.g. relative to the datum flight path. When this is so, independent longitudinal and lateral motions can exist with no coupling between them.

The conditions for separating out the longitudinal equations are more stringent than those for the lateral equations, and the latter may therefore sometimes be immediately soluble when the longitudinal are not. It is then possible to substitute the solutions for the lateral variables into the longitudinal equations thereby making the coupling terms known functions and rendering these equations consequently soluble as a group.

The lateral equations of (10.9) can be treated separately if the cross derivatives such as Y_u, L_w, N_q are zero, with the additional requirement that either Φ_e and Ψ_e must be zero or the longitudinal perturbations u', w', q', θ must be zero if v_0 or v_E enter into the calculation.

The longitudinal equations can always be considered separately if the lateral perturbations v', p', r', ϕ, ψ are zero. When, however, lateral perturbations are permitted, we can separate the longitudinal equations of (10.9) only if the datum condition is symmetric (v_e and Φ_e zero) and if cross derivatives such as X_v, Z_p, M_r are zero, with the additional requirement that Ψ_e must be zero if u_0, w_0, or w_E enter into the calculation.

---

*In practice no confusion is likely if the dashes are omitted when linearized equations are being used exclusively after they are established.

10.2.3. *Linearized equations for small disturbances from straight symmetric equilibrium flight.* If the steady state is one of straight symmetric flight then the equations of motion are separable into longitudinal and lateral groups as explained in Section 10.2.2, since $v_e = p_e = q_e = r_e = \Phi_e = \Psi_e = 0$. From equations (10.9) and (5.15) the longitudinal equations are

$$\left.\begin{aligned} &\dot{u}' + g_1\theta + x_u u' + x_w w' + (x_q + w_e)\,q' + x_{\dot{w}}\dot{w}' + x_v v' + x_\eta \eta' + x_\kappa \kappa' = 0\,, \\ &\dot{w}' + g_2\theta + z_u u' + z_w w' + (z_q - u_e)\,q' + z_{\dot{w}}\dot{w}' + z_v v' + z_\eta \eta' + z_\kappa \kappa' = 0\,, \\ &\dot{q}' + m_u u' + m_w w' + m_q q' + m_{\dot{w}}\dot{w}' + m_v v' + m_\eta \eta' + m_\kappa \kappa' = 0\,, \end{aligned}\right\} \tag{10.10}$$

and the lateral equations are

$$\left.\begin{aligned} &\dot{v}' - g_1\phi - g_2\psi + y_v v' + (y_p - w_e)\,p' + (y_r + u_e)\,r' + y_\xi \xi' + y_\delta \delta' + y_\zeta \zeta' = 0\,, \\ &\dot{p}' + e_x\dot{r}' + l_v v' + l_p p' + l_r r' + l_\xi \xi' + l_\delta \delta' + l_\zeta \zeta' = 0\,, \\ &\dot{r}' + e_z\dot{p}' + n_v v' + n_p p' + n_r r' + n_\xi \xi' + n_\delta \delta' + n_\zeta \zeta' = 0\,. \end{aligned}\right\} \tag{10.11}$$

It will be observed that in equations (10.10) and (10.11) the only 'dot' derivatives included are those with respect to w, as is usual, but more could be included if they were found to be significant. Derivatives with respect to all six equivalent motivator deflections ξ, η, ζ, ν, δ, κ have been included for generality.

Letting D denote d/dt and rearranging the above equations, we obtain

$$\left.\begin{aligned} &(D + x_u)\,u' + (x_{\dot{w}}D + x_w)\,w' + (x_q + w_e)\,q' + g_1\theta + x_v v' + x_\eta \eta' + x_\kappa \kappa' = 0\,, \\ &z_u u' + [(1 + z_{\dot{w}})\,D + z_w]\,w' + (z_q - u_e)\,q' + g_2\theta + z_v v' + z_\eta \eta' + z_\kappa \kappa' = 0\,, \\ &m_u u' + (m_{\dot{w}}D + m_w)\,w' + (D + m_q)\,q' + m_v v' + m_\eta \eta' + m_\kappa \kappa' = 0\,, \end{aligned}\right\} \tag{10.12}$$

and

$$\left.\begin{aligned} &(D + y_v)\,v' + (y_p - w_e)\,p' - g_1\phi + (y_r + u_e)\,r' - g_2\psi + y_\xi \xi' + y_\delta \delta' + y_\zeta \zeta' = 0\,, \\ &l_v v' + (D + l_p)\,p' + (e_x D + l_r)\,r' + l_\xi \xi' + l_\delta \delta' + l_\zeta \zeta' = 0\,, \\ &n_v v' + (e_z D + n_p)\,p' + (D + n_r)\,r' + n_\xi \xi' + n_\delta \delta' + n_\zeta \zeta' = 0\,. \end{aligned}\right\} \tag{10.13}$$

The linearized kinematic equations required for solving the equations of motion are given in Sections 5.5 and 7, but are noted here for convenience:

$$\left.\begin{aligned} D\theta &= q' \\ Dz_E^+ &= w_E^+ = -u'\sin\alpha_e + w'\cos\alpha_e - V_e\theta \\ Dh' = -Dz_0^+ &= -w_0^+ = u'\sin\Theta_e - w'\cos\Theta_e + V_e\theta\cos\gamma_e \end{aligned}\right\} \tag{10.14}$$

$$\left.\begin{aligned} D\phi &= p' \\ D\psi &= r' \\ Dy_E^+ &= v_E^+ = v' - V_e\phi \sin\alpha_e + V_e\psi\cos\alpha_e \\ &= v' - w_e\phi + u_e\psi \end{aligned}\right\} \tag{10.15}$$

The equations for v_E^+ and w_E^+ are needed when the motivator deflections depend on the distances y_E^+ and z_E^+ of the centre of gravity of the aircraft to the right of and below its datum flight path respectively, as for example when an automatic pilot controls the distances from an airport approach beam. The equations for $\dot\phi$, $\dot\theta$, $\dot\psi$ are always required. The equation for $\dot h'$ is required for disturbances from level flight in a non-uniform atmosphere, or when the motivator deflections depend on h'—which may arise in level flight ($\gamma_e = 0$), or in climbing or diving flight in an atmosphere that can be treated as uniform. For the non-uniform atmosphere cases, altitude-dependent terms $x_h h'$, $z_h h'$, $m_h h'$ must be added in equations (10.12).

The above equations may be modified so as to contain terms in Φ', Θ', Ψ' instead of ϕ, θ, ψ. From equations (5.14) we have for $\Phi_e = 0$

$$g_x' = -g_1\theta = -g_1\Theta',$$

$$g_y' = g_1\phi + g_2\psi = g_1\Phi',$$

$$g_z' = -g_2\theta = -g_2\Theta',$$

and to make these substitutions is the only modification required in equations (10.12) and (10.13), and also in (10.14), but (10.15) must be replaced by

$$\left.\begin{aligned} D\Phi' &= p' + r'\tan\Theta_e, \\ D\Psi' &= r'\sec\Theta_e, \\ Dy_E^+ &= v' - V_e\Phi'\sin\alpha_e + V_e\Psi'\cos\gamma_e. \end{aligned}\right\} \tag{10.16}$$

10.3. *Equations of Motion Expanded in Terms of Aerodynamic-Coefficient Derivatives.*

As explained in Section 10.2, expansions of the forces and moments in the form

$$Z = Z_e + Z_u u' + \dots$$

are not suitable when the undisturbed flight involves varying conditions such as climbing or diving in a non-uniform atmosphere, or flight at substantially varying speed. Alternative expansions may be used, however, of the form

$$\begin{aligned} Z &= \tfrac{1}{2}\rho V^2 S\, C_Z \\ &= \tfrac{1}{2}\rho V^2 S\left[C_{Ze} + \left(T + \frac{T^2}{2!} + \frac{T^3}{3!} + \dots\right) C_Z\right], \end{aligned} \tag{10.17}$$

where T is a differential operator given by

$$T \equiv \left(\alpha'\frac{\partial}{\partial\alpha} + \beta'\frac{\partial}{\partial\beta}\right) + \left(p_\gamma'\frac{\partial}{\partial p_\gamma} + q_\gamma'\frac{\partial}{\partial q_\gamma} + r_\gamma'\frac{\partial}{\partial r_\gamma}\right) + \left(\dot V_\gamma'\frac{\partial}{\partial \dot V_\gamma} + \dot\alpha_\gamma'\frac{\partial}{\partial\dot\alpha_\gamma} + \dot\beta_\gamma'\frac{\partial}{\partial\dot\beta_\gamma}\right) + \dots,$$

C_Z is a non-dimensional coefficient, ρ denotes the air density, S a representative area, and the normalised variables α, β, p_γ, q_γ, r_γ, $\dot{V}_\gamma$, $\dot{\alpha}_\gamma$, $\dot{\beta}_\gamma, \ldots$ are non-dimensional parameters corresponding to w, v, p, q, r, $\dot{u}$, $\dot{w}$, $\dot{v}, \ldots$ respectively. Thus α represents an incidence angle, and q_γ is equal to ql/V, where l is the representative length introduced when forming the moment coefficients, such as $C_m = \mathscr{M}/\frac{1}{2}\rho V^2Sl$. Further details are given in Part 4, but it is worth noting here that different values of l may be used for longitudinal and lateral moment coefficients, and that if we have $C_l = \mathscr{L}/\frac{1}{2}\rho V^2Sl_2$, $C_m = \mathscr{M}/\frac{1}{2}\rho V^2Sl_1$, then the normalised variables should be correspondingly defined: $p_\gamma = pl_2/V$, $q_\gamma = ql_1/V$.

If we write $C_{Z\alpha}$ for $\partial C_Z/\partial\alpha$, and C_{Zq} for $\partial C_Z/\partial q_\gamma$, etc., equation (10.17) may be written

$$Z = \tfrac{1}{2}\rho V^2S(C_{Ze}+C_{Z\alpha}\alpha'+C_{Z\beta}\beta'+C_{Zq}q'_\gamma+\ldots),$$

in which $\frac{1}{2}\rho V^2S$ may be replaced by $\frac{1}{2}PM^2\gamma S$, where P denotes the air pressure, γ the ratio of specific heats, and M the Mach number.

The reference value C_{Ze} and the derivatives $C_{Z\alpha}, \ldots$ are evaluated at the datum values of the parameters α, β, $p_\gamma, \ldots$. There is no objection to writing C_{Z_α} instead of $C_{Z\alpha}$, except for the possibility of confusion with the American C_{Z_α}, which does not have exactly the same meaning and is more analogous to our $\mathring{Z}_w$. Similar remarks apply to all the coefficient derivatives.

We give below a typical set of equations in terms of expansions of the form (10.17), and corresponding to equations (10.6). As before we retain only a selection of variables and first-order derivatives.

$$\left.\begin{aligned}
\dot{u}+qw-rv-g_x &= \frac{\frac{1}{2}\rho V^2S}{m}\left[C_{Xe}+C_{X\alpha}\alpha'+C_{X\beta}\beta'+C_{Xp}p'_\gamma+C_{Xq}q'_\gamma+C_{X\dot{\alpha}}\dot{\alpha}'_\gamma+C_{X\eta}\eta'+\ldots\right]\\
\dot{v}+ru-pw-g_y &= \frac{\frac{1}{2}\rho V^2S}{m}\left[C_{Ye}+C_{Y\alpha}\alpha'+C_{Y\beta}\beta'+C_{Yp}p'_\gamma+C_{Yr}r'_\gamma+C_{Y\xi}\xi'+C_{Y\zeta}\zeta'+\ldots\right]\\
\dot{w}+pv-qu-g_z &= \frac{\frac{1}{2}\rho V^2S}{m}\left[C_{Ze}+C_{Z\alpha}\alpha'+C_{Z\beta}\beta'+C_{Zp}p'_\gamma+C_{Zq}q'_\gamma+C_{Z\dot{\alpha}}\dot{\alpha}'_\gamma+C_{Z\eta}\eta'+\ldots\right]\\
\dot{p}+e_x(\dot{r}+pq)+b_xqr &= \frac{\frac{1}{2}\rho V^2Sl_2}{I_x}\left[C_{le}+C_{l\alpha}\alpha'+C_{l\beta}\beta'+C_{lp}p'_\gamma+C_{lr}r'_\gamma+C_{l\xi}\xi'+C_{l\zeta}\zeta'+\ldots\right]\\
\dot{q}+e_y(r^2-p^2)+b_yrp &= \frac{\frac{1}{2}\rho V^2Sl_1}{I_y}\left[C_{me}+C_{m\alpha}\alpha'+C_{m\beta}\beta'+C_{mq}q'_\gamma+C_{m\dot{\alpha}}\dot{\alpha}'_\gamma+C_{m\eta}\eta'+\ldots\right]\\
\dot{r}+e_z(\dot{p}-qr)+b_zpq &= \frac{\frac{1}{2}\rho V^2Sl_2}{I_z}\left[C_{ne}+C_{n\alpha}\alpha'+C_{n\beta}\beta'+C_{np}p'_\gamma+C_{nr}r'_\gamma+C_{n\xi}\xi'+C_{n\zeta}\zeta'+\ldots\right]
\end{aligned}\right\} \quad (10.18)$$

In these equations the reference values (C_{Xe}, etc.) and the derivatives of the aerodynamic coefficients are funtions of M, R, $z_{0\gamma}$, as explained in Section 17, and it may be possible in particular cases to represent them over a limited range by means of simpler functions or power series.

To solve equations (10.18) we require kinematic relationships and a knowledge of how the motivator deflections depend on the other variables. In addition we need relations between the variables on the left-hand sides and those on the right-hand sides. The increments, such as α', are obtained by subtracting the datum values from the current values, which are given by

$$\left.\begin{aligned}
\alpha &= \sin^{-1}(w/V)\,,\\
\beta &= \sin^{-1}(v/V)\,,\\
p_\gamma &= pl_2/V\,,\\
q_\gamma &= ql_1/V\,,\\
r_\gamma &= rl_2/V\,,\\
\dot{\alpha}_\gamma &= \frac{(V\dot{w}-\dot{V}w)\,l_1}{V^2(V^2-w^2)^{\frac{1}{2}}}\,,\\
\dot{\beta}_\gamma &= \frac{(V\dot{v}-\dot{V}v)\,l_2}{V^2(V^2-v^2)^{\frac{1}{2}}}\,,
\end{aligned}\right\} \tag{10.19}$$

where $V^2 = u^2+v^2+w^2$, and

$$M = V/a\,, \tag{10.20}$$

where a represents the speed of sound. Both ρ and a depend on the altitude h for a given atmosphere. The gravity components g_x, g_y, g_z may be represented in any form, for example as in equations (5.14), and h is determined from the equation

$$\dot{h} = -w_0 = ul_3+vm_3+wn_3\,, \tag{10.21}$$

where l_3, m_3, n_3 are the direction cosines of the downward vertical relative to the body axes.

The set of equations (10.18) to (10.21) together with the control equations and necessary kinematic relationships would usually be solved by means of a computing machine. The output in terms of u, v, w, p, q, r can if required be transformed to components along earth axes by applying the matrix relationships given in Section 5.6.

11. *Normalised Equations of Motion.*

11.1. *Basis of Normalising Procedure.*

It has been explained in Section 2 that physical quantities can be normalised by a process of dividing each quantity by some unit (constant in time) that has a physical significance in the particular context. It is convenient to choose the units to be mutually consistent in the same way as units are consistent within an 'ordinary' system, such as the ft slug sec or the Standard International m kg sec system, based on statutory units. A normalised quantity may then be interpreted as the value of the quantity expressed in normalised units. Some users may prefer to regard the normalising process as a non-dimensionalising one and to interpret the units merely as divisors. It is probably helpful to preserve a dual point of view: the more important feature of the normalising procedure proposed here is that the divisors, units, or scaling factors—call them what you will—are mutually consistent. A complete set may then always be derived by dimensional analysis once any three independent ones have been specified. In general, a non-dimensionalising or normalising process does not have this restriction on the divisors (which may even vary with time), and the system of units viewpoint may therefore help to emphasise the restriction when it applies. Another advantage of this interpretation is that only one symbol need be employed for denoting a physical quantity, although distinguishing marks may be required when, as in this Report, a quantity is expressed in more than one system.

It does not seem practicable to have one set of normalising units that will satisfy the needs of people who are chiefly interested in aircraft dynamics as well as people who are primarily interested in aerodynamic forces alone. This Report suggests the adoption of two systems to meet the conflicting requirements. One is called the *dynamic-normalised system* and the other the *aero-normalised system.* Dynamical investigations will most often make use of equations of motion that have been normalised according to the dynamic system. The coefficients of the various terms of aerodynamic origin, however, will depend on aerodynamic data that are best normalised according to the aero-system. At this stage we postpone the detailed development of the aero-system until Part 4 and merely note that any quantity Z expressed in the aero-system may be denoted by $\check{Z}$ (Z 'dip'), and that $\hat{Z}$ (Z 'cap') may be used to denote the same quantity expressed in the dynamic system. If the need should arise for emphasising that Z is being expressed in ordinary units, the symbol $\mathring{Z}$ (Z 'ord') could be used.

The normalised forms for presenting the equations and related aerodynamic data are essentially devised to deal with small disturbances from a datum condition, and corresponding units in the two systems are chosen to be simply related, or where possible identical. To this end a speed, such as u, is normalised by dividing its value by a unit of speed equal in magnitude to V_e, where V_e represents a datum value of the aircraft speed (V). We may thus write $\hat{u} = \check{u} = u/V_e$. Both u and V_e are usually expressed in ordinary units (o.u.), and the normalised value of u is the same whatever system of ordinary units is employed—as would be expected from a process which can be regarded as non-dimensionalising.

Similarly, the unit of force is made equal to $\frac{1}{2}\rho_e V_e^2 S$ (o.u.), where ρ_e is a datum value of ρ, and the normalised forces may thus be directly connected with the non-dimensional force coefficients, such as C_Z, which are formed by means of a divisor $\frac{1}{2}\rho V^2 S$. There are consequent connections, often fairly simple, between normalised derivatives of forces and moments and the derivatives of the corresponding force and moment coefficients (*see* Section 19).

The natural choice for datum values would be within the range of variation, and if the latter were small then the simple connections just mentioned would be realised. If, however, the range of variation were large there would be no advantage in this type of normalisation.

To complete the dynamic-normalised system we specify the third unit to be the unit of mass, and equal to the aircraft mass* m (o.u.). It follows then, for example, that the units of length and time are of magnitude $m_e/\frac{1}{2}\rho_e S$ and $m_e/\frac{1}{2}\rho_e V_e S$ respectively, which can also be written as μl_0 and $\mu l_0/V_e$ if we define the *relative density parameter* to be

$$\mu = \frac{m_e}{\frac{1}{2}\rho_e S l_0}, \tag{11.1}$$

where l_0 is a representative length.

As already stated the aero-normalised system is based on the same units of speed and force as the dynamic system, but we complete the system in this case by specifying the unit of length to be equal to the representative length used in defining μ. It follows that the units of mass and time are of magnitude $\frac{1}{2}\rho_e S l_0 = m_e/\mu$ and l_0/V_e respectively.

When an aircraft is hovering or flying at very low speed the datum speed used in the normalised procedure should be chosen on a different basis. For example, if there is a rotor of radius R and angular

*Some datum mass (m_e) must be taken if the mass of the aircraft varies, for example owing to fuel consumption. There should not be any confusion between this symbol and the concise quantity corresponding to $\mathcal{M}_e$.

velocity Ω, the datum tip speed $\Omega_e R$ should replace V_e—the representative length l_0 would then be taken equal to R (*see* Appendix J).

An important consequence of the above choice of units for the two normalised systems is that each of the three units usually considered to be basic, namely those of length, mass, and time, is μ times as large in the dynamic system as it is in the aero-system. It is found that the length μl_0 (o.u.) and the time $\mu l_0/V_e$ (o.u.) are usually convenient in relation to deviations in flight path and to dynamic behaviour, whereas the length l_0 (o.u.) and the time l_0/V_e (o.u.) are much too small to be generally suitable for work in dynamics. This is so because the representative length will be chosen to be convenient for aerodynamic work: it will normally be equal to the representative length (l) used to form some of the aerodynamic moment coefficients.

It is not always possible to find just one representative length that will be convenient for all aerodynamic work even on one aircraft. Thus longitudinal and lateral aerodynamic coefficients will often be in terms of different lengths l_1 and l_2, so that the choice of the aero-normalised unit of length becomes arbitrary. So long as the equations of motion may be separated into longitudinal and lateral groups there is no difficulty in using l_1 for longitudinal studies and l_2 for lateral, but care must be taken when dealing with 'coupled' motions.

When the equations are in terms of concise quantities, and normalised according to the dynamic system, the coefficients are independent of the choice of representative length, and it is merely necessary to be sure that in the determination of the coefficients l_1 is consistently used in association with longitudinal terms, and l_2 with lateral terms. This is explained further in Section 11.2. When, however, the equations of motion are normalised according to the aero-system (or are in terms of aerodynamic-coefficient derivatives), one or the other of l_1 and l_2 should be taken as the representative length for the dynamical work. Either the lateral or longitudinal terms will then have to be multiplied by factors such as l_2/l_1. Such a factor might be called the 'lateral length ratio', and is analogous to the 'tail volume' ratio that has up to now been employed to form derivatives based on wing dimensions from those based on tail dimensions.

