Functional Analysis

Adrian Perez Keilty

HTN3

Problem 1

Let $X = \{a, b, c, d\}$

a) 2 Hausdorff topologies on X:

This is impossible since the only Hausdorff topology on a finite set is the discrete one, i.e, $\tau(X) = \mathcal{P}(X)$.

(*proof*: If X is finite and τ is Hausdorff, then, given any $x \in X$ and a neighborhood $O_x \in \tau$, if $O_x \neq x$, let $x_1 \in O_x \setminus \{x\}$. By hypothesis there exists neighborhoods O_{x_1} and O_x^1 such that $O_{x1} \cap O_x^1 = \emptyset$. By relabeling $O_x^1 = O_x^1 \cap O_x$, if in turn $O_x^1 \neq \{x\}$ we repeat this process, obtaining $|O_x| > |O_x^1| > |O_x^2|$... and since X is finite, in a finite number of steps we will obtain $O_x^n = \{x\}$ for some $n \in \mathbb{N}$ showing that τ is in fact the discrete topology on X.)

b) 2 non Hausdorff topologies on X:

$$\tau_1 = \{X, \emptyset\}$$

$$\tau_2 = \{X, \emptyset, \{a\}\}$$

Problem 2

Consider $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a, b\}, \{b, c\}, \{b\}\}$ and the following functions:

$$\begin{aligned} f\colon X &\longrightarrow X \quad , & g\colon X &\longrightarrow X \\ a &\longmapsto c & a &\longmapsto c \\ b &\longmapsto c & b &\longmapsto a \\ c &\longmapsto a & c &\longmapsto b \end{aligned}$$

It is clear that $f^{-1}(\{b, c\}) = \{a, c\} \notin \tau$ and $g^{-1}(\{a, b\}) = \{c\} \notin \tau$, so f and g are not continuous on (X, τ).