Functional Analysis

Adrian Perez Keilty

Exam problems

a) Example of a completely continuous (compact) operator A on the real Hilbert space l_2 with infinite dimensional range:

In problem 2 of HTN10 we found that a sufficient condition on the sequence $\{a_n\}$ in order to make A completely continuous was that $\{a_n\} \in l_2$ so by choosing $\{a_n\} = \{\frac{1}{n}\}$ we obtain the following completely continuous operator:

$$A\colon l_2 \longrightarrow l_2$$

$$(x_1, x_2, ..., x_n, ...) \longmapsto (x_1, \tfrac{x_2}{2}, ..., \tfrac{x_n}{n}, ...)$$

Moreover, since $A(l_2)$ contains every canonical vector e_n for all $n \in \mathbb{N}$ ($A^{-1}(e_n) = n \cdot e_n \in l_2$) we have that A has also infinite dimensional range.

b) Example of an operator on l_2 which is not completely continuous:

The identity operator $I \in \mathcal{L}(l_2)$ is continuous but not completely continuous, since by taking the sequence $\{e_n\}$ which is bounded ($\|e_n\| = 1$) we cannot find any convergent subsequence since $\|e_i - e_j\| = \sqrt{2}\ \forall i \neq j$ (note: perhaps with a bit more time I can find more interesting non completely continuous operators on l_2)

c) Spectrum and corresponding eigenvalues of A from a) and Hilbert-Schmidt expansion of A:

By the Banach theorem on the inverse operator, A does not possess a continuous spectrum (see HTN8 problem 10) and thereby to find the spectrum of A it suffices to find all its eigenvalues. Now, A is a self adjoint operator since

$$(Ax, y) = \sum_{k=1}^{\infty} \frac{x_k}{k} \cdot y_k = \sum_{k=1}^{\infty} x_k \cdot \frac{y_k}{k} = (x, Ay)\ \forall x, y \in l_2$$

so all eigenvalues of A are real and the conditions of theorem 24.3.7 are satisfied, so from

$$Ax = \lambda x \Leftrightarrow \lambda x_1 = x_1, \lambda x_2 = \frac{x_2}{2}, ..., \lambda x_n = \frac{x_n}{n}$$

it is clear that the set of eigenvalues is $\{\lambda_n = \frac{1}{n}\}_{n\geq 1}$ with the corresponding eigenvectors $\{e_n\}_{n\geq 1}$. Thereby, by theorem 24.3.7 and since the Fourier coefficients can be expressed as $c_n = (x, e_n) = x_n$, the image of x under A can be written as

$$Ax = \sum_{n=1}^{\infty} \frac{x_n}{n} \cdot e_n$$