As in ordinary systems of units, it is difficult to invent names for more than a few units, and it is therefore suggested that the abbreviations d.n.u. (dyanamic-normalised units) and a.n.u. (aero-normalised units) may be useful. Thus a speed u (o.u.) would be equal to $\check{u}$ (a.n.u.). Although the units of speed and force are the same for the two normalising systems, they have an aerodynamic flavour, and it may be considered helpful to express the normalised speed as $\check{u}$ aerospeeds and the normalised force $\check{Z}$ (a.n.u.) as $\check{Z}$ aeroforces. This may appeal in particular to those who want to emphasise the non-dimensional nature of the normalised quantities. Similarly a length x (o.u.) and a time t (o.u.) would be equal to $\check{x}$ aerolengths or $\hat{x}$ dylengths, and $\check{t}$ aeroseconds or $\hat{t}$ dyseconds, respectively. There does not seem to be a need for naming any other units.

11.2. *The Dynamic-Normalised Equations.*

Since the normalising units form a consistent set, the formal appearance of the equations of motion will be unchanged. The terms on the left-hand side of equations (10.3) and (10.4) may be written in terms of $\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}, \hat{r}$ instead of u, v, w, p, q, r; of $\hat{g}_x, \hat{g}_y, \hat{g}_z$ instead of g_x, g_y, g_z; and of $\hat{D}$ instead of D, the differential operator d/dt.

If we denote the normalising time unit by $\tau = m_e/\frac{1}{2}\rho_e V_e S$, we may write

$$\hat{D} = \tau D,$$

where $\hat{D} \equiv d/d\hat{t}$, and the normalising divisors for the other quantities mentioned above are as follows.

Quantity	Normalising divisor
u, v, w	V_e
p, q, r	$1/\tau$
g_x, g_y, g_z	$\frac{1}{2}\rho_e V_e^2 S/m_e$

For example, $\hat{u} = u/V_e$, $\hat{p} = p\tau$, $\hat{g}_x = m_e g_x/\frac{1}{2}\rho_e V_e^2 S$.

When the right-hand sides of equations (10.3) and (10.4) are expanded in terms of force and moment derivatives, it is preferable to normalise these derivatives according to the aero-system (*see* Section 17). To form a dynamic-normalised concise derivative, such as $\hat{m}_w$, from the corresponding aero-normalised moment derivative, such as $\breve{M}_w$, it is convenient to define *inertia parameters* i_x, i_y, i_z. These are given by

$$(i_x, i_y, i_z) = \frac{1}{m_e l_0^2}(I_x, I_y, I_z),$$

and are therefore equal to the squares of the aero-normalised radii of gyration. The symbols suggested for the radii of gyration are r_x, r_y, r_z, so that $i_x = \breve{r}_x^2$, etc.

The multiplying factors required to form the various concise quantities from the appropriate derivatives (in a.n.u.) are given in Table 8. Factors from the third column will yield concise quantities for use with the dynamic-normalised equations. Factors from the fourth column will give concise quantities appropriate for the equations of motion expressed in ordinary units. For example,

$$\hat{m}_w = -\frac{\mu \breve{M}_w}{i_y}, \qquad \overset{o}{m}_w = -\frac{\mu \breve{M}_w}{i_y V_e \tau^2} = \frac{\hat{m}_w}{V_e \tau^2}.$$

The relevant inertia parameter must be used for a moment or a moment derivative: thus,

$$\hat{l}_v = -\frac{\mu \breve{L}_v}{i_x}, \qquad \hat{n}_v = -\frac{\mu \breve{N}_v}{i_z}$$

An examination of the Table reveals a pattern that can provide a general basis for determining the multiplying factors for any sort of derivative, including higher-order ones. It is easy to show that for any variable, denoted here by the symbol ω, any aero-normalised force derivative such as $\breve{Z}_\omega$ will be related to the corresponding dynamic-normalised concise derivative $\hat{z}_\omega$ according to the equation

$$\frac{\hat{z}_\omega}{\breve{Z}_\omega} = -\frac{\breve{\omega}}{\hat{\omega}}.$$

TABLE 8

Factors for Converting Forces, Moments, and their Derivatives in Aero-Normalised Units to corresponding Concise Quantities in Different Units

$\mu = m_e/\tfrac{1}{2}\rho_e S l_0$ $\qquad\qquad$ $\tau = m_e/\tfrac{1}{2}\rho_e V_e S = \mu l_0/V_e$

Quantity in aero-normalised units	Example	Multiplying factor for forming concise quantity in	
		dynamic-normalised units	ordinary units
Force	$\breve{Z}_e$	-1	$-V_e/\tau$
Moment	$\breve{\mathcal{M}}_e$	$-\mu/i_y$	$-\mu/i_y\tau^2$
Force derivative with respect to			
linear displacement	$\breve{Z}_h$	$-\mu$	$-\mu/\tau^2$
linear velocity	$\breve{Z}_w$	-1	$-1/\tau$
linear acceleration	$\breve{Z}_{\dot{w}}$	$-1/\mu$	$-1/\mu$
angular displacement	$\breve{Z}_\eta$	-1	$-V_e/\tau$
angular velocity	$\breve{Z}_q$	$-1/\mu$	$-V_e/\mu$
Moment derivative with respect to			
linear displacement	$\breve{M}_h$	$-\mu^2/i_y$	$-\mu^2/i_y V_e\tau^3$
linear velocity	$\breve{M}_w$	$-\mu/i_y$	$-\mu/i_y V_e\tau^2$
linear acceleration	$\breve{M}_{\dot{w}}$	$-1/i_y$	$-1/i_y V_e\tau$
angular displacement	$\breve{M}_\eta$	$-\mu/i_y$	$-\mu/i_y\tau^2$
angular velocity	$\breve{M}_q$	$-1/i_y$	$-1/i_y\tau$

Similarly

$$\frac{\hat{m}_\omega}{\breve{M}_\omega} = -\frac{\mu}{i_y}\frac{\breve{\omega}}{\hat{\omega}}.$$

The ratio $\breve{\omega}/\hat{\omega}$ depends only on the dimensions of ω, and is a power of μ as given by the following Table (second column).

It can also be shown that the concise derivatives in ordinary units are related to the corresponding dynamic-normalised derivatives as follows:

$$\frac{\mathring{z}_\omega}{\hat{z}_\omega} = \frac{V_e}{\tau}\frac{\hat{\omega}}{\mathring{\omega}}, \qquad \frac{\mathring{m}_\omega}{\hat{m}_\omega} = \frac{1}{\tau^2}\frac{\hat{\omega}}{\mathring{\omega}},$$

where the ratio $\hat{\omega}/\mathring{\omega}$ depends only on the dimensions of ω, as listed in the third column of the following Table.

Dimensions of ω	$\breve{\omega}/\hat{\omega}$	$\hat{\omega}/\mathring{\omega}$
Length	μ	$1/V_e\tau$
Linear velocity	1	$1/V_e$
Linear acceleration	$1/\mu$	τ/V_e
Angle	1	1
Angular velocity	$1/\mu$	τ
Angular acceleration	$1/\mu^2$	τ^2

Relations between higher derivatives, involving (say) the variables $\omega_1, \omega_2, \ldots$, are very similar. For example,

$$\frac{\hat{z}_{\omega_1\omega_2\cdots}}{\breve{Z}_{\omega_1\omega_2\cdots}} = -\frac{\breve{\omega}_1}{\hat{\omega}_1}\frac{\breve{\omega}_2}{\hat{\omega}_2}\cdots,$$

$$\frac{\hat{m}_{\omega_1\omega_2\cdots}}{\breve{M}_{\omega_1\omega_2\cdots}} = -\frac{\mu}{i_y}\frac{\breve{\omega}_1}{\hat{\omega}_1}\frac{\breve{\omega}_2}{\hat{\omega}_2}\cdots.$$

Although Table 8 is designed for the force and moment terms in the usual equations of motion (10.3) and (10.4), it may also be used for constructing concise quantities arising in the equations of motion for extra degrees of freedom. For example, if the moment of inertia of an elevator about its hinge line is I_η, and the aerodynamic moment about that line is B, we may equate the concise derivative $\hat{b}_w$ with $-\mu\breve{B}_w/i_\eta$, where $B_w \equiv \partial B/\partial w$ and i_η is defined as $I_\eta/m_e l_0^2$. This is not necessarily the most convenient way of expressing $\hat{b}_w$, and it might be better to have it in terms of $C_{B\alpha} \equiv \partial C_B/\partial\alpha$, where C_B is based on a representative area and length of the elevator itself. Similarly, we might prefer to define an inertia parameter i_η in terms of local dimensions and perhaps local mass. Each extra degree of freedom should be examined from this point of view, and Section M.9 mentions a possible approach for aero-elastic degrees of freedom.

As the dynamic-normalised units do not depend on the choice of a representative length, the concise quantities such as $\hat{m}_w$ are also independent of the representative length although the constituent factors μ, $\breve{M}_w$, and i_y do contain l_0. It follows that compatible concise quantities will be obtained even if different representative lengths (l_1 and l_2) are used for longitudinal and lateral terms, provided that we consistently use l_1 or l_2 in each constituent factor contributing to a particular concise quantity. For example,

$$\hat{m}_w = -\frac{\mu_1\breve{M}_w}{i_y},$$

where

$$\mu_1 = \frac{m_e}{\frac{1}{2}\rho_e S l_1}, \quad i_y = \frac{I_y}{m_e l_1^2}, \quad \breve{M}_w = \frac{M_w}{\frac{1}{2}\rho_e V_e S l_1};$$

and

$$\breve{n}_v = -\frac{\mu_2 \breve{N}_v}{i_z},$$

where

$$\mu_2 = \frac{m_e}{\frac{1}{2}\rho_e S l_2}, \quad i_z = \frac{I_z}{m_e l_2^2}, \quad \breve{N}_v = \frac{N_v}{\frac{1}{2}\rho_e V_e S l_2}.$$

The above remarks also apply to the concise quantities in ordinary units, but not to concise quantities normalised according to the aero-system.

11.3. *The Aero-Normalised Equations.*

As stated previously the aero-normalising units of length and time are not generally suitable for dynamic analysis. It may occasionally, however, be desired to employ the equations of motion normalised according to the aero-system, and these are obtained as follows.

The terms on the left hand sides of equations (10.3) and (10.4) may be written in terms of $\breve{u}$, $\breve{v}$, $\breve{w}$, $\breve{p}$, $\breve{q}$, $\breve{r}$, $\breve{g}_x$, $\breve{g}_y$, $\breve{g}_z$, and of the operator $\breve{D}$. Since the unit of time is of magnitude l_0/V_e, we have $V_e\breve{D} = l_0 D$, where $\breve{D} \equiv d/d\breve{t}$, and the normalising divisors for the other quantities are as in the following Table.

Quantity	Normalising Divisor
u, v, w	V_e
p, q, r	$V_e/l_0 = \mu/\tau$
g_x, g_y, g_z	$V_e^2/l_0 = \mu\rho_e V_e^2 S/2m_e$

For example, $\breve{u} = u/V_e\,(= \hat{u})$, $\breve{p} = pl_0/V_e\,(= \hat{p}/\mu)$, $\breve{g}_x = m_e g_x/\frac{1}{2}\rho_e V_e^2 S\mu\,(= \hat{g}_x/\mu)$.

When the aircraft mass and moments of inertia are constant their values in aero-normalised units are $\breve{m}_e = \mu$, and $\breve{I}_x \doteq \mu i_x$, etc. It follows that all concise quantities concerned with aerodynamic forces can be obtained by dividing by $-\mu$, and all concise quantities concerned with aerodynamic moments can be formed by dividing by $-\mu i_x$, $-\mu i_y$, or $-\mu i_z$, whichever is appropriate. For example,

$$\breve{x}_e = -\breve{X}_e/\mu\,, \quad \breve{m}_w = -\breve{M}_w/\mu i_y\,,$$

$$\breve{z}_w = -\breve{Z}_w/\mu\,, \quad \breve{n}_r = -\breve{N}_r/\mu i_z\,.$$

When different representative lengths, say l_1 and l_2, are used for longitudinal and lateral aerodynamic coefficients, and for their derivatives, one of these should be chosen as the aero-length for normalising the equations as a whole. This naturally includes the expressions for μ and the inertia parameters i_x, i_y, i_z, etc.

12. *Stability and Response Quantities.*

It is convenient to have symbols for representing the complicated determinantal expressions that arise when we are studying the stability and response of an aircraft on the basis of small perturbations and linearized equations of motion. As examples we have taken the separate groups of longitudinal and lateral equations as set out in Section 10.2.3, and in each case we first consider the *natural* aircraft, that is the aircraft in a state where motivators are not actuated by any automatic control device.

We distinguish between the operator $D \equiv d/dt$ and the Laplace transform or Heaviside parameter. It has been traditional to represent the latter as p in non-aeronautical work, although the strong influence of American literature on automatic control is resulting in the use of s. It is not advisable to use p in aeronautical work since it represents rate of roll, and the use of s is generally endorsed. There may occasionally be a clash with another s, for example the pilot's control output (*see* Section 8), and in principle it seems unnecessary as a rule to depart from the traditional λ, for its original meaning of a root of the stability equation is consistent with Laplace parameter usage. In this report we use λ but the freedom of choice exists to suit the circumstances. The incidence-plane angle is also denoted by λ, but this would rarely cause any confusion.

12.1. *Longitudinal Stability and Response.*

If only one type of motivator is in operation (say the elevator), equations (10.12) and (10.14) may be reduced to the operational form

$$\frac{u'}{\Delta_{u\eta}} = \frac{w'}{\Delta_{w\eta}} = \frac{q'}{\Delta_{q\eta}} = \frac{\theta}{D^{-1}\Delta_{q\eta}} = \frac{h'}{\Delta_{h\eta}} = -\frac{\eta'}{\Delta_0}, \tag{12.1}$$

where the denominators are polynomials in D, and are equal (apart from sign) to the determinants formed from the array below by deleting each column in turn. As it stands the array is valid for level flight ($\gamma_e = 0$), but can also be applied to non-level flight in a uniform atmosphere by putting $x_h = z_h = m_h = 0$, and replacing h' in equations (12.1) by $-z_E^+$, $\Delta_{h\eta}$ becoming $-\Delta_{z\eta}$.

$$\begin{pmatrix} D+x_u & x_{\dot{w}}D+x_w & (x_q+w_e)\,D+g_1 & x_h & x_\eta \\ z_u & (1+z_{\dot{w}})\,D+z_w & (z_q-u_e)\,D+g_2 & z_h & z_\eta \\ m_u & m_{\dot{w}}D+m_w & D^2+m_qD & m_h & m_\eta \\ -\sin\alpha_e & \cos\alpha_e & -V_e & D & 0 \end{pmatrix}$$

It is useful to express each denominator of equations (12.1) in the form

$$\Delta_{u\eta} = U_x x_\eta + U_z z_\eta + U_m m_\eta\,, \tag{12.2}$$

since the twelve polynomials U_x, U_z, U_m, W_x, etc. recur in the analysis of aircraft stability with and without automatic control. $\Delta_{u\eta}$ may be called the *complete response polynomial* of u with respect to η, and it has component parts U_x, U_z, U_m, which are *response polynomials.*

The function Δ_0, obtained from the first four columns, defines the stability of the aircraft when the motivator is fixed, that is $\eta' = 0$. If in the usual way we assume that u', w', q', θ, h' are proportional to $\exp(\lambda t)$, and we replace D by λ, the equation $\Delta_0 = 0$ becomes the *characteristic* or *stability equation* for the natural aircraft, and Δ_0 is called the *stability determinant* or (after expansion) the *stability polynomial**. The polynomial is in general a quintic, and the determinant may be expanded in various ways, such as

$$\begin{aligned} \Delta_0 &= U_x(\lambda+x_u)+W_x(x_{\dot{w}}\lambda+x_w)+Q_x[x_q+w_e+g_1\lambda^{-1}]+H_x x_h \\ &= U_z z_u+W_z[(1+z_{\dot{w}})\lambda+z_w]+Q_z[z_q-u_e+g_2\lambda^{-1}]+H_z z_h \\ &= U_m m_u+W_m(m_{\dot{w}}\lambda+m_w)+Q_m(\lambda+m_q)+H_m m_h\,. \end{aligned}$$

---

*The stability determinant for any system may be denoted by Δ. The suffix o is added here in order to avoid confusion later in this Section, but may be omitted when no confusion arises. Similarly, Δ_u may be written instead of $\Delta_{u\eta}$ when possible, and expressed as a polynomial with coefficients U_i: $\Delta_u = \sum_i U_i D^i$.

If

$$\Delta_0 = K_5\lambda^5+K_4\lambda^4+K_3\lambda^3+K_2\lambda^2+K_1\lambda+K_0\,,$$

and U_{mi}, etc. are the coefficients of λ^i in the polynomials U_m, etc., distinct expressions for the coefficients K_i may be obtained from the various expansions of Δ_0. The third expansion, for instance, leads to the following expressions.

$$\begin{aligned}
K_5 &= Q_{m4}\\
K_4 &= Q_{m3}+Q_{m4}m_q+W_{m3}m_{\dot w}\\
K_3 &= Q_{m2}+Q_{m3}m_q+W_{m2}m_{\dot w}+W_{m3}m_w+U_{m3}m_u\\
K_2 &= Q_{m1}+Q_{m2}m_q+W_{m1}m_{\dot w}+W_{m2}m_w+U_{m2}m_u+H_{m2}m_h\\
K_1 &= \qquad Q_{m1}m_q+W_{m0}m_{\dot w}+W_{m1}m_w+U_{m1}m_u+H_{m1}m_h\\
K_0 &= \qquad\qquad W_{m0}m_w+U_{m0}m_u+H_{m0}m_h
\end{aligned}$$

A complete list of the coefficients U_{mi}, etc. is given below. Extensive use is made of short-hand expressions for second-order determinants: for example,

$$x_u z_w - x_w z_u \equiv \overset{\frown}{x_u z_w}\,,$$

and it is suggested that the overscript $\frown$ be called a 'slur'.

$$\begin{aligned}
U_{x4} &= 1+z_{\dot w}\\
U_{x3} &= m_q+z_w+u_e m_{\dot w}+\overset{\frown}{m_q z_{\dot w}}\\
U_{x2} &= u_e m_w+m_q z_w-g_2 m_{\dot w}-z_h\cos\alpha_e\\
U_{x1} &= -g_2 m_w+V_e(m_h z_{\dot w}+w_e m_h)+m_h z_q\cos\alpha_e\\
U_{x0} &= V_e\overset{\frown}{m_h z_w}+g_2 m_h\cos\alpha_e
\end{aligned}$$

$$\begin{aligned}
U_{z4} &= -x_{\dot w}\\
U_{z3} &= -x_w-\overset{\frown}{m_q x_{\dot w}}+w_e m_{\dot w}\\
U_{z2} &= \overset{\frown}{m_w x_q}+w_e m_w+g_1 m_{\dot w}+x_h\cos\alpha_e\\
U_{z1} &= g_1 m_w+V_e\overset{\frown}{m_{\dot w} x_h}+(\overset{\frown}{m_q x_h}-w_e m_h)\cos\alpha_e\\
U_{z0} &= V_e\overset{\frown}{m_w x_h}-g_1 m_h\cos\alpha_e
\end{aligned}$$

$$U_{m3} = -x_q - w_e + \widehat{z_q x}_{\dot w} - u_e x_{\dot w} - w_e z_{\dot w}$$

$$U_{m2} = \widehat{x_w z}_q - g_1(1 + z_{\dot w}) + g_2 x_{\dot w} - u_e x_w - w_e z_w$$

$$U_{m1} = -g_1 z_w + g_2 x_w + V_e(\widehat{x_{\dot w} z}_h - w_e x_h) + (\widehat{x_q z}_h + w_e z_h) \cos \alpha_e$$

$$U_{m0} = V_e \widehat{x_w z}_h + (g_1 z_h - g_2 x_h) \cos \alpha_e$$

$$W_{x3} = -z_u$$

$$W_{x2} = -\widehat{m_q z}_u - u_e m_u - z_h \sin \alpha_e$$

$$W_{x1} = g_2 m_u + (\widehat{m_h z}_q - u_e m_h) \sin \alpha_e$$

$$W_{x0} = V_e \widehat{m_u z}_h + g_2 m_h \sin \alpha_e$$

$$W_{z4} = 1$$

$$W_{z3} = x_u + m_q$$

$$W_{z2} = \widehat{x_u m}_q - w_e m_u + x_h \sin \alpha_e$$

$$W_{z1} = -g_1 m_u + V_e u_e m_h + \widehat{m_q x}_h \sin \alpha_e$$

$$W_{z0} = V_e \widehat{m_h x}_u - g_1 m_h \sin \alpha_e$$

$$W_{m3} = u_e - z_q$$

$$W_{m2} = u_e x_u + w_e z_u + \widehat{z_u x}_q - g_2$$

$$W_{m1} = g_1 z_u - g_2 x_u - V_e u_e z_h + (\widehat{x_q z}_h + u_e x_h) \sin \alpha_e$$

