Functional Analysis

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HTN10

Problem 1 (p.251 Kolmogorov, Fomin)

Let H_1 be any subspace of a Hilbert space H and let $H_2 = H \ominus H_1$ so that every element $h \in H$ can be represented as $h = h_1 + h_2$. Then the projection operator $Ph = h_1$ is completely continuous if and only if the subspace H_1 is finite-dimensional.

Proof

- $\boxed{\Leftarrow}$ Since $P(H) = H_1$, the image of P is finite dimensional and therefore P has finite rank which implies that P is completely continuous (theorem C).

- $\boxed{\Rightarrow}$ For convenience, suppose that $H = l_2$ (no loss of generality since $H \cong l_2$) and that H_1 is infinite-dimensional. Then, since $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$ is a base for l_2 there exists a subset $\{e_j\}_{j\in J} \subset \mathcal{B}$ which is a base for H_1. Now, since $\{e_j\}_{j\in J} \subset H_1$, $\{e_j\}_{j\in J}$ is fixed under P, i.e, $\{P(e_j)\}_{j\in J} = \{e_j\}_{j\in J}$ and is a bounded sequence that cannot contain any convergent subsequence since $\|e_j - e_k\| = \sqrt{2}$ for all $j \neq k$. Thereby P is not completely continuous.

Problem 2 (p.251 Kolmogorov, Fomin)

The operator

$$A\colon l_2 \longrightarrow l_2$$

$$(x_1, x_2, ..., x_n, ...) \longmapsto (x_1, \tfrac{x_2}{2}, ..., \tfrac{x_n}{2^{n-1}}, ...)$$

is completely continuous.

Proof As hinted, since every bounded set is contained in a sphere it suffices to check that the images of spheres are relatively compact, and by the linearity of A this reduces to the case of the unit sphere.
For all $x \in B_1(0) = \{x \in l_2 : \|x\| < 1\}$ we have

$$|(Ax)_i| = |\frac{x_i}{2^{i-1}}| < \frac{1}{2^{i-1}} \implies \big[A(B_1(0))\big] = \Pi \equiv \text{ Hilbert cube}$$

Now, by theorem 11.3.3, a subset of a complete metric space (such as l_2) is relatively compact if and only if it is totally bounded, i.e, for all $\epsilon > 0$ there exists a finite ϵ-net associated to it.
Now, given $\epsilon > 0$ choose n such that

$$\frac{1}{2^{n-1}} < \frac{\epsilon}{2}$$

and to each $x \in \Pi$ associate the point $x^* = (x_1, ..., x_n, 0, 0, ...)$ so that

$$\rho(x, x^*) = \sqrt{\sum_{k=n+1}^{\infty} x_k^2} \leq \sqrt{\sum_{k=n}^{\infty} \frac{1}{4^k}} = \frac{1}{\sqrt{3}\cdot 2^{n-1}} < \frac{\epsilon}{2}$$

The set $\Pi^* = \{x^* : x \in \Pi\}$ is then a bounded set in a n-dimensional space so it is also totally bounded and thereby it possesses a finite $\frac{\epsilon}{2}$-net which is in fact a ϵ-net for Π since for all $x \in \Pi$ there exists a $\tilde{x}$ belonging to this $\frac{\epsilon}{2}$-net such that

$$\rho(\tilde{x}, x) \leq \rho(\tilde{x}, x^*) + \rho(x^*, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thereby, A is completely continuous (see ex 5 p.98).

A sufficient condition on the sequence $a = \{a_n\}$ so that $Ax = (a_1x_1, ..., a_nx_n)$ is completely continuous is that $\{a_n\} \in l_2$ since in this case given $\epsilon > 0$ we can choose $n \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} a_k^2 < \frac{\epsilon^2}{4}$$

and similarly as before associate to each $x \in \left[A(B_1(0))\right]$ the element $x^* = (a_1x_1, ..., a_nx_n, 0, 0, ...)$ to obtain

$$\rho(x, x^*) = \sqrt{\sum_{k=n+1}^{\infty} a_k^2 x_k^2} \leq \sqrt{\sum_{k=n+1}^{\infty} a_k^2} < \frac{\epsilon}{2}$$

For all $x \in B_1(0)$, $\|Ax\| \leq \|a\|$ so the set $\Pi^* = \{x^* : x \in \left[A(B_1(0))\right] \subset \left[A(B_1(0))\right]\}$ is bounded and finite dimensional so totally bounded. Same as before, by taking a $\frac{\epsilon}{2}$-net of Π^* we obtain a ϵ-net for $\left[A(B_1(0))\right]$ which proves it is relatively compact.

Problem 3 (p.251 Kolmogorov, Fomin)

The integral operator

$$\begin{aligned} A\colon C_{[-1,1]} &\longrightarrow C_{[-1,1]} \\ \phi(y) &\longmapsto (A\phi)(x) = \textstyle\int_{-1}^{x} \phi(y)dy \end{aligned}$$

maps the closed unit sphere in $C_{[-1,1]}$ into a non-compact set.