$$W_{m0} = V_e \widehat{z_u x}_h + (g_1 z_h - g_2 x_h) \sin \alpha_e$$

$$Q_{x3} = -m_u + \widehat{m_{\dot w} z}_u$$

$$Q_{x2} = \widehat{m_w z}_u + (\widehat{m_{\dot w} z}_h - m_h) \sin \alpha_e$$

$$Q_{x1} = \widehat{m_u z}_h \cos \alpha_e + \widehat{m_w z}_h \sin \alpha_e$$

$$Q_{z4} = -m_{\dot w}$$

$$Q_{z3} = -m_w - \widehat{m_{\dot w} x}_u$$

$$Q_{z2} = -\widehat{m_w x}_u + m_h \cos \alpha_e + \widehat{m_h x}_{\dot w} \sin \alpha_e$$

$$Q_{z1} = \widehat{m_h x}_u \cos \alpha_e + \widehat{m_h x}_w \sin \alpha_e$$

$$Q_{m4} = 1+z_{\dot{w}}$$

$$Q_{m3} = x_u+z_w+\widehat{x_u z_{\dot{w}}}$$

$$Q_{m2} = \widehat{x_u z_w}-z_h \cos\alpha_e+(x_h+\widehat{z_{\dot{w}} x_h}) \sin\alpha_e$$

$$Q_{m1} = \widehat{z_u x_h} \cos\alpha_e+\widehat{z_w x_h} \sin\alpha_e$$

$$H_{x3} = (1+z_{\dot{w}}) \sin\alpha_e$$

$$H_{x2} = z_u \cos\alpha_e+(m_q+z_w+u_e m_{\dot{w}}+\widehat{m_q z_{\dot{w}}}) \sin\alpha_e$$

$$H_{x1} = -V_e(w_e m_u+\widehat{m_u z_{\dot{w}}})+\widehat{m_q z_w} \cos\alpha_e +(\widehat{m_q z_w}+u_e m_w-g_2 m_{\dot{w}}) \sin\alpha_e$$

$$H_{x0} = V_e \widehat{m_w z_u}-g_2(m_u \cos\alpha_e+m_w \sin\alpha_e)$$

$$H_{z3} = -\cos\alpha_e-x_{\dot{w}} \sin\alpha_e$$

$$H_{z2} = V_e m_{\dot{w}}-(x_u+m_q) \cos\alpha_e+(\widehat{m_{\dot{w}} x_q}-x_w+w_e m_{\dot{w}}) \sin\alpha_e$$

$$H_{z1} = -V_e(m_w+\widehat{m_{\dot{w}} x_u})+(w_e m_u-\widehat{x_u m_q}) \cos\alpha_e +(\widehat{m_w x_q}+w_e m_w+g_1 m_{\dot{w}}) \sin\alpha_e$$

$$H_{z0} = - V_e \widehat{x_u z_w}+g_1(m_u \cos\alpha_e+m_w \sin\alpha_e)$$

$$H_{m2} = V_e(w_e+z_{\dot{w}})+z_q \cos\alpha_e-(x_q+\widehat{x_q z_{\dot{w}}}+w_e+u_e x_{\dot{w}}+w_e z_{\dot{w}}) \sin\alpha_e$$

$$H_{m1} = V_e(z_w+w_e x_u+\widehat{x_u z_{\dot{w}}})+(\widehat{x_u z_q}-w_e z_u+g_2) \cos\alpha_e -(\widehat{z_w x_q}+u_e x_w+w_e z_w+g_1+g_1 z_{\dot{w}}-g_2 x_{\dot{w}}) \sin\alpha_e$$

$$H_{m0} = V_e \widehat{x_u z_w}+(g_2 x_u-g_1 z_u) \cos\alpha_e+(g_2 x_w-g_1 z_w) \sin\alpha_e$$

It should be noted that $u_e = V_e \cos\alpha_e$ and $w_e = V_e \sin\alpha_e$, and that α_e is zero when aerodynamic-body axes are used. Also, in either the dynamic- or aero-normalised system V_e will by definition be unity. Furthermore, if derivatives with respect to altitude can be neglected the stability quintic reduces to a quartic, since U_{x0}, W_{x0}, Q_{x1}, U_{z0}, etc. are all zero, and hence K_0 is zero. It might then be preferable to rewrite Δ_0 as $(K_4\lambda^4+\ldots+K_0)$, where the new K_4 corresponds to the K_5 quoted earlier in this section, and so on. The renumbering of the suffixes might also be carried out for the coefficients of the response polynomials.

Various relationships exist between the coefficients of the polynomials given above. For instance, by using the second of equations (10.14) we obtain

$$H_{xi} = U_{x,i+1} \sin\alpha_e - W_{x,i+1} \cos\alpha_e + V_e Q_{x,i+2},$$

and similar equalities for H_{zi} and H_{mi}.

Automatic control.

Consider an automatic pilot which produces pitcher, proaptor, and catanator deflections (*see* Section 8). Ideally simplified linearized *control equations* would be of the form

$$\eta' = G_u u' + G_{\dot{u}} D u' + G_{\bar{u}} \frac{1}{D} u' + G_w w' + \ldots\ldots,$$

$$v' = A_u u' + A_{\dot{u}} D u' + A_{\bar{u}} \frac{1}{D} u' + A_w w' + \ldots\ldots,$$

$$\kappa' = C_u u' + C_{\dot{u}} D u' + C_{\bar{u}} \frac{1}{D} u' + C_w w' + \ldots\ldots,$$

where G_u, A_u, C_u, etc. are autopilot parameters* (called gains or gearings), which are assumed constant for small deviations from particular steady state flight conditions, but which may be varied as functions of the steady state flight conditions. The operator $1/D$ is defined to be $\int_0^t \ldots dt$. It is sufficient to consider the abridged control equations.

$$\begin{aligned} \eta' &= G_u u' + G_w w' + G_\theta \theta + G_h h', \\ v' &= A_u u' + A_w w' + A_\theta \theta + A_h h', \\ \kappa' &= C_u u' + C_w w' + C_\theta \theta + C_h h', \end{aligned} \tag{12.3}$$

since the effect of other gearings can be deduced as will be explained.

When the motivator deflections are functions of the variables it is possible to derive many expressions on the lines of (12.1) giving the ratios of pairs of variables, and it does not seem necessary to devise a notation that would cover all possibilities. It is, however, useful to consider the stability polynomial, which from equations (10.11), (10.13), and (12.3), is equal to the determinant Δ given by

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & x_v & x_\eta & x_\kappa \\ b_1 & b_2 & b_3 & b_4 & z_v & z_\eta & z_\kappa \\ c_1 & c_2 & c_3 & c_4 & m_v & m_\eta & m_\kappa \\ d_1 & d_2 & d_3 & d_4 & 0 & 0 & 0 \\ -A_u & -A_w & -A_\theta & -A_h & 1 & 0 & 0 \\ -G_u & -G_w & -G_\theta & -G_h & 0 & 1 & 0 \\ -C_u & -C_w & -C_\theta & -C_h & 0 & 0 & 1 \end{vmatrix}$$

where $|a_1 b_2 c_3 d_4|$ would be the stability determinant (Δ_0) when $v' = \eta' = \kappa' = 0$, as set out earlier in this Section. The general expression for Δ has the form

---

*Other suffixes (or no suffixes) may be used when control equations include awkward transfer functions or special groupings. For example, $\eta' = \dfrac{1+n\tau D}{1+\tau D} G\theta$ or $\eta' = G_1(\theta + kh') + \dfrac{1}{1+\tau D} G_2 \dot{\theta}$.

$$\Delta = \Delta_0 + A_u \Delta_{uv} + A_w \Delta_{wv} + A_\theta \Delta_{qv} \lambda^{-1} + A_h \Delta_{hv} +$$

$$+ G_u \Delta_{u\eta} + G_w \Delta_{w\eta} + G_\theta \Delta_{q\eta} \lambda^{-1} + G_h \Delta_{h\eta} +$$

$$+ C_u \Delta_{u\kappa} + C_w \Delta_{w\kappa} + C_\theta \Delta_{q\kappa} \lambda^{-1} + C_h \Delta_{h\kappa} +$$

$$+ (\text{double product terms such as } A_u G_w \ldots) +$$

$$+ (\text{triple product terms such as } A_u G_w C_\theta \ldots),$$

and the first three rows may also be put in the form

$$\Delta_0 + x_v (A_u U_x + A_w W_x + A_\theta Q_x \lambda^{-1} + A_h H_x) +$$

$$+ z_v (A_u U_z + A_w W_z + A_\theta Q_z \lambda^{-1} + A_h H_z) +$$

$$+ m_v (A_u U_m + A_w W_m + A_\theta Q_m \lambda^{-1} + A_h H_m) +$$

$$+ x_\eta (G_u U_x + G_w W_x + G_\theta Q_x \lambda^{-1} + G_h H_x) +$$

$$+ z_\eta (G_u U_z + G_w W_z + G_\theta Q_z \lambda^{-1} + G_h H_z) +$$

$$+ m_\eta (G_u U_m + G_w W_m + G_\theta Q_m \lambda^{-1} + G_h H_m) +$$

$$+ x_\kappa (C_u U_x + C_w W_x + C_\theta Q_x \lambda^{-1} + C_h H_x) +$$

$$+ z_\kappa (C_u U_z + C_w W_z + C_\theta Q_z \lambda^{-1} + C_h H_z) +$$

$$+ m_\kappa (C_u U_m + C_w W_m + C_\theta Q_m \lambda^{-1} + C_h H_m)$$

When there is only one kind of control all the double and triple product terms are zero as well as many of the simpler terms, since only A's or only G's or only C's are involved.

The triple product terms are all contained in the product of two determinants:

$$\begin{vmatrix} x_\eta & x_v & x_\kappa \\ z_\eta & z_v & z_\kappa \\ m_\eta & m_v & m_\kappa \end{vmatrix} \times \begin{vmatrix} -\sin\alpha_e & \cos\alpha_e & -V_e & \lambda \\ -G_u & -G_w & -G_\theta & -G_h \\ -A_u & -A_w & -A_\theta & -A_h \\ -C_u & -C_w & -C_\theta & -C_h \end{vmatrix}$$

The contribution to the stability polynomial is therefore equal to

$$\begin{aligned} &\lambda(m_\eta \widehat{x_v z_\kappa} + z_\eta \widehat{m_v x_\kappa} + x_\eta \widehat{z_v m_\kappa})(G_u \widehat{A_w C_\theta} + G_w \widehat{A_\theta C_u} + G_\theta A_u C_w) + \\ &+ V_e(\quad ,, \qquad ,, \qquad ,, \quad)(G_u \widehat{A_h C_w} + G_h \widehat{A_w C_u} + G_w \widehat{A_u C_h}) + \\ &+ \cos\alpha_e(\quad ,, \qquad ,, \qquad ,, \quad)(G_u \widehat{A_\theta C_h} + G_\theta \widehat{A_h C_u} + G_h \widehat{A_u C_\theta}) + \\ &+ \sin\alpha_e(\quad ,, \qquad ,, \qquad ,, \quad)(G_w \widehat{A_\theta C_h} + G_\theta \widehat{A_h C_w} + G_h \widehat{A_w C_\theta}) . \end{aligned}$$

The double products are conveniently listed in terms of second-order minor determinants of Δ_0. The eighteen minors required are those containing elements of the fourth row, and each minor occurs in three double product terms. The 54 terms may be obtained from Table 4. Thus, the contribution to the stability polynomial is given by Table 4A:

TABLE 4A

$\overset{\frown}{m_\eta z_\nu}$	$\overset{\frown}{m_\eta x_\nu}$	$\overset{\frown}{x_\nu z_\eta}$	$\overset{\frown}{z_\nu m_\kappa}$	$\overset{\frown}{x_\nu m_\kappa}$	$\overset{\frown}{x_\nu z_\kappa}$	$\overset{\frown}{m_\eta z_\kappa}$	$\overset{\frown}{m_\eta x_\kappa}$	$\overset{\frown}{z_\kappa x_\eta}$	
$\overset{\frown}{G_hA_\theta}$			$\overset{\frown}{A_\theta C_h}$			$\overset{\frown}{C_\theta G_h}$			$\overset{\frown}{a_1d_2}$
$\overset{\frown}{G_hA_w}$			$\overset{\frown}{A_wC_h}$			$\overset{\frown}{C_wG_h}$			$\overset{\frown}{a_3d_1}$
$\overset{\frown}{G_\theta A_w}$			$\overset{\frown}{A_wC_\theta}$			$\overset{\frown}{C_wG_\theta}$			$\overset{\frown}{a_1d_4}$
$\overset{\frown}{G_uA_h}$			$\overset{\frown}{A_hC_u}$			$\overset{\frown}{C_hG_u}$			$\overset{\frown}{a_3d_2}$
$\overset{\frown}{G_uA_\theta}$			$\overset{\frown}{A_\theta C_u}$			$\overset{\frown}{C_\theta G_u}$			$\overset{\frown}{a_2d_4}$
$\overset{\frown}{G_wA_u}$			$\overset{\frown}{A_uC_w}$			$\overset{\frown}{C_uG_w}$			$\overset{\frown}{a_3d_4}$
	$\overset{\frown}{G_\theta A_h}$			$\overset{\frown}{A_hC_\theta}$			$\overset{\frown}{C_hG_\theta}$		$\overset{\frown}{b_1d_2}$
	$\overset{\frown}{G_wA_h}$			$\overset{\frown}{A_hC_w}$			$\overset{\frown}{C_hG_w}$		$\overset{\frown}{b_3d_1}$
	$\overset{\frown}{G_wA_\theta}$			$\overset{\frown}{A_\theta C_w}$			$\overset{\frown}{C_\theta G_w}$		$\overset{\frown}{b_1d_4}$
	$\overset{\frown}{G_hA_u}$			$\overset{\frown}{A_uC_h}$			$\overset{\frown}{C_uG_h}$		$\overset{\frown}{b_3d_2}$
	$\overset{\frown}{G_\theta A_u}$			$\overset{\frown}{A_uC_\theta}$			$\overset{\frown}{C_uG_\theta}$		$\overset{\frown}{b_2d_4}$
	$\overset{\frown}{G_wA_u}$			$\overset{\frown}{A_uC_w}$			$\overset{\frown}{C_uG_w}$		$\overset{\frown}{b_4d_3}$
		$\overset{\frown}{G_hA_\theta}$			$\overset{\frown}{A_\theta C_h}$			$\overset{\frown}{C_hG_\theta}$	$\overset{\frown}{c_1d_2}$
		$\overset{\frown}{G_hA_w}$			$\overset{\frown}{A_wC_h}$			$\overset{\frown}{C_hG_w}$	$\overset{\frown}{c_3d_1}$
		$\overset{\frown}{G_\theta A_w}$			$\overset{\frown}{A_wC_\theta}$			$\overset{\frown}{C_\theta G_w}$	$\overset{\frown}{c_1d_4}$
		$\overset{\frown}{G_uA_h}$			$\overset{\frown}{A_hC_u}$			$\overset{\frown}{C_uG_h}$	$\overset{\frown}{c_3d_2}$
		$\overset{\frown}{G_uA_\theta}$			$\overset{\frown}{A_\theta C_u}$			$\overset{\frown}{C_uG_\theta}$	$\overset{\frown}{c_2d_4}$
		$\overset{\frown}{G_wA_u}$			$\overset{\frown}{A_uC_w}$			$\overset{\frown}{C_wG_u}$	$\overset{\frown}{c_3d_4}$

$$
\begin{aligned}
&\overset{\frown}{m_\eta z_\nu}\left(\overset{\frown}{G_hA_\theta}\overset{\frown}{a_1d_2}+\overset{\frown}{G_hA_w}\overset{\frown}{a_3d_1}+\overset{\frown}{G_\theta A_w}\overset{\frown}{a_1d_4}+\overset{\frown}{G_uA_h}\overset{\frown}{a_3d_2}+\overset{\frown}{G_uA_\theta}\overset{\frown}{a_2d_4}+\overset{\frown}{G_wA_u}\overset{\frown}{a_3d_4}\right)+\\
&+\overset{\frown}{m_\eta x_\nu}\left(\overset{\frown}{G_\theta A_h}\overset{\frown}{b_1d_2}+\overset{\frown}{G_wA_h}\overset{\frown}{b_3d_1}+\,\ldots\ldots\right)+\\
&+\overset{\frown}{x_\nu z_\eta}\left(\overset{\frown}{G_hA_\theta}\overset{\frown}{c_1d_2}+\overset{\frown}{G_hA_w}\overset{\frown}{c_3d_1}+\,\ldots\ldots\right)+\\
&+\overset{\frown}{z_\nu m_\kappa}\left(\overset{\frown}{A_\theta C_h}\overset{\frown}{a_1d_2}+\overset{\frown}{A_wC_h}\overset{\frown}{a_3d_1}+\,\ldots\ldots\right)+\\
&+\ldots\ldots\ldots+\\
&+\overset{\frown}{z_\kappa x_\eta}\left(\overset{\frown}{C_hG_\theta}\overset{\frown}{c_1d_2}+\ldots\ldots\ldots+\overset{\frown}{C_wG_u}\overset{\frown}{c_3d_4}\right)\text{ are the required terms.}
\end{aligned}
$$

It should be noted that each gearing product such as $\overset{\frown}{G_hA_\theta}$ occurs once in each block. Table 4B gives the complete expressions for the minors. For example,

$$\overset{\frown}{a_1d_4} = \lambda^2+x_u\lambda+x_h\sin\alpha_e\,.$$

As an example consider an autopilot whose equations are

$$\eta' = G_\theta\theta\ , \qquad \nu' = A_uu'\ , \qquad \kappa' = C_hh'\,.$$

TABLE 4B

	λ^3 coeff.	λ^2 coeff.	λ coeff.	Const. coeff.
$\widehat{a_1 d_2}$	.	.	$\cos\alpha_e + x_{\dot{w}} \sin\alpha_e$	$x_u \cos\alpha_e + x_w \sin\alpha_e$
$\widehat{a_3 d_1}$	.	.	$V_e - (w_e + x_q) \sin\alpha_e$	$V_e x_u - g_1 \sin\alpha_e$
$\widehat{a_1 d_4}$	.	1	x_u	$x_h \sin\alpha_e$
$\widehat{a_3 d_2}$	.	.	$V_e x_{\dot{w}} + (w_e + x_q) \cos\alpha_e$	$V_e x_w + g_1 \cos\alpha_e$
$\widehat{a_2 d_4}$	.	$x_{\dot{w}}$	x_w	$-x_h \cos\alpha_e$
$\widehat{a_3 d_4}$	.	$w_e + x_{\dot{q}}$	g_1	$V_e x_h$
$\widehat{b_1 d_2}$	.	.	$(1 + z_{\dot{w}}) \sin\alpha_e$	$z_u \cos\alpha_e + z_w \sin\alpha_e$
$\widehat{b_3 d_1}$	.	.	$(u_e - z_q) \sin\alpha_e$	$V_e z_u - g_2 \sin\alpha_e$
$\widehat{b_1 d_4}$	.	.	z_u	$z_h \sin\alpha_e$
$\widehat{b_3 d_2}$	.	.	$V_e(1 + z_{\dot{w}}) - (u_e - z_q) \cos\alpha_e$	$V_e z_w + g_2 \cos\alpha_e$
$\widehat{b_2 d_4}$	.	$1 + z_{\dot{w}}$	z_w	$-z_h \cos\alpha_e$
$\widehat{b_4 d_3}$	.	$u_e - z_q$	$-g_2$	$-V_e z_h$
$\widehat{c_1 d_2}$	.	.	$m_{\dot{w}} \sin\alpha_e$	$m_u \cos\alpha_e + m_w \sin\alpha_e$
$\widehat{c_3 d_1}$	.	$-\sin\alpha_e$	$-m_q \sin\alpha_e$	$V_e m_u$
$\widehat{c_1 d_4}$	.	.	m_u	$m_h \sin\alpha_e$
$\widehat{c_3 d_2}$	.	$\cos\alpha_e$	$V_e m_{\dot{w}} + m_q \cos\alpha_e$	$V_e m_w$
$\widehat{c_2 d_4}$	.	$m_{\dot{w}}$	m_w	$-m_h \cos\alpha_e$
$\widehat{c_3 d_4}$	1	m_q	.	$V_e m_h$

The stability polynomial is

$$
\begin{aligned}
\Delta = \Delta_0 &+ G_\theta \lambda^{-1}(Q_m m_\eta + Q_z z_\eta + Q_x x_\eta) + \\
&+ \quad A_u(U_m m_\nu + U_z z_\nu + U_x x_\nu) + \\
&+ \quad C_h(H_m m_\kappa + H_z z_\kappa + H_x x_\kappa) + \\
&+ G_\theta A_u\big(\widehat{m_\eta x_\nu}\widehat{b_2 d_4} - \widehat{m_\eta z_\nu}\widehat{a_2 d_4} - \widehat{x_\nu z_\eta}\widehat{c_2 d_4}\big) + \\
&+ A_u C_h\big(-\widehat{x_\nu z_\kappa}\widehat{c_3 d_2} - \widehat{z_\nu m_\kappa}\widehat{a_3 d_2} + \widehat{x_\nu m_\kappa}\widehat{b_3 d_2}\big) + \\
&+ G_\theta C_h\big(-\widehat{m_\eta z_\kappa}\widehat{a_1 d_2} + \widehat{m_\eta x_\kappa}\widehat{b_1 d_2} + \widehat{z_\kappa x_\eta}\widehat{c_1 d_2}\big) - \\
&- G_\theta A_u C_h \cos\alpha_e\big(m_\eta \widehat{x_\nu z_\kappa} + z_\eta \widehat{m_\nu x_\kappa} + x_\eta \widehat{z_\nu m_\kappa}\big).
\end{aligned}
$$

If the pitch control equation is instead

$$\eta' = G_\theta \theta + G_q q',$$

we merely have to pick out all the G_θ terms in the expression for the stability polynomial and write $(G_\theta + G_q\lambda)$ wherever there is a G_θ. Similarly if the control equation contains a term $G_{\bar{\theta}} \frac{1}{D}\theta$, we must add $G_{\bar{\theta}}\lambda^{-1}$ wherever there is a G_θ.

12.2. *Lateral Stability and Response.*

If only one type of motivator is in operation (say the ailerons), equations (10.13) and (10.15) may be reduced to the operational form

$$\frac{v'}{\Delta_{v\xi}} = \frac{p'}{\Delta_{p\xi}} = \frac{\phi}{D^{-1}\Delta_{p\xi}} = \frac{r'}{\Delta_{r\xi}} = \frac{\psi}{D^{-1}\Delta_{r\xi}} = \frac{y_E^+}{\Delta_{y\xi}} = -\frac{\xi'}{\Delta_0}, \tag{12.4}$$

where the denominators are polynomials in D, and are equal (apart from sign) to the determinants formed from the array below by deleting each column in turn.

$$\begin{pmatrix} D+y_v & (y_p-w_e)D-g_1 & (y_r+u_e)D-g_2 & 0 & y_\xi \\ l_v & D^2+l_pD & e_xD^2+l_rD & 0 & l_\xi \\ n_v & e_zD^2+n_pD & D^2+n_rD & 0 & n_\xi \\ -1 & V_e\sin\alpha_e & -V_e\cos\alpha_e & D & 0 \end{pmatrix}$$

As in Section 12.1 we express the denominators of equations (12.4) in the form

$$\Delta_{v\xi} = V_y y_\xi + V_l l_\xi + V_n n_\xi, \tag{12.5}$$

where $\Delta_{v\xi}$ is the *complete response polynomial* of v with respect to ξ, and its component parts V_y, V_l, V_n are *response polynomials.* The footnote on page 18 also applies here, so that for example we may write $\Delta_{v\xi} = \Delta_v = \sum_i V_i D^i$ if there is no ambiguity.

The stability determinant Δ_0 is obtained from the first four columns. The stability polynomial is in general a sextic with a factor λ^2 corresponding to the neutral stability of an aircraft (when no control is applied) in respect of

(a) lateral deviations from the datum flight path,

(b) angular deviations in heading from the azimuth datum.

The determinant Δ_0 may be expanded in various ways, such as

$$\begin{aligned}\Delta_0 &= V_y(\lambda+y_v)+V_l l_v+V_n n_v \\ &= P_y(y_p-w_e-g_1\lambda^{-1})+P_l(\lambda+l_p)+P_n(e_z\lambda+n_p) \\ &= R_y(y_r+u_e-g_2\lambda^{-1})+R_l(e_x\lambda+l_r)+R_n(\lambda+n_r).\end{aligned}$$

If

$$\Delta_0 = J_6\lambda^6+J_5\lambda^5+J_4\lambda^4+J_3\lambda^3+J_2\lambda^2,$$

and V_{yi}, etc. are the coefficients of λ^i in the polynomials V_y, etc., distinct expressions for the coefficients J_i may be obtained from the various expansions of Δ_0. The first expansion, for example, leads to the following expressions.

$$J_6 = V_{y5}$$

$$J_5 = V_{y4} + V_{y5} y_v$$

$$J_4 = V_{y3} + V_{y4} y_v + V_{l4} l_v + V_{n4} n_v$$

$$J_3 = \quad + V_{y3} y_v + V_{l3} l_v + V_{n3} n_v$$

$$J_2 = \quad V_{l2} l_v + V_{n2} n_v$$

A complete list of the coefficients V_{yi}, etc. is given below, and as before the 'slur' is used to shorten the writing of expressions such as

$$l_v n_r - l_r n_v \equiv \widehat{l_v n_r}\,.$$

$$V_{y5} = 1 - e_x e_z$$
$$V_{y4} = l_p + n_r - e_x n_p - e_z l_r$$
$$V_{y3} = \widehat{l_p n_r}$$

$$V_{l4} = w_e - y_p + e_z(u_e + y_r)$$
$$V_{l3} = u_e n_p + w_e n_r + \widehat{y_r n_p} + g_1 - e_z g_2$$
$$V_{l2} = g_1 n_r - g_2 n_p$$

$$V_{n4} = -(u_e + y_r) - e_x(w_e - y_p)$$
$$V_{n3} = -u_e l_p - w_e l_r - \widehat{y_r l_p} + g_2 - e_x g_1$$
$$V_{n2} = g_2 l_p - g_1 l_r$$

$P_{y4} = -l_v + e_x n_v$	$R_{y4} = -n_v + e_z l_v$
$P_{y3} = \widehat{l_r n_v}$	$R_{y3} = \widehat{l_v n_p}$
$P_{l5} = 1$	$R_{l5} = -e_z$
$P_{l4} = y_v + n_r$	$R_{l4} = -n_p - e_z y_v$
$P_{l3} = \widehat{y_v n_r} - u_e n_v$	$R_{l3} = \widehat{y_p n_v} - w_e n_v$
$P_{l2} = g_2 n_v$	$R_{l2} = -g_1 n_v$
$P_{n5} = -e_x$	$R_{n5} = 1$
$P_{n4} = -l_r - e_x y_v$	$R_{n4} = y_v + l_p$
$P_{n3} = \widehat{y_r l_v} + u_e l_v$	$R_{n3} = \widehat{y_v l_p} + w_e l_v$
$P_{n2} = -g_2 l_v$	$R_{n2} = g_1 l_v$

$$Y_{y4} = 1 - e_x e_z = V_{y5}$$
$$Y_{y3} = l_p + n_r - e_x n_p - e_z l_r = V_{y4}$$
$$Y_{y2} = \widehat{l_p n_r} - u_e(n_v - e_z l_v) + w_e(l_v - e_x n_v)$$
$$V_{y1} = u_e \widehat{l_v n_p} + w_e l_v n_r$$

$$
\begin{aligned}
Y_{l3} &= -y_p + e_z y_r \\
Y_{l2} &= \widehat{y_r n_p} - y_v(w_e + e_z u_e) + g_1 - e_z g_2 \\
Y_{l1} &= -u_e \widehat{y_v n_p} - w_e \widehat{y_v n_r} + g_1 n_r - g_2 n_p \\
Y_{l0} &= -V_e g n_v \cos\gamma_e = -n_v(g_1 u_e + g_2 w_e)
\end{aligned}
$$

$$
\begin{aligned}
Y_{n3} &= -y_r + e_x y_p \\
Y_{n2} &= \widehat{y_p l_r} + y_v(u_e + e_x w_e) + g_2 - e_x g_1 \\
Y_{n1} &= u_e \widehat{y_v l_p} + w_e \widehat{y_v l_r} - g_1 l_r + g_2 l_p \\
Y_{n0} &= V_e g l_v \cos\gamma_e = l_v(g_1 u_e + g_2 w_e)
\end{aligned}
$$

It should be noted that if aerodynamic-body axes are used then u_e may be replaced by V_e, and w_e equated to zero, since $u_e = V_e \cos\alpha_e$, $w_e = V_e \sin\alpha_e$. Also, in either the dynamic- or aero-normalised system V_e will by definition be unity.

Various relationships exist between the coefficients of the twelve polynomials. For instance, from the third of equations (10.15) it follows that

$$Y_{yi} = V_{y,i+1} - w_e P_{y,i+2} + u_e R_{y,i+2},$$

and similar equalities are found for Y_{li} and Y_{ni}.

Automatic control.

Consider an automatic pilot which produces roller, dexilator, and yawer deflections (*see* Section 8). Ideally simplified linearized control equations would be of the form

$$
\begin{aligned}
\xi' &= F_v v' + F_{\dot v}\dot v' + F_{\bar v}\frac{1}{D}v' + F_\phi \phi + \ldots\ldots, \\
\delta' &= B_v v' + B_{\dot v}\dot v' + B_{\bar v}\frac{1}{D}v' + B_\phi \phi + \ldots\ldots, \\
\zeta' &= H_v v' + H_{\dot v}\dot v' + H_{\bar v}\frac{1}{D}v' + H_\phi \phi + \ldots\ldots,
\end{aligned}
$$

where F_v, B_v, H_v, etc. are autopilot parameters* (called gains or gearings), which are assumed constant for small deviations from particular steady state flight conditions, but which may be varied as functions of the steady state flight conditions. The operator $1/D$ is defined to be $\int_0^t \ldots dt$. It is sufficient to consider the abridged control equations

$$
\left.
\begin{aligned}
\xi' &= F_v v' + F_\phi \phi + F_\psi \psi + F_y y_E^+, \\
\delta' &= B_v v' + B_\phi \phi + B_\psi \psi + B_y y_E^+, \\
\zeta' &= H_v v' + H_\phi \phi + H_\psi \psi + H_y y_E^+,
\end{aligned}
\right\} \tag{12.6}
$$

since the effect of other gearings can be deduced as explained in Section 12.1.

---

*See footnote on page 22.

As in the longitudinal case all possible relations similar to equations (12.4) are not considered when the control applications are functions of the variables, and only the stability polynomial is set out. The stability determinant Δ is

$$\begin{vmatrix} a_1 & a_2 & a_3 & 0 & y_\xi & y_\delta & y_\zeta \\ b_1 & b_2 & b_3 & 0 & l_\xi & l_\delta & l_\zeta \\ c_1 & c_2 & c_3 & 0 & n_\xi & n_\delta & n_\zeta \\ d_1 & d_2 & d_3 & d_4 & 0 & 0 & 0 \\ -F_v & -F_\phi & -F_\psi & -F_y & 1 & 0 & 0 \\ -B_v & -B_\phi & -B_\psi & -B_y & 0 & 1 & 0 \\ -H_v & -H_\phi & -H_\psi & -H_y & 0 & 0 & 1 \end{vmatrix},$$

where $|a_1 b_2 c_3 d_4|$ would be the stability determinant Δ_0 when $\xi' = \delta' = \zeta' = 0$, as set out earlier in Section 12.2. The general expression for Δ has the form

$$\begin{aligned} \Delta = \Delta_0 &+ F_v\Delta_{v\xi} + F_\phi\Delta_{p\xi}\lambda^{-1} + F_\psi\Delta_{r\xi}\lambda^{-1} + F_y\Delta_{y\xi} + \\ &+ B_v\Delta_{v\delta} + B_\phi\Delta_{\phi\delta}\lambda^{-1} + B_\psi\Delta_{r\delta}\lambda^{-1} + B_y\Delta_{y\delta} + \\ &+ H_v\Delta_{v\zeta} + H_\phi\Delta_{p\zeta}\lambda^{-1} + H_\psi\Delta_{r\zeta}\lambda^{-1} + H_y\Delta_{y\zeta} + \\ &+ (\text{double product terms such as } F_v B_\phi \ldots) + \\ &+ (\text{triple product terms such as } F_v B_\phi H_\psi \ldots), \end{aligned}$$

and the first three rows may also be put in the form

$$\begin{aligned} \Delta_0 &+ y_\xi(F_v V_y + F_\phi P_y\lambda^{-1} + F_\psi R_y\lambda^{-1} + F_y Y_y) + \\ &+ l_\xi(F_v V_l + F_\phi P_l\lambda^{-1} + F_\psi R_l\lambda^{-1} + F_y Y_l) + \\ &+ n_\xi(F_v V_n + F_\phi P_n\lambda^{-1} + F_\psi R_n\lambda^{-1} + F_y Y_n) + \\ &+ y_\delta(B_v V_y + B_\phi P_y\lambda^{-1} + B_\psi R_y\lambda^{-1} + B_y Y_y) + \\ &+ l_\delta(B_v V_l + B_\phi P_l\lambda^{-1} + B_\psi R_l\lambda^{-1} + B_y Y_l) + \\ &+ n_\delta(B_v V_n + B_\phi P_n\lambda^{-1} + B_\psi R_n\lambda^{-1} + B_y Y_n) + \\ &+ y_\zeta(H_v V_y + H_\phi P_y\lambda^{-1} + H_\psi R_y\lambda^{-1} + H_y Y_y) + \\ &+ l_\zeta(H_v V_l + H_\phi P_l\lambda^{-1} + H_\psi R_l\lambda^{-1} + H_y Y_l) + \\ &+ n_\zeta(H_v V_n + H_\phi P_n\lambda^{-1} + H_\psi R_n\lambda^{-1} + H_y Y_n). \end{aligned}$$

Where there is only one kind of control, all the double and triple product terms are zero as well as many of the simpler terms, since only F's or only B's or only H's are involved.

The double products are conveniently listed in terms of second-order minor determinants of $|a_1 b_2 c_3 d_4|$. The eighteen minors required are those containing elements of the fourth row, and each minor occurs in three double product terms. The 54 terms may be obtained from Table 5. The contribution to the stability polynomial is given by Table 5A.

TABLE 5A

$\widehat{l_\xi n_\zeta}$	$\widehat{y_\zeta n_\xi}$	$\widehat{y_\zeta l_\xi}$	$\widehat{l_\xi n_\delta}$	$\widehat{y_\delta n_\xi}$	$\widehat{y_\delta l_\xi}$	$\widehat{l_\delta n_\zeta}$	$\widehat{y_\delta n_\zeta}$	$\widehat{y_\delta l_\zeta}$	
$\widehat{F_\psi H_y}$			$\widehat{F_\psi B_y}$			$\widehat{B_\psi H_y}$			$\widehat{a_1 d_2}$
$\widehat{F_\phi H_y}$			$\widehat{F_\phi B_y}$			$\widehat{B_\phi H_y}$			$\widehat{a_3 d_1}$
$\widehat{F_\phi H_\psi}$			$\widehat{F_\phi B_\psi}$			$\widehat{B_\phi H_\psi}$			$\widehat{a_1 d_4}$
$\widehat{F_y H_v}$			$\widehat{F_y B_v}$			$\widehat{B_y H_v}$			$\widehat{a_3 d_2}$
$\widehat{F_\psi H_v}$			$\widehat{F_\psi B_v}$			$\widehat{B_\psi H_v}$			$\widehat{a_2 d_4}$
$\widehat{F_v H_\phi}$			$\widehat{F_v B_\phi}$			$\widehat{B_v H_\phi}$			$\widehat{a_3 d_4}$
	$\widehat{F_\psi H_y}$			$\widehat{F_\psi B_y}$			$\widehat{B_y H_\psi}$		$\widehat{b_1 d_2}$
	$\widehat{F_\phi H_y}$			$\widehat{F_\phi B_y}$			$\widehat{B_y H_\phi}$		$\widehat{b_3 d_1}$
	$\widehat{F_\phi H_\psi}$			$\widehat{F_\phi B_\psi}$			$\widehat{B_\psi H_\phi}$		$\widehat{b_1 d_4}$
	$\widehat{F_v H_y}$			$\widehat{F_y B_v}$			$\widehat{B_v H_y}$		$\widehat{b_3 d_2}$
	$\widehat{F_\psi H_v}$			$\widehat{F_\psi B_v}$			$\widehat{B_v H_\psi}$		$\widehat{b_2 d_4}$
	$\widehat{F_v H_\phi}$			$\widehat{F_v B_\phi}$			$\widehat{B_\phi H_v}$		$\widehat{b_3 d_4}$
		$\widehat{F_y H_\psi}$			$\widehat{F_y B_\psi}$			$\widehat{B_\psi H_y}$	$\widehat{c_1 d_2}$
		$\widehat{F_\phi H_y}$			$\widehat{F_\phi B_y}$			$\widehat{B_y H_\phi}$	$\widehat{c_1 d_3}$
		$\widehat{F_\phi H_\psi}$			$\widehat{F_\phi B_\psi}$			$\widehat{B_\psi H_\phi}$	$\widehat{c_4 d_1}$
		$\widehat{F_y H_v}$			$\widehat{F_y B_v}$			$\widehat{B_v H_y}$	$\widehat{c_2 d_3}$
		$\widehat{F_v H_\psi}$			$\widehat{F_v B_\psi}$			$\widehat{B_\psi H_v}$	$\widehat{c_2 d_4}$
		$\widehat{F_\phi H_v}$			$\widehat{F_\phi B_v}$			$\widehat{B_v H_\phi}$	$\widehat{c_3 d_4}$

$$
\begin{aligned}
&\widehat{l_\xi n_\zeta}\ (\widehat{F_\psi H_y}\widehat{a_1 d_2}+\widehat{F_\phi H_y}\widehat{a_3 d_1}+\widehat{F_\phi H_\psi}\widehat{a_1 d_4}+\widehat{F_y H_v}\widehat{a_3 d_2}+\widehat{F_\psi H_v}\widehat{a_2 d_4}+\widehat{F_v H_\phi}\widehat{a_3 d_4})+\\
&+\widehat{y_\zeta n_\xi}\ (\widehat{F_\psi H_y}\widehat{b_1 d_2}+\widehat{F_\phi H_y}\widehat{b_3 d_1}+\ldots\ldots)+\\
&+\widehat{y_\zeta l_\xi}\ (\widehat{F_y H_\psi}\widehat{c_1 d_2}+\widehat{F_\phi H_y}\widehat{c_1 d_3}+\ldots\ldots)+\\
&+\widehat{l_\xi n_\delta}\ (\widehat{F_\psi B_y}\widehat{a_1 d_2}+\widehat{F_\phi B_y}\widehat{a_3 d_1}+\ldots\ldots)+\\
&+\ldots\ldots\ldots+\\
&+\widehat{y_\delta l_\zeta}\ (\widehat{B_\psi H_y}\widehat{c_1 d_2}+\ldots\ldots+\widehat{B_v H_\phi}\widehat{c_3 d_4})\text{ are the required terms.}
\end{aligned}
$$

It should be noted that each gearing product such as $\widehat{F_\psi H_y}$ occurs once in each block. Table 5B gives the complete expressions for the minors. For example,

$$\widehat{b_1 d_2} = \lambda^2 + l_p\lambda + l_v V_e \sin\alpha_e\,.$$

TABLE 5B

	λ^3 coeff.	λ^2 coeff.	λ coeff.	Const. coeff.
$\widehat{a_1 d_2}$	.	.	$y_p - w_e + V_e \sin\alpha_e$	$g_1 + y_v V_e \sin\alpha_e$
$\widehat{a_3 d_1}$	.	.	$V_e \cos\alpha_e - u_e - y_r$	$y_v V_e \cos\alpha_e - g_2$
$\widehat{a_1 d_4}$	.	1	y_v	.
$\widehat{a_3 d_2}$	.	.	$(y_p - w_e)\cos\alpha_e + (y_r + u_e)\sin\alpha_e$	$g_1 \cos\alpha_e + g_2 \sin\alpha_e$
$\widehat{a_2 d_4}$	.	$y_p - w_e$	g_1	.
$\widehat{a_3 d_4}$	.	$y_r + u_e$	g_2	.
$\widehat{b_1 d_2}$	.	1	l_p	$l_v V_e \sin\alpha_e$
$\widehat{b_3 d_1}$	.	$-e_x$	$-l_r$	$l_v V_e \cos\alpha_e$
$\widehat{b_1 d_4}$	.	.	l_v	.
$\widehat{b_3 d_2}$	.	$V_e(\cos\alpha_e + e_x \sin\alpha_e)$	$V_e(l_p \cos\alpha_e + l_r \sin\alpha_e)$	.
$\widehat{b_2 d_4}$	1	l_p	.	.
$\widehat{b_3 d_4}$	e_x	l_r	.	.
$\widehat{c_1 d_2}$	.	e_z	n_p	$n_v V_e \sin\alpha_e$
$\widehat{c_1 d_3}$	.	1	n_r	$-n_v V_e \cos\alpha_e$
$\widehat{c_4 d_1}$	.	.	$-n_v$	.
$\widehat{c_2 d_3}$	.	$-V_e(\sin\alpha_e + e_z \cos\alpha_e)$	$-V_e(n_p \cos\alpha_e + n_r \sin\alpha_e)$	.
$\widehat{c_2 d_4}$	e_z	n_p	.	.
$\widehat{c_3 d_4}$	1	n_r	.	.