Proof See hint p.252. the sequence of functions in the unit sphere of $C_{[-1,1]}$

$$\phi_n(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ nx & \text{if } \quad 0 < x \leq \frac{1}{n} \\ 0 & \text{if } \quad \frac{1}{n} < x \leq 1 \end{cases} \tag{1}$$

is bounded ($\|\phi_n\|_\infty = 1$) and the sequence $\{(A(\phi_n)\}$ converges to

$$\chi(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ x & \text{if } \quad 0 < x \leq 1 \end{cases} \tag{2}$$

which having a discontinuous derivative at 0 cannot be the image under the integral operator of any function in $C_{[-1,1]}$. If the domain was the set of all functions then the pre-image under A would be

$$\tilde{\chi}(x) = \begin{cases} 0 & \text{if } -1 \leq x \leq 0 \\ 1 & \text{if } \quad 0 < x \leq 1 \end{cases} \quad \notin C_{[-1,1]} \tag{3}$$

To reconcile this with Theorem 24.1.1, according to it, the operator $(\tilde{A}\phi)(x) = \int_{-1}^{1} \phi(y)dy$ would be completely continuous since the kernel $K(x, y) = 1$ is bounded and has no discontinuities. If we look back at the preceding sequence,

$$(\tilde{A}\phi_n)(x) = \int_{-1}^{0} \phi_n(y)dy + \int_{0}^{\frac{1}{n}} \phi_n(y)dy + \int_{\frac{1}{n}}^{1} \phi_n(y)dy = ... = 1 - \frac{1}{2n} \xrightarrow[n\to\infty]{} 1 \in C_{[-1,1]}$$

Problem 5 (p.252 Kolmogorov, Fomin)

a) A linear combination of completely continuous operators on a Banach space is itself a completely continuous operator.

   Proof Let A, B completely continuous on a Banach space E and let $M \subset E$ a bounded set. Then, αM and βM are also bounded, so given $\epsilon > 0$, by hypothesis there exists finite $\frac{\epsilon}{2}$-nets Π_1 and Π_2 of $A(\alpha M)$ and

$B(\beta M)$ respectively. But then $\Pi_3 = \{x^* + \tilde{x} : x^* \in \Pi_1, \tilde{x} \in \Pi_2\}$ is a finite ϵ-net of $(\alpha A + \beta B)(M)$ since for all $x \in M$ there exists $x^* \in \Pi_1$ and $\tilde{x} \in \Pi_2$ such that

$$\|\alpha A(x) + \beta B(x) - (x^* + \tilde{x})\| \leq \|A(\alpha x) - x^*\| + \|B(\beta x) - \tilde{x})\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thereby, $(\alpha A + \beta B)(M)$ is totally bounded, or equivalently, relatively compact.

b) The set $\mathfrak{C}(E, E)$ of all completely continuous operators mapping a Banach space E into itself is a closed subspace of the linear space $\mathcal{L}(E, E)$ of all bounded linear operators mapping E into E.

Proof Since every completely continuous operator is linear and continuous, $\mathfrak{C}(E, E)$ is a a subset of $\mathcal{L}(E, E)$ and by the preceding section, $\mathfrak{C}(E, E)$ is also a linear space so it is a subspace of $\mathcal{L}(E, E)$. Furthermore, since every sequence of completely continuous operators $\{A_n\}$ that converges in norm to an operator A, i.e, $\|A - A_n\| \xrightarrow[n\to\infty]{} 0$, satisfies the property that A is itself a completely continuous operator (theorem 24.2.2), we have that $\mathfrak{C}(E, E)$ is also closed in $\mathcal{L}(E, E)$.

Problem 6 (p.252 Kolmogorov, Fomin)

a) $\mathcal{L}(E, E)$ is a ring when equipped with the usual operations of addition and multiplication of operators.

Proof We check the properties that need to be satisfied for addition and multiplication, namely:

- $A + B \in \mathcal{L}(E, E)$ for all $A, B \in \mathcal{L}(E, E)$ (additive closure)
- $O + A = A + O = A$ where O is the zero operator (additive identity element)
- $A + (-A) = (-A) + A = O$ where $-A(x) = (-1)A(x) \in \mathcal{L}(E, E)$ (additive inverse)
- $(A + B) + C = A + (B + C)$ since the sum in E is associative (additive associative property)
- $A \circ B \in \mathcal{L}(E, E)$ composition is well defined in $\mathcal{L}(E, E)$ (multiplicative closure)
- $I \circ A = A \circ I = A$ where I is the identity operator (Multiplicative identity)
- $A \circ (B \circ C) = (A \circ B) \circ C$ (multiplicative associative property)
- $A \circ (B + C)(x) = A(Bx + Cx) = A(Bx) + A(Cx) = A \circ B(x) + A \circ C(x)$ since A is linear (right distributive property)
- $(A + B) \circ C(x) = (A + B)(Cx) = A \circ C(x) + B \circ C(x)$ addition of operators used (left distributive property)

b) $\mathfrak{C}(E, E)$ is a two-sided ideal in $\mathcal{L}(E, E)$.

Proof Verifying the same properties as before we have that $\mathfrak{C}(E, E)$ is a sub-ring of $\mathcal{L}(E, E)$. Moreover, given $A \in \mathfrak{C}(E, E)$ and $B \in \mathcal{L}(E, E)$ we have that $AB \in \mathfrak{C}(E, E)$ and $AB \in \mathfrak{C}(E, E)$ (theorem 24.2.3).

Problem 11 (p.253 Kolmogorov, Fomin)

An example of a completely continuous operator A mapping a Hilbert space H into itself, such that A has no eigenvectors:

(Hint p.253) Consider the operator

$$A \colon l_2 \longrightarrow l_2$$
$$(x_1, x_2, ..., x_n, ...) \longmapsto (0, x_1, \tfrac{x_2}{2}, ..., \tfrac{x_{n-1}}{n-1}, ...)$$

which is completely continuous since the closure of $A(B_1(0))$ is totally bounded (use the exact same reasoning as in Problem 2 but given $\epsilon > 0$ choose n such that $\sum_{k=n}^{\infty} \frac{1}{k^2} < \frac{\epsilon}{2}$, associate to each $x \in [])$. Then $Ax = \lambda x$ implies that x=0 as described in the hint. To reconcile this with theorem 24.3.7 we conclude that a completely continuous operator on a Hilbert space with no eigenvectors cannot be a self adjoint operator of H.