The triple product terms are all contained in the product of two determinants:

$$\begin{vmatrix} y_\xi & y_\delta & y_\zeta \\ l_\xi & l_\delta & l_\zeta \\ n_\xi & n_\delta & n_\zeta \end{vmatrix} \times \begin{vmatrix} -1 & V_e \sin\alpha_e & -V_e \cos\alpha_e & \lambda \\ -F_v & -F_\phi & -F_\psi & -F_y \\ -B_v & -B_\phi & -B_\psi & -B_y \\ -H_v & -H_\phi & -H_\psi & -H_y \end{vmatrix}$$

The contribution to the stability polynomial is therefore equal to

$$\begin{aligned} &\lambda(y_\delta \widehat{l_\zeta n_\xi} + l_\delta \widehat{y_\xi n_\zeta} + n_\delta \widehat{y_\zeta l_\xi})(F_v \widehat{B_\phi H_\psi} + F_\phi \widehat{B_\psi H_v} + F_\psi \widehat{B_v H_\phi}) + \\ &+ V_e \cos\alpha_e(\;,, \quad ,, \quad ,,\;)(F_v \widehat{B_\phi H_y} + F_\phi \widehat{B_y H_v} + F_y \widehat{B_v H_\phi}) + \\ &+ V_e \sin\alpha_e(\;,, \quad ,, \quad ,,\;)(F_v \widehat{B_\psi H_y} + F_\psi \widehat{B_y H_v} + F_y \widehat{B_v H_\psi}) + \\ &+(\;,, \quad ,, \quad ,,\;)(F_\phi \widehat{B_\psi H_y} + F_\psi \widehat{B_y H_\phi} + F_y \widehat{B_\phi H_\psi}). \end{aligned}$$

As an example consider an autopilot whose equations are

$$\xi' = F_\phi \phi\,, \qquad \delta' = B_v v'\,, \qquad \zeta' = H_\psi \psi\,.$$

The stability polynomial is

$$\begin{aligned}
\Delta = \Delta_0 &+ F_\phi \lambda^{-1}(P_y y_\xi + P_l l_\xi + P_n n_\xi) + \\
&+ B_v(V_y y_\delta + V_l l_\delta + V_n n_\delta) + \\
&+ H_\psi \lambda^{-1}(R_y y_\zeta + R_l l_\zeta + R_n n_\zeta) + \\
&+ F_\phi B_v(-\wideparen{l_\xi n_\delta}\wideparen{a_3 d_4} - \wideparen{y_\delta n_\xi}\wideparen{b_3 d_4} + \wideparen{y_\delta l_\xi}\wideparen{c_3 d_4}) + \\
&+ B_v H_\psi(-\wideparen{l_\delta n_\zeta}\wideparen{a_2 d_4} + \wideparen{y_\delta n_\zeta}\wideparen{b_2 d_4} - \wideparen{y_\delta l_\zeta}\wideparen{c_2 d_4}) + \\
&+ F_\phi H_\psi(\wideparen{l_\xi n_\zeta}\wideparen{a_1 d_4} + \wideparen{y_\zeta n_\xi}\wideparen{b_1 d_4} + \wideparen{y_\zeta l_\xi}\wideparen{c_4 d_1}) - \\
&- F_\phi B_v H_\psi \lambda(y_\delta \wideparen{l_\zeta n_\xi} + l_\delta \wideparen{y_\xi n_\zeta} + n_\delta \wideparen{y_\zeta l_\xi}).
\end{aligned}$$

If the roll control equation is instead

$$\xi' = F_\phi \phi + F_p p'\,,$$

we must pick out all the F_ϕ terms in the expression for the stability polynomial and write $(F_\phi + F_p \lambda)$ wherever there is an F_ϕ. Similarly if the control equation contains a term $F_{\bar\phi} \frac{1}{D} \phi$, we must add $F_{\bar\phi} \lambda^{-1}$ wherever there is an F_ϕ.

13. *Some Characteristics of Linear Dynamic Systems.*

13.1. *General Nomenclature.*

The behaviour of many* linear systems can be specified in terms of a set of differential equations

$$\left.\begin{aligned}
\Delta_0 x &= \Delta_x^a \delta_a + \Delta_x^b \delta_b + \ldots + F_x\,, \\
\Delta_0 y &= \Delta_y^a \delta_a + \Delta_y^b \delta_b + \ldots + F_y\,,
\end{aligned}\right\} \tag{13.1}$$

and so on, where $x, y, \ldots$ are the variables; $\delta_a, \delta_b, \ldots$ represent control inputs such as motivator deflections; $F_x, F_y, \ldots$ are disturbances; and the Δ's are polynomials in the differential operator $D \equiv d/dt$ and having constant real coefficients. Any one of these equations may be derived by eliminating the unwanted variables from the *original equations* of motion, and the same polynomial Δ_0 will always be obtained on the left-hand side. Certain coefficients are given special names, and before considering the *overall equations*** of motion (13.1) we deal with a simple second-order equation

$$a\ddot{x} + b\dot{x} + cx = F\,, \tag{13.2}$$

---

*A report on notation does not seem a suitable place for dealing with the characteristics of a *general* linear system. There are many books covering this ground, for example Brown [32]. There appears to be no standard term for linear differential equations with constant coefficients, but the name 'panlinear' has been proposed[33].

**This term and others defined later were introduced in Ref. 34 in order to clarify the interpretation of certain properties of panlinear systems.

where x represents a linear or angular displacement. By analogy with a system comprising a mass, dashpot, and spring, we may call the three terms on the left-hand side the *inertia term*, *damping term*, and *stiffness term*. The coefficients a, b, c, respectively are named in the same way.

When we have two second-order equations in x and y, we can give useful names provided the variables have been chosen in a special way, and it is usual for this to be done. Consider

$$\left.\begin{aligned} A\ddot{x}+B\dot{x}+Cx+l\ddot{y}+m\dot{y}+ny &= F_1, \\ a\ddot{x}+b\dot{x}+cx+L\ddot{y}+M\dot{y}+Ny &= F_2. \end{aligned}\right\} \tag{13.3}$$

In general these equations are called the F_1 and F_2 equations, and F_1, F_2 might for example represent forces in two particular directions, whereas x, y might represent deflections in other directions. This procedure is unusual, and it is normally profitable to associate F_1 with the x direction, and F_2 with the y. This is also convenient for nomenclature, for we may then call A and L the *direct inertia coefficients*, and a, l the *cross inertia coefficients*. Similarly, B and M are *direct damping coefficients*, b and m are *cross damping coefficients*, C and N are *direct stiffness coefficients*, c and n are *cross stiffness coefficients*.

When F_1 is associated with the x direction the terms $A\ddot{x}$, $B\dot{x}$, Cx and $l\ddot{y}$, $m\dot{y}$, ny in the F_1 equation are called *direct terms* and *cross terms* respectively, and other equations are treated similarly. The presence of cross terms in *each* equation implies *coupling*, but cross terms are not synonymous with coupling terms except in special cases. Thus all coupling terms are cross terms when the coupling is simple, but in general additional coupling terms can appear as contributions to direct terms (*see* Ref. 33). It should be noted that cross terms, coupling terms, damping terms, stiffness terms are phrases that can also be used for non-linear systems, but it may not always be possible to define corresponding coefficients.

The definitions given above are not quite precise when x, y represent displacements in a mechanical system and are measured with respect to rotating axes. A term such as $B\dot{x}$ will contain two parts, one which represents a genuine force (e.g. due to a dashpot), and one which is kinematic in origin (e.g. a Coriolis term). It is desirable to restrict the meaning of the basic words 'damping' and 'stiffness', and to use them only when referring to forces produced by physical elements like dashpots and springs. It is proposed therefore that when kinematic terms are included in $B\dot{x}$ they should be called *virtual damping* terms, the total term being an *equivalent damping* term. Similar nomenclature applies to stiffness terms, and, with a slight difference, to inertia terms. A *virtual inertia* term will arise when a force proportional to $\ddot{x}$ is present. Such a force usually opposes the motion and the virtual inertia then augments the real inertia to give a total *equivalent inertia**.

It should be noted that the coefficient of $\ddot{x}$ is not called an inertia coefficient just because $\ddot{x}$ is the second derivative of a variable: it is because $\ddot{x}$ represents a linear or angular acceleration, and a term $A\dot{u}$ is also an inertia term provided u represents a velocity. In fact the three varieties of terms that arise in equations describing mechanical systems could perhaps be better described as *acceleration terms*, *velocity terms*, and *displacement terms*. In systems that are not mechanical, quantities analogous to displacement, velocity, force, etc. must be defined in order to generalise this view. It may be unhelpful to try to distinguish between direct terms and cross terms in non-mechanical systems (*see* Ref. 33).

When there are more than two variables the concepts of direct terms and coupling terms and the nomenclature for coefficients will still be relevant if the forces F_1, F_2, ... are associated directly with the variables x, y, ... respectively; but, if other equations are formed by elimination, no general nomenclature is proposed for the resulting terms or coefficients unless the elimination is complete and leads to

---

*It is common to replace genuine inertia terms by equal and opposite equivalent forces in order that the problem can be treated in some respects as one in statics (D'Alembert's principle). This is a mathematical trick, and the practice of describing a system as actually being always in equilibrium is deplored. This destroys the very meaning of the word equilibrium, and it is much more useful to have it available for distinguishing a state of zero acceleration.

the overall equations (13.1). Let us first consider these when there is no control action (i.e. all the δ's are zero). We then write

$$\Delta_0 x = K_n D^n x + K_{n-1} D^{n-1} x + \ldots + K_1 Dx + K_0 x = F_x(t), \tag{13.4}$$

and similarly for $\Delta_0 y$, etc. For simplicity we assume that the derivatives of $F_x(t)$ are all zero at $t = 0$, but this restriction will be removed later.

If the disturbance F_x is a step input to the system at rest, the initial response in x will be $D^n x = F_x/K_n$, and the final steady deviation will be $x = F_x/K_0$. We may therefore call K_n the *overall inertia of the system*, and K_0 the *overall stiffness of the system.* The ratio K_0/K_n is also significant since it specifies the initial response ($D^n x$) in relation to the ultimate intended change in x. It applies equally to the case where the system is displaced and held with a steady value of x, and then released. It is proposed that K_0/K_n be called the *effective stiffness of the system*, and the system is statically stable or unstable according as K_0/K_n is positive or negative.

The general solution of any one of the overall equations (13.1) is well known[35]. It depends amongst other things on the roots of an algebraic polynomial equation called variously the auxiliary, *stability*, or *characteristic, equation*:

$$K_n \lambda^n + K_{n-1} \lambda^{n-1} + \ldots\ldots + K_1 \lambda + K_0 = 0. \tag{13.5}$$

For instance, if there are no control inputs and a step disturbance is applied, the response will consist solely of the natural response of the system, and it may be expressed as the sum of terms representing (natural) *modes of motion*:

$$x = A_a \exp(-k_a t) + A_b \exp(-k_b t) + \ldots\ \ldots + A_r \exp(-k_r t) \sin(v_r t + \varepsilon_{xr}) + \ldots,$$
$$y = B_a \exp(-k_a t) + B_b \exp(-k_b t) + \ldots\ \ldots + B_r \exp(-k_r t) \sin(v_r t + \varepsilon_{yr}) + \ldots,$$
$$\ldots\ldots,$$

where $-k_a, -k_b, \ldots, -k_r \pm iv_r, \ldots$ are the n roots of (13.5), often called the *stability roots.* Modes a, b are two of the *exponential modes**, and mode r is one of the *oscillatory modes.* The quantities $A_a, B_a, \ldots; A_b, B_b, \ldots; A_r, B_r, \ldots$ are the *modal coefficients of amplitude***, and the *phase angles* $\varepsilon_{xr}, \varepsilon_{yr}, \ldots$ specify the phase of $x, y, \ldots$ in the mode r. All these are inter-related according to the initial conditions and to the values of the roots.

If some of the roots are repeated the modes become more complicated[36], but in practical problems the only likely case is $k_a = k_b = k$, and then modes a and b combine to form a single mode represented as $(A + A_1 t) \exp(-kt)$. Alternatively this may be regarded as the limit of an oscillatory mode as $k_r \to k$ and $v_r \to 0$. In the present discussion no other type of repeated root is considered. If k is positive, such a mode is said to be *critically damped.*

---

*A mode associated with a real root has traditionally been termed 'aperiodic', but this simply means 'non-periodic' and is not really suitable in view of the standard mathematical definition of a periodic function. For systems in general it seems more useful to distinguish between oscillatory and non-oscillatory quantities, the former being allowed to have a varying frequency and possessing a period only when the frequency is constant. The further restriction of a constant amplitude is required for a quantity to be called periodic. (*See* Ref. 33).

**It will often be convenient to use symbols corresponding to the variable. For example $A_a, A_b, \ldots$ can be replaced by $x_a, x_b, \ldots$, and $B_a, B_b, \ldots$ by $y_a, y_b, \ldots$. Numerical suffixes may be disliked because $x_1, x_2, \ldots$ are commonly used to denote the values of x at $t = t_1, t_2$, etc.

The exponential mode $A \exp(-kt)$ associated with a simple real linear factor $(\lambda+k)$, is called a *subsidence* or a *divergence* according as k is positive or negative (stable or unstable mode). When there is a simple real quadratic factor $(\lambda^2+2k\lambda+\omega^2)$ with ω positive, it is convenient to write it also as $(\lambda^2+2E\omega\lambda+\omega^2)$, where E is real. When $|E|>1$ the quadratic factor has two distinct real factors corresponding to two separate exponential modes. When $|E| = 1$ the factor is a perfect square $(\lambda\pm\omega)^2, k = \pm\omega$, but there is just one corresponding mode as already mentioned, and such a mode is critically damped if $k = +\omega$. When $|E| < 1$ the quadratic factor has two conjugate complex factors $(\lambda+k+iv)$ and $(\lambda+k-iv)$, which taken together correspond to an oscillatory mode which is stable or unstable according as k (and hence E) is positive or negative. A stable oscillation is variously described as damped, decaying, convergent, decreasing, etc., and an unstable one as negatively damped, growing, divergent, increasing, and so on. When its damping is zero ($k = 0$), an oscillatory mode has a constant amplitude, and it is said to be undamped or steady: in a panlinear system the frequency will naturally be constant.

The custom of using the words 'damped' and 'damping' when describing the response of a system would seem to clash with their application to terms in the original equations of motion. For a second-order equation there is no conflict, and this is why the dual usage has come about. It seems impracticable at present to abandon this usage*, but ambiguity can easily be removed by adding the word 'effective' when we refer to the response, although for measures of damping (*see* Section 13.2) this is unnecessary – it is unlikely that phrases like damping index or damping angle would be interpreted as referring to terms in the original equations.

It has been stated earlier that the effective damping of a mode, that is the stability of a mode, is directly connected with a root or roots of the stability equation. It is reasonable to define the quantity K_{n-1}/K_n as the *total effective damping of the system*, since this is equal to the sum of the roots with signs reversed. It is thus equal to the sum of $k_a, k_b, \ldots, 2k_r, \ldots$, which are the individual effective dampings of the modes. The system is stable provided that $k_a, k_b, \ldots, k_r, \ldots$ are all positive, whereas it is unstable if any of these are negative. The total effective damping is the amount available to be shared among the modes, and the amount taken up by an oscillatory mode is $2k_r$, since the corresponding factors are $(\lambda+k_r+iv_r)(\lambda+k_r-iv_r)$.

Oscillatory modes and pairs of exponential modes correspond to second-order equations, but a single exponential mode, which will always be present if the stability polynomial is of odd degree, corresponds to a first-order equation. In a mechanical system we have in general elements corresponding to inertia, damping, and stiffness, and a first-order equation cannot give a true representation of the dynamics. In some circumstances, however, a first-order equation may give an adequate approximation. Thus, when the stiffness is negligible, equation (13.2) reduces to

$$a\dot{u}+bu = F, \tag{13.6}$$

where $u = \dot{x}$, and when the inertia is negligible the equation becomes

$$b\dot{x}+cx = F. \tag{13.7}$$

The effective damping of the system corresponding to (13.6) is b/a, and for (13.7) it is c/b. The general concepts of overall inertia and overall stiffness are still valid but the terms should be used with discretion. There is no difficulty when the equation is of order three, five, and so on.

Now consider the overall equations (13.1) when there is just one control action and no disturbance. We then have

$$\left.\begin{aligned} \Delta_0 x &= \Delta_x \delta, \\ \Delta_0 y &= \Delta_y \delta, \end{aligned}\right\} \tag{13.8}$$

and so on, which may also be written

$$\frac{x}{\Delta_x} = \frac{y}{\Delta_y} = \ldots = \frac{\delta}{\Delta_0}. \tag{13.9}$$

*This matter is discussed in Ref. 33.

It may be convenient to expand the denominators as

$$\Delta_x = X_l D^l + \ldots + X_1 D + X_0, \text{ etc.,}$$

and

$$\Delta_0 = K_n D^n + \ldots + K_1 D + K_0 .$$

Equations like (13.9) have been introduced in Sections 12.1 and 12.2, and the polynomials Δ_x, Δ_y, ... called *complete response polynomials*, in order that component parts could be called *response polynomials.*

When discussing steady-state relationships we need only the constant coefficients, and we have

$$\left[\frac{x}{\delta}\right]_{\text{steady state}} = \frac{X_0}{K_0}.$$

The denominator has already been defined as the overall stiffness of the system, and the numerator is termed the *trim power* of the motivator δ as regards the variable x. Further aspects of steady states and quasi-steady states of manoeuvring are discussed in Section 14.2.

If, on the other hand, we examine the initial response of the system (previously at rest with δ zero) to a step change in δ, we have

$$\left[\frac{D^{n-l}x}{\delta}\right]_{t=0} = \frac{X_l}{K_n}.$$

This could be termed the 'coefficient of initial response', and $(n-l)$ would be the 'order of the initial response'. X_l could be called the 'initial power' of the motivator as regards x.

It is customary also to relate the outputs $x, y, \ldots$ to an input such as δ by means of transfer functions. If we write

$$x = T_x \delta, \quad y = T_y \delta, \ldots , \tag{13.10}$$

then $T_x(D)$, $T_y(D)$, etc. are called *transfer functions.* Thus for panlinear systems and in our notation

$$T_x = \frac{\Delta_x}{\Delta_0}$$

$$= \frac{X_l D^l + \ldots X_1 D + X_0}{K_n D^n + \ldots K_1 D + K_0}$$

$$= \frac{X_l\,(D+b_1)\,(D+b_2)\ldots(D^2+2b_s D+c_s^2)\ldots}{K_n\,(D+k_a)\,(D+k_b)\ldots(D^2+2k_r D+\omega_r^2)\ldots}. \tag{13.11}$$

It is not implied by the example that numerical and alphabetical suffixes are always preferred for the numerator and denominator respectively. The American practice of writing a quadratic factor as $(D^2+2\zeta\omega D+\omega^2)$, which is the equivalent of our $(D^2+2E\omega D+\omega^2)$, regardless of whether it belongs to a response polynomial or to the stability polynomial does not seem to have any advantage, as the coefficients b_s, c_s cannot in general be associated with physical characteristics like damping and frequency. However, for some purposes it is convenient to have a notation for the factors of $(D^2+2b_s D+c_s^2)$, and these may be written as $(D+b_s+id_s)\,(D+b_s-id_s)$.

In particular, when for control purposes the input δ is made to depend on a response variable x, the factors of the numerator of T_x may indicate limiting values of the stability roots. Thus, if $\delta = \delta_0 - Gx$, where δ_0 is an independent input and G a scaler parameter, we have

$$(\Delta_0 + G\Delta_x)x = \Delta_x\delta_0,$$

so that

$$x = \frac{\Delta_x}{\Delta_0 + G\Delta_x}\delta_0\,,$$

and the stability polynomial for the closed loop is $\Delta_0 + G\Delta_x$. When G becomes very large, some of the stability roots then approach the roots of the numerator Δ_x of the open-loop transfer function.

It should be noted that if we take the (unilateral* or one-sided) Laplace transform of equations (13.8), and we write $\bar{x}$ for the transform of x, and λ for the Laplace parameter, then

$$\Delta_0(\lambda)\,\bar{x} = \Delta_x(\lambda)\,\bar{\delta} + \text{initial condition terms}\,,$$

and therefore $\bar{x}/\bar{\delta}$ is not always equal to $T_x(\lambda)$. The latter is sometimes called the transform transfer function[37], although one should expect such a name to refer to $\bar{x}/\bar{\delta}$. $T(i\omega)$ may be called the harmonic transfer ratio (*see* Ref. 33), where ω is the angular frequency of a sinusoidal input.

The notation above can be extended to include a disturbance by writing

$$\Delta_0 x = \Delta_x\delta + \Gamma_x\gamma\,,$$

where the disturbance F_x is expressed in terms of some variable γ, and Γ_x is a polynomial in D. We can also write

$$x = T_x^\delta\delta + T_x^\gamma\gamma\,,$$

or

$$x = T_{x\delta}\delta + T_{x\gamma}\gamma\,,$$

where T_x^δ is the control transfer function, and T_x^γ is the disturbance transfer function. If it were necessary to include several control inputs $\delta_a, \delta_b, \ldots$, and several disturbance variables $\gamma_A, \gamma_B, \ldots$, we would have to write something like

$$\Delta_0 x = \Delta_x^a\delta_a + \Delta_x^b\delta_b + \ldots + \Gamma_x^A\gamma_A + \Gamma_x^B\gamma_B + \ldots\,,$$

$$x = T_x^a\delta_a + T_x^b\delta_b + \ldots + T_x^A\gamma_A + T_x^B\gamma_B + \ldots.$$

13.2. *Measures of Damping and Frequency.*

For both exponential and oscillatory modes the quantity $A\exp(-kt)$ is the *amplitude* at time t. The contribution of a mode $A\exp(-kt)$ to the value of x at $t = 0$ is just A, the initial amplitude, but the corresponding contribution of an oscillatory mode $A\exp(-kt)\sin(\nu t + \varepsilon)$ is $A\sin\varepsilon$, where ε is the phase angle.

The quantity k is the *damping index*, and is a measure of the amplitude reduction per unit time. The quantity ν is the *angular frequency*, and ω is the *undamped angular frequency*. Each of these may sometimes be described as a *natural ..... frequency* to distinguish it from the frequency of an oscillatory input, which may be described as having a *forcing frequency*. It is often convenient to work in terms of the ratio (forcing frequency)/(natural frequency), and this may be called a relative frequency.

It is unfortunate that for many years the terms 'damping coefficient' and 'damping factor' have been

---

**See* Brown[32]

used for both k and E, the latter defining damping as a fraction of critical damping*. Confusion should be avoided by the adoption of the term *relative damping ratio* for E. The shortened names *relative damping* or *damping ratio* would be admissible. The latter is already in widespread use in servomechanism literature, but it is suggested that the other shortened name is preferable. The relative damping ratio E is related to the quantities k, ω, v by the following relations:

$$E = k/\omega\,,$$

$$v = \omega\sqrt{1-E^2}\,.$$

When $E < 1$, the quantity δ defined as $\sin^{-1}E$ is called the *damping angle*. It is related to the other measures of damping as follows:

$$\delta = \sin^{-1}(k/\omega) = \tan^{-1}(k/v)\,.$$

The quantity P, defined by

$$P = 2\pi/v\,,$$

is the *period* of the oscillation.

For an oscillatory mode it can be convenient to express time in terms of the *time angle* (φ), or the *number of cycles* (C), where

$$\varphi = vt\,,$$

$$C = vt/2\pi\,.$$

The corresponding analytical expressions for the mode are

$$A\exp(-kt)\sin(vt+\varepsilon) = A\exp(-\kappa\varphi)\sin(\varphi+\varepsilon) = A\exp(-\Delta C)\sin(2\pi C+\varepsilon)$$

$$= A\exp(-\kappa vt)\sin(vt+\varepsilon) = A\exp(-\Delta t/P)\sin(2\pi t/P+\varepsilon)\,,$$

where κ is the *angular damping index*, and Δ the *logarithmic decrement* (cyclic damping index) or '*log dec*'. These two measures of damping have been much used in the past, although κ has not previously been given a proper name—it was sometimes referred to as the 'R-over-J ratio', where $-R$ and J denoted the real and imaginary parts of complex roots. The three damping indices are related as follows:

$$k/v = \kappa = \Delta/2\pi = \tan\delta\,.$$

The ratio of the amplitude at any time t to its value at time $(t-1)$ is called the *persistence* (G), and is equal to $\exp(-k)$. A mode is stable or unstable, the amplitude reducing or growing, according as the persistence is less or greater than unity. When the amplitude is constant the persistence is unity.

Corresponding angular and cyclic persistencies exist in principle, as for the damping index. However,

---

*Many writers have used the symbol D or ζ to represent the fraction of critical damping. In aeronautical work it would be acceptable to use one of these provided there were no clash with D denoting the operator d/dt or with ζ denoting rudder deflection or equivalent.

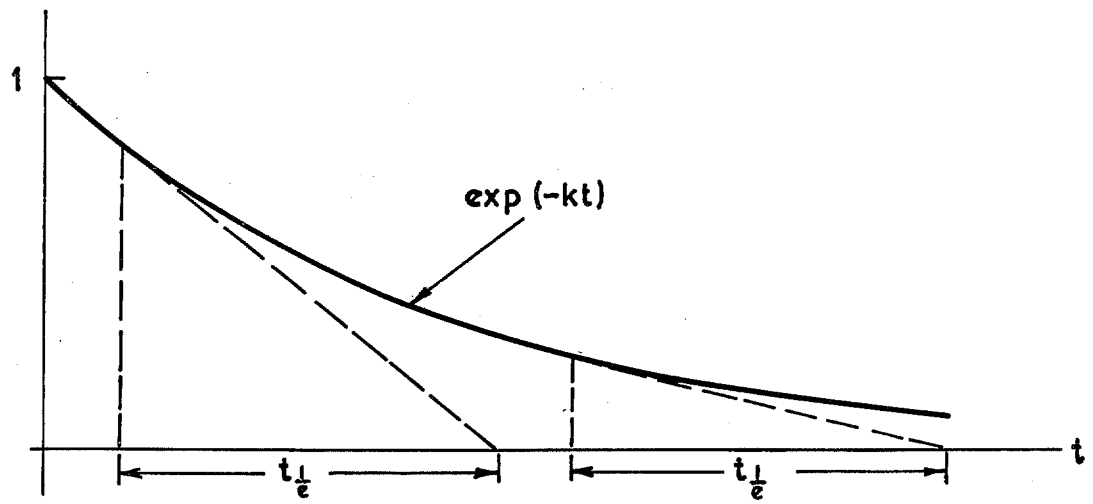
an angular measure is not likely to be used, and for a cyclic measure it seems more useful to define a related quantity, the *swing ratio** (Γ). This is the ratio of a peak excursion to the preceding one of opposite sign, and the oscillation is stable or unstable according as Γ is less or greater than unity. It should be noted that the cyclic persistence and swing ratio (or a sequence of swing ratios) will exist for any form of oscillatory response. The present section is essentially concerned with a simple linear mode, and in this case the swing ratio is equal to the persistence per half-cycle, which is given by $\Gamma = \exp(-\frac{1}{2}\Delta)$. Even for a single linear mode the swing ratio cannot in general be interpreted as an overshoot factor, since the latter usually refers to the *first* overshoot and therefore depends on the initial conditions.

The time ($t_{\frac{1}{e}}$) taken for the amplitude to diminish by a factor of e (2·718...) has often been called the time constant or damping time. The latter is not a good term because it might be interpreted as the time taken for the amplitude to decrease to some specified value, but the only objection to the former is that 'time constant' is also used extensively for specifying parameters of isolated elements with first-order lag. Thus an element with transfer function $1/(1+\tau D)$ is said to have a time constant of τ.

It is proposed that $t_{\frac{1}{e}}$ be called the *time to one-over-e*, but the symbol T may be used instead, particularly when additional suffixes are required. It is suggested that this quantity should not be called a time constant unless it is clear that it refers to a mode of motion rather than an element of the dynamic system. When the latter is a first-order one, it does not matter which term or symbol is used. In general, however, the values of $t_{\frac{1}{e}}$ for the various modes are complicated implicit functions of any τ time constants of elements of the system.

Corresponding to $t_{\frac{1}{e}}$ there are angular and cyclic measures of damping: the number of *radians to one-over-e* ($r_{\frac{1}{e}}$) and *cycles to one-over-e* ($C_{\frac{1}{e}}$). It is obvious that $t_{\frac{1}{e}} = 1/k$, $r_{\frac{1}{e}} = 1/\kappa$, $C_{\frac{1}{e}} = 1/\Delta$.

If a tangent is drawn to the curve $\exp(-kt)$, then the intercept on the time axis is always equal to $t_{\frac{1}{e}}$, as illustrated.



---

*In the past the term 'attenuation ratio per half-cycle' has sometimes been used. The term 'amplitude ratio' is not considered available as it is much used for other purposes. If angular and cyclic persistencies were defined, Γ and Γ_c would be appropriate symbols, and Γ_s would denote the swing ratio. The term *persistence index* seems reasonable for the quantity $\sigma = -k$ (*see* Ref. 33).

Section 13.2

In practice many people prefer to deal with amplitude changes of 1/2 rather than $1/e$. This may be interpreted as a preference for writing $2^{-k_B t}$ instead of $\exp(-kt)$, where

$$k_B = \frac{k}{\log_e 2},$$

and may be called the *binary damping index**. The *time to half-amplitude* ($t_{\frac{1}{2}}$) is equal to $1/k_B$.

The corresponding angular and cyclic quantities may be defined but the angular ones are not likely to be used. We thus define the *binary log dec* (Δ_B), and its reciprocal the number of *cycles to half-amplitude* ($C_{\frac{1}{2}}$), where

$$\Delta_B = \frac{\Delta}{\log_e 2} = \frac{1}{C_{\frac{1}{2}}}.$$

Occasionally unstables modes may be described in terms of $t_{\frac{2}{1}}$ or $C_{\frac{2}{1}}$, the time or number of cycles to double-amplitude.

Of the symbols discussed above, only a few are required when the time-vector method[38,39] is used for representing the contributions of an oscillatory mode in each equation of motion, and each kinematic equation, by a polygon of vectors. The undamped angular frequency (ω) and the damping angle (δ) are required for all response variables, and the relative phases of the latter are denoted by ε with suitable suffixes as described below.

A quantity $u = u_1 \exp(-kt) \sin vt$ is represented as a line of length u_1 at zero phase. Since

$$\begin{aligned} \dot{u} &= \omega u_1 \exp(-kt) \cos(vt+\delta) \\ &= \omega u_1 \exp(-kt) \sin(vt+\tfrac{1}{2}\pi+\delta), \end{aligned}$$

$\dot{u}$ will be represented by a line of length ωu_1 at phase $(\frac{1}{2}\pi+\delta)$. Similarly,

$$\ddot{u} = \omega^2 u_1 \exp(-kt) \sin(vt+\pi+2\delta),$$

and

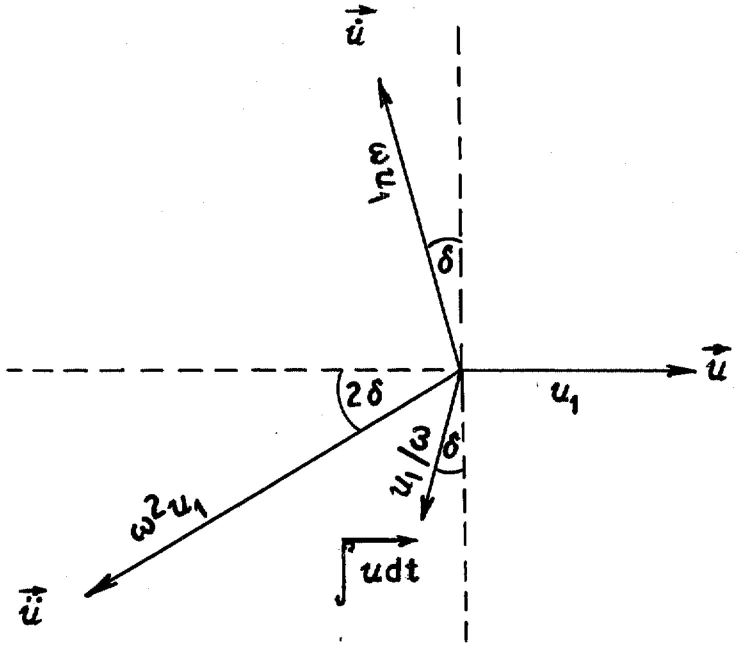
$$\int u\,dt = \frac{1}{\omega} u_1 \exp(-kt) \sin(vt-\tfrac{1}{2}\pi-\delta),$$

---

*It is suggested in Ref. 20 that 'exp' can be given more generality, so that in our case we could write

$$\exp(-kt) = \exp_2(-k_B t).$$

so that the four vectors representing u, $\dot{u}$, $\ddot{u}$, $\int u\,d\,t$ are drawn as follows:



If

$$u = u_1 \exp(-kt) \sin(\nu t + \varepsilon_u),$$

then ε_u is called the phase of u, and all vectors associated with u are rotated through ε_u anti-clockwise.
The terms and symbols introduced in this Section are summarised in Tables 9 and 10.

TABLE 9

Measures of Damping.

Name	Symbol	Definition
Damping index	k	Amplitude $= A\exp(-kt)$
Angular damping index	κ	Amplitude $= A\exp(-k\nu t)$: therefore $\kappa = k/\nu = \tan\delta$
Logarithmic decrement (cyclic damping index)	Δ	Amplitude $= A\exp(-\Delta t/P)$: therefore $\Delta = kP = 2\pi\kappa$
Relative damping ratio (*relative damping*, or *damping ratio*)	E (or ζ)	Amplitude $= A\exp(-E\omega t)$: therefore $E = k/\omega = \sin\delta$ = fraction of critical damping
Damping angle	δ	$\delta = \sin^{-1}E = \tan^{-1}(k/\nu)$
Binary damping index	k_B	Amplitude $= A2^{-k_Bt}$: therefore $k_B = \dfrac{k}{\log_e 2} = \dfrac{1}{t_{\frac{1}{2}}}$
Binary log. dec.	Δ_B	Amplitude $= A2^{-\Delta_B t/P}$: therefore $\Delta_B = \dfrac{\Delta}{\log_e 2} = \dfrac{1}{C_{\frac{1}{2}}}$
Time to $1/e$	$t_{\frac{1}{e}}$ (or T)	$t_{\frac{1}{e}} = 1/k$
Radians to $1/e$	$r_{\frac{1}{e}}$	$r_{\frac{1}{e}} = 1/\kappa$
No. of cycles to $1/e$	$C_{\frac{1}{e}}$	$C_{\frac{1}{e}} = 1/\Delta$
Time to half-amplitude	$t_{\frac{1}{2}}$	$t_{\frac{1}{2}} = \dfrac{\log_e 2}{k}$
Radians to half-amplitude	$r_{\frac{1}{2}}$	$r_{\frac{1}{2}} = \dfrac{\log_e 2}{\kappa}$
No. of cycles to half-amplitude	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}} = \dfrac{\log_e 2}{\Delta}$

TABLE 9—*continued*

Measures of Damping

Name	Symbol	Definition
Persistence	G	Ratio of amplitude to its value one unit of time earlier $G = \exp(-k) = \exp(\sigma)$
Persistence per cycle	Γ_c	Ratio of amplitude to its value one period earlier $\Gamma_c = \exp(-\Delta)$
Swing ratio	Γ (or Γ_s)	Ratio of a peak excursion to the preceding one of opposite sign $\Gamma = \exp(-\frac{1}{2}\Delta)$

TABLE 10

Measures of Frequency.

Name	Symbol	Definition
Angular frequency	ν	No. of radians per unit time $\nu = (\omega^2 - k^2)^{\frac{1}{2}}$ $= \omega \cos \delta$
Undamped angular frequency	ω	Value of angular frequency if damping were zero
(Cyclic) *Frequency*	ν_c	No. of cycles per unit time $\nu_c = \nu/2\pi$
Period	P	$P = 2\pi/\nu = 1/\nu_c$
Undamped period		Value of period if damping were zero

13.3. *A Discussion of the Merits of Various Measures of Damping.*

When the variations in stability of a complicated system are to be depicted as a parameter is altered, it is usual to show the values of the damping index and the angular frequency of each mode of motion. Transitions of oscillatory modes into two exponential modes and *vice-versa* are then shown directly, transition points occurring when the frequency becomes zero and there are two coincident values of damping index.

When attention is confined to oscillatory modes, it is often preferable to represent the damping per cycle, and it is difficult to say which of the measures listed in Table 9 is the most convenient. Perhaps the most universally recognised quantity is the relative damping ratio, but it does suffer from the disadvantage that much experience is necessary before one can correlate a given value of relative damping with the actual dynamic response of the system. Relative damping is a much more sensitive measure of changes in the response when the damping is low than when damping is more than half the critical value. Relative damping is a convenient algebraic paramcter, but it cannot be directly or simply derived from a record of the damped oscillation. The range of values of relative damping is very suitable, zero and unity corresponding to zero and critical damping respectively.

The above remarks are also relevant to the log dec (Δ) and the angular damping index (κ). The latter can be more convenient algebraically than relative damping, but the scale of values is markedly non-linear and critical damping corresponds to $\kappa = \infty$.

It will be remembered that $E = \sin\delta$ and $\kappa = \tan\delta$, so that $E \approx \kappa \approx \delta$ when the damping is low. The log dec is equal to $2\pi\kappa$ and therefore seems to be redundant except on historical grounds.

The time to half-amplitude and the cycles to half-amplitude have often been used, but there is a tendency now to use their reciprocals instead, that is the binary damping index and the binary log dec. It can be argued that the alleged convenience of the binary measures is superficial, since tables of $\exp(-x)$ are more readily available than tables of the function 2^{-x}. It is interesting that the binary log dec is approximately equal to ten times the relative damping up to values of about $E = 0{\cdot}4\,(\Delta_B = 9{\cdot}0647 \tan\delta, E = \sin\delta)$.

The persistence and the swing ratio are by definition directly related to the dynamic response presented in the form of a time-based record, and are unsuitable for algebraic work. One convenient way of estimating the persistence is to draw the 'envelope' of the record and then plot the logarithm of the amplitude (which is half the width of the envelope) against time. By taking two points on the resultant straight line (in practice the mean straight line) the persistence or the swing ratio can be read directly off the logarithmic amplitude scale. One point is taken at the intersection of one of the principal (multiple of 10) grid lines (*see* Fig. 6), and the other is taken unit time (or half a period) later. The log dec is almost as easy to determine. Once the swing ratio has been found, the log dec can be calculated from the relation $\Gamma = \exp(-\frac{1}{2}\Delta)$, either by consulting exponential tables, or taking logarithms, or by measurement of the scale factor used in plotting the amplitude. For this purpose, setting off a full period and first determining the persistence per cycle would be slightly more convenient, as $\Gamma_c = \Gamma^2 = \exp(-\Delta)$.

If the mean of the plotted points is very different from a straight line then the system dynamics is complex or markedly non-linear, and the question of what quantity is a valid measure of dynamic performance is raised. Even in a linear system it may be difficult to plot the amplitude of one particular mode when other modes are present, but a reasonable estimate may be made when the other modes are well damped and the mode being investigated is not well damped.

The damping angle and the relative damping are probably the most convenient of all the oscillation damping measures for use generally in the theoretical work, although it may be preferable to convert to other measures in order to interpret results. All the other measures, including relative damping, are elementary functions of the damping angle.

Corresponding numerical values of the oscillation damping measures are given in Fig. 7. An extended version of this diagram is given in Ref. 64, and covers a range of negative damping as well as positive.

14. *Centres of Force and their Relation to Control Discriminants.*

14.1. *Centres of Pressure and Aerodynamic Centres.*

It is sometimes convenient to pretend that some components of the aerodynamic moment are entirely due to forces acting at specific points, and these centres of force are traditionally called *centres of pressure.*

This is a well-known technique which replaces the forces and some of the moments acting on a rigid object by an equivalent system of forces. Increments of the moments from datum values may likewise be assumed to be the result of incremental force components acting at points called *aerodynamic centres*: it is advantageous to generalise the term 'aerodynamic centre' which in the past has been restricted to a particular incremental force, namely the increment in lift due to an incremental change in angle of attack. The positions of the centres of pressure and the aerodynamic centres may vary with the flight conditions and the mass distribution of the aircraft.

Consider the definition of centres of pressure in terms of configuration body axes (*see* Section 4.1). For this purpose a force equal to X is usually assumed to act along the longitudinal axis, and the rolling moment L is not replaced by an equivalent force. The pitching moment M and the yawing moment N are then assumed to be produced by forces equal to Z, Y respectively acting at centres of pressure on the longitudinal axis, and their distances aft of the reference point (the origin) are given by

$$H = \frac{M}{Z}, \quad I = -\frac{N}{Y}, \tag{14.1}$$

respectively. The reference point is usually taken to be in fore-and-aft alinement with the leading edge or other point on the second mean chord (*see* Section 16).

Centres of pressure and aerodynamic centres may be defined for parts of an aircraft as well as the whole: from now on, however, we shall consider the whole aircraft only.

The position of the centre of gravity may be specified by its coordinates x_G, y_G, z_G, but it can be convenient to write H_G for $-x_G$. The out-of-balance moments on the aircraft are given by

$$L - mg(z_G m_3 + y_G n_3),$$

$$ZH + mg(H_G n_3 + z_G l_3),$$

$$YI - mg(H_G m_3 + y_G l_3),$$

where l_3, m_3, n_3 are the direction cosines of the downward vertical relative to the configuration body axes. For a horizontal attitude l_3 and m_3 are zero while $n_3 = 1$, and if also y_G, z_G are zero the moments are equal to

$$L, \quad ZH + mgH_G, \quad YI,$$

so that when $Z = -mg$ the pitching moment becomes $mg(H_G - H)$.

In a similar way an increment in the pitching moment (for example $M_w w'$) is assumed to be due to the corresponding z-component of incremental force ($Z_w w'$) acting at an aerodynamic centre on the longitudinal axis. The distances of such aerodynamic centres aft of the reference point – which might be called 'aerodynamic arms'—are given by

$$K_u = \frac{M_u}{Z_u}, \quad K_w = \frac{M_w}{Z_w}, \quad K_\eta = \frac{M_\eta}{Z_\eta},$$

and so on. Similarly we may specify 'yaw' aerodynamic centres on the longitudinal axis in terms of aerodynamic arms such as

$$J_v = -\frac{N_v}{Y_v}, \quad J_r = -\frac{N_r}{Y_r},$$

aft of the reference point.

Control discriminants[34], which are discussed in Section 14.2, can often be expressed conveniently in terms of various aerodynamic arms. For example, a static trim margin for level flight can be defined as

$$K_s = K_w - K_u .$$

The distances H, H_G, K_u, ..., I, J_v, ... are usually normalised in terms of the appropriate aerodynamic representative length, in which case small letters are used:

$$(h, h_G, k_u, \ldots) = \frac{1}{l_1}(H, H_G, K_u, \ldots),$$

$$(i, j_v, \ldots) = \frac{1}{l_2}(I, J_v, \ldots).$$

Alternatively, $\breve{H}$, $\breve{K}_u$, $\breve{J}_v$, ... may be used instead of h, k_u, j_v, ..., so long as it is clear that the aero-normalised unit of length is l_1 for longitudinal quantities and l_2 for lateral.

When the terminology 'alpha and beta planes', 'alpha and beta incidence angles', is introduced (*see* Section 6.2), care should be taken, as the name 'alpha aerodynamic centre' could be misinterpreted (α might be used as an alternative variable to w).

14.2. *Control Discriminants.*

The ease with which an aircraft can be controlled through a particular motivator may be partly assessed by evaluating certain functions called *control discriminants.* It is assumed from the start that the aircraft can maintain any one of a range of steady flight conditions, each requiring its particular combination of motivator deflections. If we consider two steady states such that only one motivator has to be deflected by an increment δ' from its position in one state in order to maintain the other steady state, in which the other variables differ by increments x', y', ..., and if also we can establish that

$$\delta' = f_x x' = f_y y' = \ldots,$$

then $f_x, f_y, \ldots$ are the control discriminants for the motivator δ with respect to $x, y, \ldots$.

It is desirable that there should be no changes in sign of a discriminant when different pairs of steady states are taken for one aircraft. Furthermore, it may be possible from flight experience to prescribe favoured ranges of values for some discriminants.

When each of the steady states is an equilibrium condition, and the increments in δ and $x, y, \ldots$ are so small as to permit the use of linearized equations, we may write, from equations (13.9) and (13.11),

$$\frac{x'}{X_0} = \frac{y'}{Y_0} = \ldots = \frac{\delta}{K_0},$$

since all the derivatives $\dot{x}, \ddot{x}, \ldots, \dot{y}, \ddot{y}, \ldots$ are zero. The control discriminants are therefore equal to K_0/X_0, $K_0/Y_0, \ldots$, and since these are related through K_0 to the classical concepts[40,41,34] of static margin and static stability, they may be called *static trim discriminants*, although the term *system trim discriminant* might be preferred, particularly when systems other than aircraft are considered. There should be no confusion in merely calling them *trim discriminants.*

Similarly when one of the steady states is an equilibrium condition and the other consists of a manoeuvre (involving for example a steady angular velocity or a steady centripetal acceleration), the discriminants may be called *manoeuvre discriminants* by analogy with the classical term 'manoeuvre margin'[42,43].

Many implications of this approach to the formulation of control criteria have yet to be explored, and it is premature to recommend a definite notation in this Report. It can be noted, however, that the quantity defined in Section 13.1 as the overall stiffness of the system is the same as the K_0 involved in the static trim discriminant. Strictly a system can have only one K_0, but in the analysis of a manoeuvre the overall equations of motion can often be replaced by a simpler approximate set. The K_0 for this set is called the *manoeuvre stiffness*. An alternative definition is often feasible based on approximate expressions for the transfer functions T_x, T_y, etc.

For example, in the well-known theory of the manoeuvre margin it is assumed that the speed is constant during a pitching manoeuvre, and the longitudinal equations of motion for aerodynamic-body axes reduce to the approximate form

$$(1+z_{\dot{w}})\,\dot{w}' + z_w w' + (z_q - u_e)\,q' + z_\eta \eta' = 0\,,$$

$$m_{\dot{w}}\dot{w}' + m_w w' + \dot{q}' + m_q q' + m_\eta \eta' = 0\,,$$

and the constant coefficient in the stability polynomial is then equal to $(u_e m_w + z_w m_q - z_q m_w)$. This is quite different from the K_0 given in Section 12.1 for the full equations, but it is often approximately equal to the coefficient K_3 given there*.

A factor of K_0 may be found to deserve special attention and definition. Thus, in the example just mentioned, if we divide by $-Z_w(u_e + Z_q/m)/I_y$ we obtain the factor

$$K_m = \frac{M_w}{Z_w} - \frac{M_q}{mu_e + Z_q}\,,$$

which represents a length and can be interpreted as the distance between two points on the x-axis. It may be possible also to find significant factors of X_0, Y_0, ...

Examples are given below for the static trim discriminant and manoeuvre discriminant corresponding to the flight conditions usually assumed for the definition of static and manoeuvre margins. The significant factors of the overall stiffness and of the manoeuvre stiffness are then lengths whose aero-normalised values are equal to, or nearly equal to, the classical static and manoeuvre margins. For example, the aero-normalised value of K_m (defined in the previous paragraph) is given by

$$\breve{K}_m \approx -\left(\frac{\breve{M}_w}{C_{L\alpha}} + \frac{\breve{M}_q}{\mu}\right),$$

if we assume that $\breve{Z}_w \approx -C_{L\alpha}$ and $\mu + \breve{Z}_q \approx \mu$. In old notation the manoeuvre margin was equal to $-\dfrac{2l}{\bar{c}}\left(\dfrac{m_w}{\partial C_L/\partial\alpha} + \dfrac{m_q}{\mu_1}\right)$, but the new notation eliminates the factor $2l/\bar{c}$.

It is proposed that in general the significant factor of the overall stiffness will be called the *static trim margin* or *(system) trim margin* and denoted by K_s, and that the significant factor of a particular manoeuvre stiffness will be called the *manoeuvre margin* for that manoeuvre and denoted by K_m.

The significant factors of the 'trim powers' X_0, Y_0, . . . are of the form $(aM_\delta + bZ_\delta + cX_\delta)$ and for most variables a will be made unity for a pitch motivator, b unity for a catanator, and c unity for a proaptor. The factors may be called 'trim power margins' for the motivator δ and the variables $x, y, \ldots$ when belonging to the static trim discriminants, and 'manoeuvre power margins' when belonging to a manoeuvre discriminant. Power margins for $x, y, \ldots$ could be represented by X_s, Y_s, . . . or X_m, Y_m, . . . as appropriate.

---

*In old notation the corresponding quantity $(\omega - z_w v)$ was approximately equal to the coefficient of λ^2 in the stability quartic. A quartic rather than a quintic was obtained because it was usual to assume that the atmosphere was uniform. It was also usual to neglect z_q.

It may be convenient also to have a name for the significant factor of the whole control discriminant, and 'control margin' is suggested. For example, the 'trim control margin' for the variable x would be equal to K_s/X_s, and the 'manoeuvre control margin' equal to K_m/X_m.

The inverse of a control discriminant (i.e. X_0/K_0) gives the steady-state value of x produced by unit deflection of the motivator and may therefore be called the 'trim sensitivity' or 'manoeuvre sensitivity' as appropriate. A significant factor would be called 'trim sensitivity margin' $\left(\frac{X_s}{K_s}\right)$ or 'manoeuvre sensitivity margin' $\left(\frac{X_m}{K_m}\right)$.

It is suggested that the symbols for denoting control discriminant and sensitivity discriminant should be δ_x and x_δ respectively since they can be interpreted as $\partial\delta/\partial x$ and $\partial x/\partial\delta$ respectively. Similarly the symbols for a control margin and sensitivity margin could be K_X and X_K (J_X and X_J for lateral). When necessary a second suffix (s or m) could distinguish the cases of static trim and manoeuvre. Typical symbols arising in longitudinal examples would be η_u, u_η K_U, U_K (or η_{us}, $u_{\eta s}$, ...) and η_q, q_η, K_Q, Q_K or η_{qm}, ...). Lateral examples would be ξ_p, p_ξ, J_P, P_J (or ξ_{pm}, ...).

After the simple examples of Section 14.2.1 the general derivation of significant factors of K_0, X_0, Y_0, ... is discussed briefly in Section 14.2.2. It is concluded that specific definitions should not be laid down in this Report, although some indication of the notation can be given.

The control discriminants are functions of the position of the centre of gravity of the aircraft. It is sometimes possible to locate the points for which discriminants are zero, and such points would be called *the neutral point* or *manoeuvre points*, corresponding to the static trim discriminant or manoeuvre dis-discriminant respectively. If these points lie on the x-axis the distance of the pitch neutral point aft of the reference point is denoted by H_n, and the distance of a pitch manoeuvre point by H_m. Similarly the position of the yaw neutral pointis specified by a distance I_n, and yaw manoeuvre points by I_m. Normalised values are denoted by h_n, h_m, i_n, i_m, or by $\breve{H}_n$, $\breve{H}_m$, $\breve{I}_n$, $\breve{I}_m$, as is done for centres of pressure and aerodynamic centres Normalised values of the static trim margin and a manoeuvre margin are likewise denoted by k_s, k_m or $\breve{K}_s$, $\breve{K}_m$.

When motivators are allowed to act freely without direct inputs from the pilot or control system actuators, the ideas developed above can be extended by treating the motivator deflection as one of the response variables x, y, ... For example, for a free elevator an additional system equation governing the elevator hinge moment is required, and tab deflection τ may take the place of δ in the analysis above. The additional suffix f could be used for distinguishing static trim margins and the like in the motivator-free case, for example as in K_{sf}. This is preferable to using K_s' in the Gates and Lyon style, since in the present notation K_s' could represent an increment in K_s.

14.2.1. *Control discriminants for an aircraft in straight unbanked level flight.*

(a) *Static trim discriminants.*

Consider two equilibrium flight conditions, one of which is straight and level, the other being a neighbouring condition with small perturbations u', w', θ, η' relative to the first, which is taken as a datum. It is assumed for simplicity that in both conditions the bank angle is zero and that perturbations in all other variables are zero. In addition body axes are chosen such that the centre of gravity lies on the x-axis.

In each condition the aerodynamic forces are in equilibrium with the gravitational force mg, and the resultant moment is zero. In the datum condition X_e and M_e are zero, and $Z_e = -mg$: in the perturbed condition X' and $(Z_e + Z')$ must balance mg, and M' must be zero. In other words $X' = mg\theta$, and Z', M' are zero, since the components of gravity along the x and z-axes in the perturbed position are $-g\theta$ and g respectively to first-order accuracy.

Fig. 8 illustrates the incremental forces acting on the aircraft in its neighbouring equilibrium condition. The forces proportional to u', w', η' are assumed to act along lines that intersect the x-axis at the points U, W, E which are at distances K_u, K_w, K_η from the origin. They can therefore be represented by the components $X_u u'$, $Z_u u'$ acting at U, and so on. For equilibrium,

$$X_u u' + X_w w' + X_\eta \eta' = mg\theta\,,$$

$$Z_u u' + Z_w w' + Z_\eta \eta' = 0\,,$$

$$M_u u' + M_w w' + M_\eta \eta' = 0\,,$$

where M_u, M_w, M_η may be replaced by $K_u Z_u$, $K_w Z_w$, $K_\eta Z_\eta$, respectively.

The algebraic solution of these equations is

$$\frac{\eta'}{\Delta} = \frac{w'}{\Delta_w} = \frac{u'}{\Delta_u} = \frac{\theta}{\Delta_\theta}, \tag{14.2}$$

where

$$\Delta = mg\overset{\frown}{Z_u M_w} = mg(Z_u M_w - Z_w M_u)$$

$$= mgZ_u Z_w (K_w - K_u)\,,$$

$$\Delta_w = mg\overset{\frown}{M_u Z_\eta}\,,$$

$$\Delta_u = mg\overset{\frown}{Z_w M_\eta}\,,$$

$$\Delta_\theta = M_\eta \overset{\frown}{X_u Z_w} + Z_\eta \overset{\frown}{M_u X_w} + X_\eta \overset{\frown}{Z_u M_w}\,.$$

The static trim discriminants are Δ/Δ_w, Δ/Δ_u, Δ/Δ_θ, and they are also equal to

$$\frac{-Z_w(K_w - K_u)}{M_\eta - K_u Z_\eta},\quad \frac{Z_u(K_w - K_u)}{M_\eta - K_w Z_\eta},\quad \frac{mgZ_u Z_w (K_w - K_u)}{\Delta_\theta},$$

respectively.

In order to relate these discriminants to the static margin we must take the expressions as given in aero-normalised units. If then we say that $(\breve{K}_w - \breve{K}_u)$ is common to all three discriminants, and is furthermore the only factor likely to change sign, we may claim that it is of particular significance and that it alone may form a basis for a criterion of static trim. This assumption is implied in the definition of static margin* by Gates and Lyon[41], which however is not identical with $(\breve{K}_w - \breve{K}_u)$ even after allowing for differences in normalising divisors. They defined static margin in such a way that K_n was in effect equal to $\dfrac{\breve{Z}_u \breve{Z}_w (\breve{K}_w - \breve{K}_u)}{C_L C_{L\alpha}}$, for *in their notation* $K_n = \dfrac{z_u}{C_L} \dfrac{2z_w}{\partial C_L/\partial\alpha} \dfrac{l}{\bar{c}} \left(\dfrac{m_w}{z_w} - \dfrac{m_u}{z_u} \right)$ when $C_R = C_L$.

This illustrates the arbitrary nature of the definition of a significant factor of the control discriminants, but nevertheless we can propose some definite notation. It must be possible to extract a factor which is a length, and this may serve as the static trim margin K_s although it would not in general be identical with the classical static margin. The latter was defined for a particular system with a prescribed set of equations and the control discriminants were therefore completely specified: this must be so once the equations are established. The usefulness of the quantity K_s will be much reduced unless it too has a unique definition for any particular system, but this can be achieved only by agreement. These remarks apply equally to the yaw static trim margin (J_s), and to any manoeuvre margins that may be defined (K_m, J_m).

---

*The old symbol was K_n.

Returning to the derivation of a static trim margin, we consider now an alternative approach to the algebraic development. We may treat the problem as one of statics, and take moments about U and W in Fig. 8. We then have

$$(K_\eta - K_u)\, Z_\eta \eta' = -(K_w - K_u)\, Z_w w',$$

and

$$(K_\eta - K_w)\, Z_\eta \eta' = (K_w - K_u)\, Z_u u',$$

and hence

$$\frac{\eta'}{K_w - K_u} = \frac{-Z_w w'}{M_\eta - K_u Z_\eta} = \frac{Z_u u'}{M_\eta - K_w Z_\eta}. \tag{14.3}$$

The expression for θ is a little more complicated and may be obtained either by taking moments about the point of intersection of the forces proportional to u' and w', or by substituting for u', w' in terms of η' in the X-force equation. The static trim margin is given by

$$K_s = K_w - K_u.$$

The point W is approximately the same as the traditional aerodynamic centre and is identical with it when aerodynamic-body axes are used. If there are no compressibility or distortion effects and if the thrust line passes through the centre of gravity, the aerodynamic centre U is practically coincident with G. In this simple case K_s is virtually equal to the distance of G from the neutral point, but this is not true in general. Complications in fact arise as soon as we take the datum condition to be non-level, quite apart from variations in the positions of aerodynamic centre and the centre of gravity. Expressions for static trim discriminants in the general case are given in Section 14.2.2.

(b) *Pitching manoeuvre at constant speed.*

Consider an equilibrium flight condition which is straight and level, and a neighbouring steady state in which a constant rate of pitch q $(=q')$ is maintained by an elevator deflection η' relative to the datum value η_e of equilibrium. Corresponding perturbations in the other longitudinal variables will in general exist (u', w', θ) and a proaptor deflection v' may be required. It is assumed that the aircraft is unbanked in each condition and that all perturbations take place in the longitudinal (vertical) plane.

Manoeuvre discriminants cannot be defined unless approximating assumptions are made, and it is usual to postulate that

(i) linearized equations for small perturbations are valid,

(ii) terms in θ (i.e. gravitational) can be neglected.

This leads to three equations in the five variables η', w', u', q', v', and if we adopt for simplicity aerodynamic-body axes with the x-axis passing through the centre of gravity the equations are

$$\left.\begin{aligned} X_\eta \eta' + X_w w' + X_u u' + X_q q + X_v v' &= 0, \\ Z_\eta \eta' + Z_w w' + Z_u u' + (Z_q + m u_e)\, q + Z_v v' &= 0, \\ M_\eta \eta' + M_w w' + M_u u' + (M_q + m u_e H_G)\, q + M_v v' &= 0. \end{aligned}\right\} \tag{14.4}$$

Some constraint must be imposed before these equations can be solved for the ratios η'/w', etc. The classical manoeuvre margin was based on the constraint $u' = 0$, and the small effects due to Z_v and M_v were ignored. Simple manoeuvre discriminants are then obtained, and a significant length along the x-axis extracted, but this is not possible if other constraints such as $v' = 0$ are assumed. The general problem is discussed in Section 14.2.2, and here we restrict ourselves to the classical assumptions already mentioned.

If we wish to reduce the problem to one of statics, we assume that forces $X_w w'$, $Z_w w'$ act at the aerodynamic centre W, and so on (Fig. 9). It is also convenient to define a centre of force C at which the resultant of the aerodynamic force $Z_q q$ and the centrifugal force $mu_e q$ is assumed to act. We define

$$K_q^* = \frac{M_q^*}{Z_q^*} = \frac{M_q + mu_e H_G}{Z_q + mu_e}. \tag{14.5}$$

M_q and Z_q are normally negative, and C is therefore forward of G.

Taking moments about C and W, we have

$$(K_\eta - K_q^*)\, Z_\eta \eta' = -(K_w - K_q^*)\, Z_w w',$$

$$(K_\eta - K_w)\, Z_\eta \eta' = (K_w - K_q^*)\, Z_q^* q,$$

and hence

$$\frac{\eta'}{K_w - K_q^*} = \frac{-Z_w w'}{M_\eta - K_q^* Z_\eta} = \frac{Z_q^* q}{M_\eta - K_w Z_\eta}. \tag{14.6}$$

This provides a basis for a manoeuvre margin K_m equal to $(K_w - K_q^*)$, and, if we allow for the difference in normalising divisors, $\breve{K}_m$ is in fact exactly equivalent to the classical expression, provided that $\breve{Z}_w$ is replaced by its approximate value $C_{L\alpha}$, Z_q is neglected in the denominator of (14.5), and the origin is at G.

14.2.2. *General interpretation of control discriminants.* When the aircraft is taken to be completely rigid and we consider purely longitudinal motion, there are just the three equations in X, Z, M and in seeking ratios η'/u', etc. we must restrict the number of variables to four. Let us assume that the four increments are unspecified to begin with, and denoted merely by x_1, x_2, x_3, x_4. The incremental forces proportional to x_1, x_2, x_3, x_4 are assumed to act at points P_1, P_2, P_3, P_4 which are at distances $K_1, \ldots$ from the origin 0 (Fig. 10). The components along body axes are $X_1 x_1$, $Z_1 x_1$, $\ldots$ and the moments are $M_1 x_1 = K_1 Z_1 x_1$, and so on, where $X_1, M_1, \ldots$ stand for $\partial X/\partial x_1$, $\partial M/\partial x_1$, etc.

It is convenient to take $Z_i x_i$ as the variables in order that the equations can be written as

$$\begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ 1 & 1 & 1 & 1 \\ K_1 & K_2 & K_3 & K_4 \end{bmatrix} \begin{bmatrix} Z_1 x_1 \\ Z_2 x_2 \\ Z_3 x_3 \\ Z_4 x_4 \end{bmatrix} = 0, \tag{14.7}$$

that is $C_1 Z_1 x_1 + C_2 Z_2 x_2 + C_3 Z_3 x_3 + C_4 Z_4 x_4 = 0$, etc. where $C_i = \cot \varepsilon_i = X_i/Z_i$. The solution of equations (14.7) is

$$\frac{Z_1 x_1}{A_1} = \frac{-Z_2 x_2}{A_2} = \frac{Z_3 x_3}{A_3} = \frac{-Z_4 x_4}{A_4}, \tag{14.8}$$

where

$$\begin{aligned} A_1 &= (K_2 - K_3)\, C_4 + (K_3 - K_4)\, C_2 + (K_4 - K_2)\, C_3 \\ &= K_{23} C_4 + K_{34} C_2 + K_{42} C_3 \\ &= -K_2 C_{34} - K_3 C_{42} - K_4 C_{23}, \end{aligned}$$

if we write $K_{23} = K_2 - K_3$, $C_{23} = C_2 - C_3$, etc.

Similarly
$$A_2 = K_{34}C_1 + K_{41}C_3 + K_{13}C_4$$
$$= -K_3C_{41} - K_4C_{13} - K_1C_{34},$$
$$A_3 = K_{41}C_2 + K_{12}C_4 + K_{24}C_1$$
$$= -K_4C_{12} - K_1C_{24} - K_2C_{41},$$
$$A_4 = K_{12}C_3 + K_{23}C_1 + K_{31}C_2$$
$$= -K_1C_{23} - K_2C_{31} - K_3C_{12}.$$

The A_i are lengths but a direct geometrical interpretation may not be illuminating. It is sometimes possible to divide equations (14.8) by a factor (say λ), and at the same time to rearrange them to give

$$\frac{x_1}{\lambda A_1} = \frac{-Z_2x_2}{\lambda A_2Z_1} = \frac{Z_3x_3}{\lambda A_3Z_1} = \frac{-Z_4x_4}{\lambda A_4Z_1}$$

i.e.
$$\frac{x_1}{A'_1} = \frac{-Z'_2x_2}{A'_2Z'_1} = \frac{Z'_3x_3}{A'_3Z'_1} = \frac{-Z'_4x_4}{A'_4Z'_1}, \tag{14.9}$$

where A'_1 is a more suitable length than A_1, and some of the other denominators in (14.9) can be expressed as

$$A'_2Z'_1 = M_1 + a_2Z_1 + b_2X_1$$
$$= M_1 + a'_2Z'_1 + b'_2X'_1, \tag{14.10}$$

and so on. If x_1 is identified with a motivator deflection η' the denominators defined as in (14.10) are modified M_η type derivatives and can be called trim power margins or manoeuvre power margins as mentioned earlier.

One way of finding a factor λ is to choose the line of action of one of the forces R_i, say R_4, as a base for centres of force and aerodynamic centres. Suppose the lines of action of R_1, R_2, R_3 meet the R_4 line at Q_1, Q_2, Q_3 and that P_4Q_1, P_4Q_2, P_4Q_3 are equal to K'_{14}, K'_{24}, K'_{34}. As illustrated at Q_1 we can assume incremental forces X'_1x_1, Z'_1x_1 to act at these points, and we can impose the condition that the moments about P_4 are unchanged, so that $Z_1K_{14} = Z'_1K'_{14}$, and so on. It is found that

$$K_{14} = K'_{14}C_{41}\sin\varepsilon_4,$$
$$Z'_1 = Z_1C_{41}\sin\varepsilon_4$$
$$= Z_1\cos\varepsilon_4 - X_1\sin\varepsilon_4,$$
$$X'_1 = X_1\cos\varepsilon_4 + Z_1\sin\varepsilon_4, \text{ etc.,}$$

and equations (14.9) are then valid if we choose

$$A'_1 = \frac{A_1}{C_{42}C_{43}\sin\varepsilon_4} = K'_{24} - K'_{34}, \tag{14.11}$$

$$\left.\begin{aligned} A_2'Z_1' &= \frac{A_2Z_1}{C_{43}} = M_1+\frac{Z_1(K_3C_4-K_4C_3)-X_1K_{34}}{C_{43}}, \\ A_3'Z_1' &= \frac{A_3Z_1}{C_{42}} = M_1+\frac{Z_1(K_2C_4-K_4C_2)-X_1K_{24}}{C_{42}}. \end{aligned}\right\} \tag{14.12}$$

The length A_1' has thus been given a direct geometrical meaning, namely the distance Q_2Q_3, and the quantities $A_2'Z_1'$ have the desired form. With this particular transformation based on the force R_4, it is not surprising that the expression for x_4 is dissimilar to the others and that $Z_4'/A_4'Z_1'$ is an indeterminate form. The transformed solution is in fact given by

$$\begin{aligned} \frac{x_1}{K_{24}'-K_{34}'} &= \frac{-Z_2'x_2}{A_2'Z_1'} = \frac{Z_3'x_3}{A_3'Z_1'} \\ &= \frac{R_4\sin(\varepsilon_3-\varepsilon_4)\sin(\varepsilon_2-\varepsilon_4)}{-Z_1A_4\sin\varepsilon_3\sin\varepsilon_2}, \end{aligned} \tag{14.13}$$

where $A_2'Z_1'$ and $A_3'Z_1'$ are given by (14.12) or alternatively by

$$\left.\begin{aligned} A_2'Z_1' &= M_1-Z_1'\left[K_{34}'+\frac{K_4}{C_{41}\sin\varepsilon_4}\right], \\ A_3'Z_1' &= M_1-Z_1'\left[K_{24}'+\frac{K_4}{C_{42}\sin\varepsilon_4}\right], \\ \text{and} \qquad -A_4Z_1 &= M_1C_{23}+Z_1(K_2C_3-K_3C_2)-X_1K_{23}. \end{aligned}\right\} \tag{14.14}$$

The cot terms such as C_2, C_{23} can of course be replaced by X_2/Z_2, $\left(\frac{X_2}{Z_2}-\frac{X_3}{Z_3}\right)$, and so on.

To illustrate this technique consider the general trimmed equilibrium condition where the datum attitude is Θ_e. In the neighbouring condition illustrated in Fig. 11, the incremental force due to gravity is $mg\theta$ along the line GQ_1, and we shall take this to be the R_4 line, the origin being chosen to be at the centre of gravity. If we define GU, GQ_3 to be K_u, K_u^0, etc. and define derivatives

$$X_u^0 = X_u\cos\Theta_e+Z_u\sin\Theta_e,$$

$$Z_u^0 = Z_u\cos\Theta_e-X_u\sin\Theta_e,$$

and so on, the equivalent of equations (14.13) becomes

$$\begin{aligned} \frac{\eta'}{K_w^0-K_u^0} &= \frac{-Z_w^0w'}{M_\eta-K_u^0Z_\eta^0} = \frac{Z_u{}^0u'}{M_\eta-K_w^0Z_\eta^0} \\ &= \frac{mg\theta}{M_\eta C_{wu}+Z_\eta(K_wC_u-K_uC_w)-X_\eta K_{wu}}, \end{aligned} \tag{14.15}$$

where as before
$$M_w = K_w Z_w = K_w^0 Z_w^0 ,$$

$$C_w = \frac{X_w}{Z_w},$$

$$C_{wu} = C_w - C_u ,$$

$$K_{wu} = K_w - K_u, \text{ etc.}$$

If we had drawn the diagram to show the datum state, the line GQ_1 would have been horizontal, the derivatives X_u^0, etc. would have been those along horizontal and vertical directions*, and the static trim margin can be interpreted as the distance between w' and u' aerodynamic centres fixed on the horizontal line through the centre of gravity.

The analogous treatment of the manoeuvre case is not so straightforward, and will not be discussed here.

Acknowledgements.

This Report includes many ideas arising out of earlier deliberations of an R.A.E. Panel (C. H. E. Warren, A. K. Weaver, S. Neumark, H. R. Hopkin) and out of more recent discussions with H. H. B. M. Thomas and others.

---

*It is important to distinguish between X_u^0 and $X_{u_0}^0$ (*see* Section 20), but the latter is not required here.

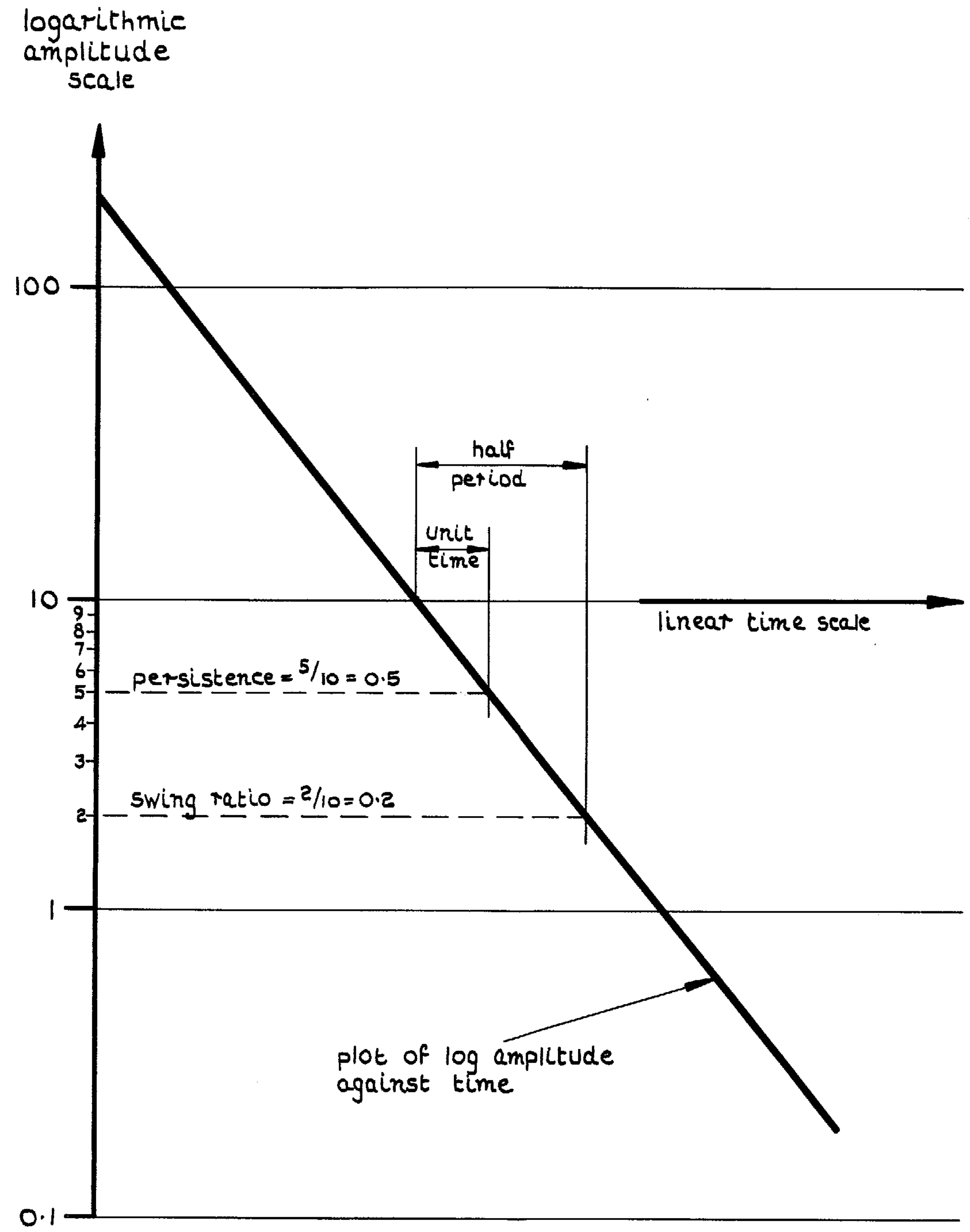


FIG. 6. Derivation of swing ratio and persistence.

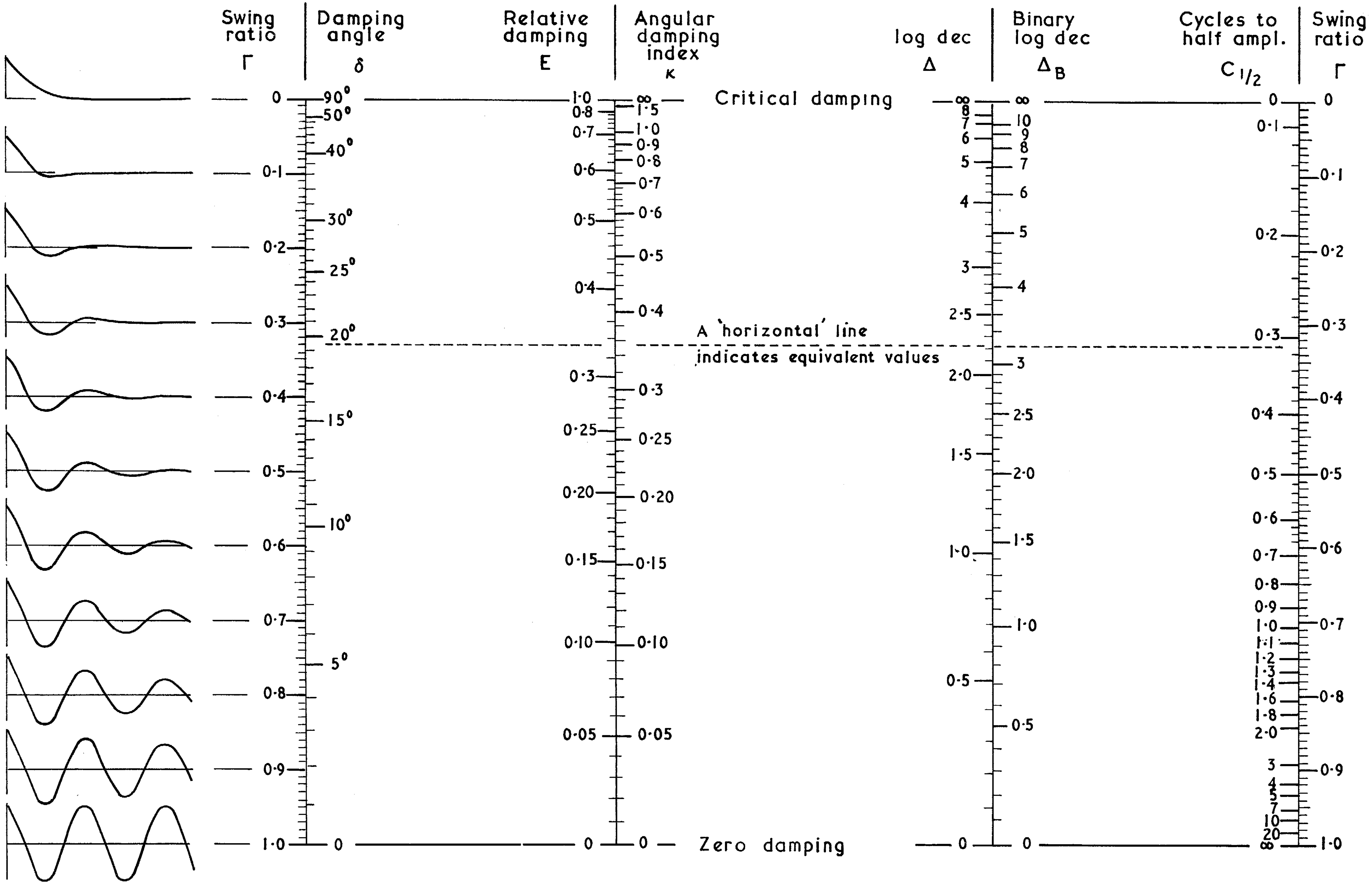


FIG. 7. Nomogram of equivalent measures of damping.

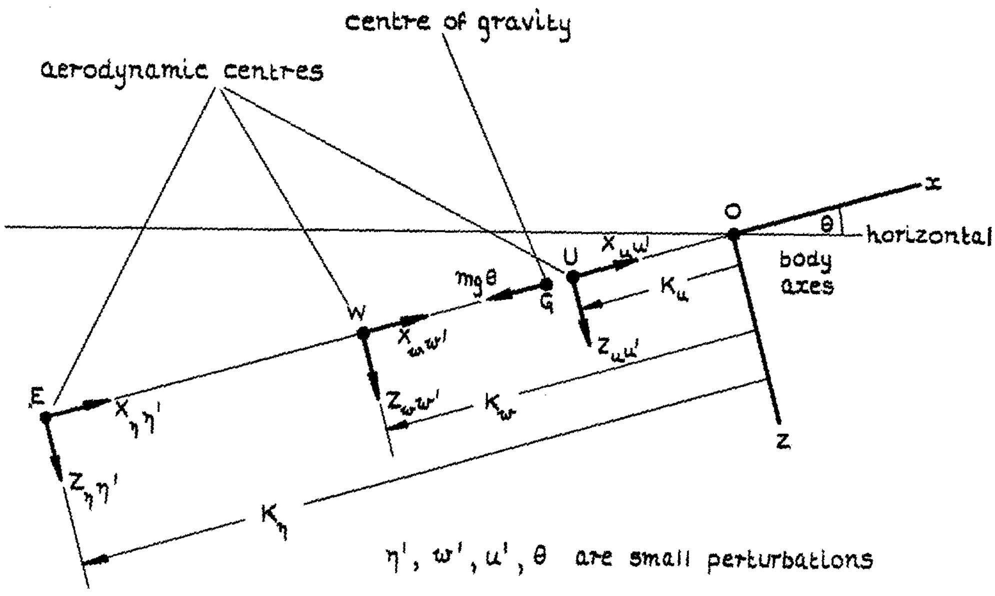


FIG. 8. Equilibrium of incremental forces (level datum).

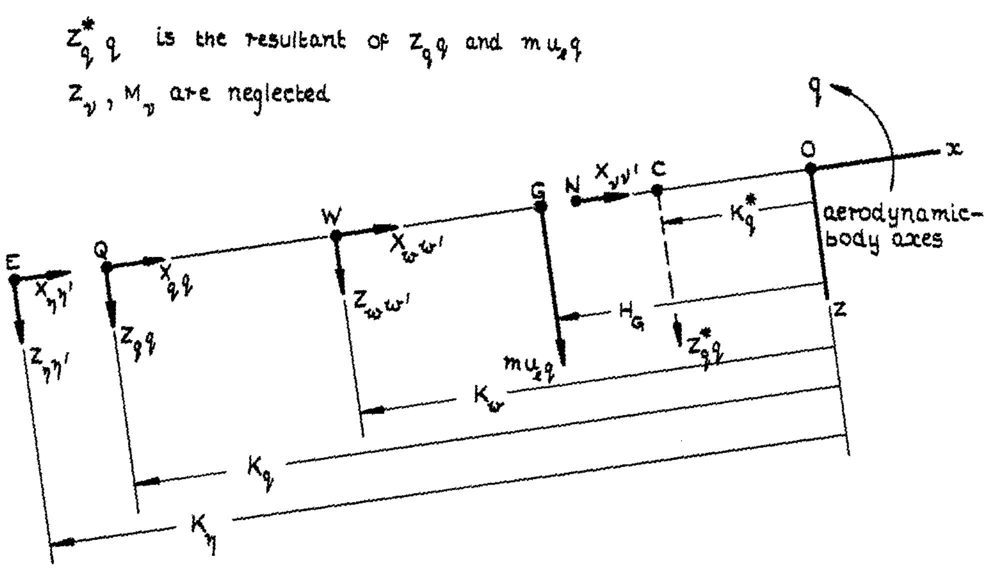


FIG. 9. Incremental forces in a steady pitching manoeuvre.

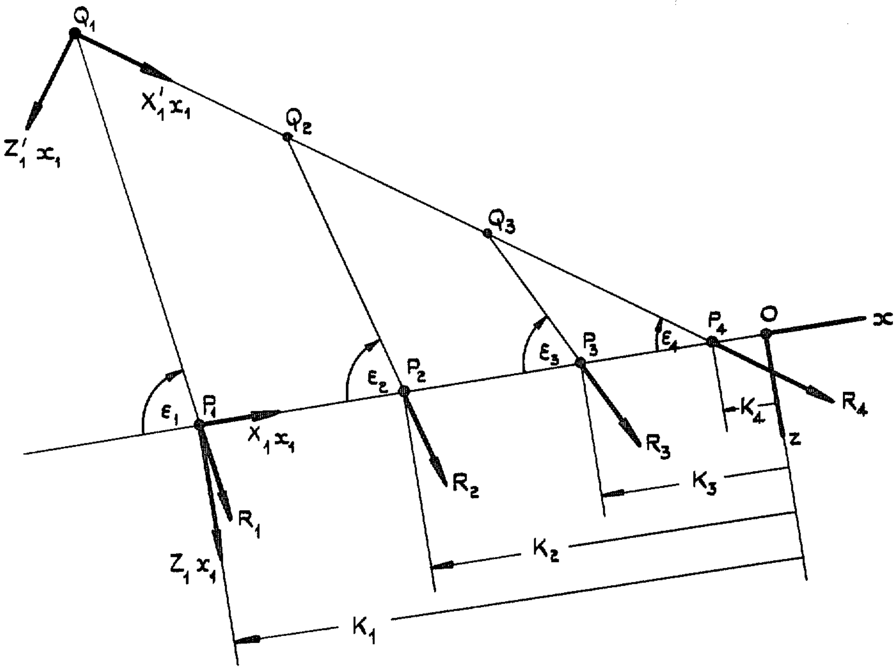


FIG. 10. Equilibrium of four incremental forces.

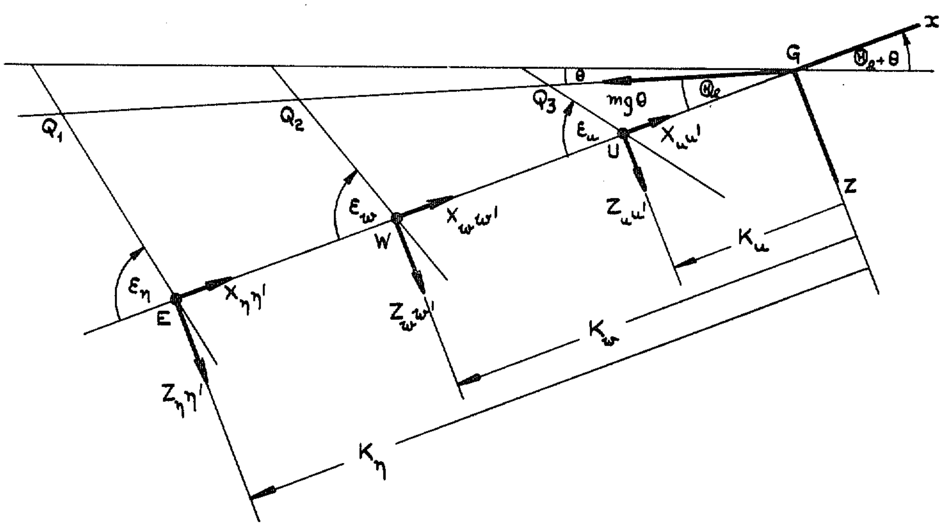


FIG. 11. Equilibrium of incremental forces. (Non-level datum).

Printed in Wales for Her Majesty's Stationery Office by Allens Printers (Wales) Limited

Dd. 135646 K 5

Published by
HER MAJESTY'S STATIONERY OFFICE

To be purchased from
49 High Holborn. London W.C.1
13a Castle Street, Edinburgh EH2 3AR
109 St. Mary Street, Cardiff CF1 1JW
Brazennose Street, Manchester M60 8AS
50 Fairfax Street, Bristol BS1 3DE
258 Broad Street, Birmingham 1
7 Linenhall Street, Belfast BT2 8AY
or through any bookseller

R. & M. No. 3562, Part 3

SBN 11 470228 4