Functional Analysis

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HTN4

1. Banach spaces

Given a norm $\|x\|$ defined on a vector space V, consider the metric induced by the norm via the identity $\rho(x, y) = \|x - y\|$. Then, as proved in HTN2, the following are examples of Banach spaces:

- $(\mathbb{R}, \|x\| = |x|)$
- $(\mathbb{R}^n, \|x\| = |\sum_{k=1}^n x_k^2|^{\frac{1}{2}})$
- $(C_{[a,b]}, \|f\|_\infty = \max_{a \le t \le b} |f(t)|)$

2. Non Banach normed spaces

- $(C_{[a,b]}, \|f\| = (\int_a^b f^2(t) dt)^{\frac{1}{2}})$ (see counterexample in HTN2)
- The vector space $\mathcal{P}[0, \frac{1}{2}]$ of polynomials defined in the interval $[0, \frac{1}{2}]$ together with the norm $\|p\|_\infty = \max_{0 \le t \le \frac{1}{2}} |p(t)|$. As a counterexample, take the sequence of polynomials $p_n(t) = \sum_{k=0}^n t^n, n \ge 1$ which is a Cauchy sequence convergent to $p(t) = \frac{1}{1-t} \notin \mathcal{P}[0, \frac{1}{2}]$.
- The vector space $C_{[a,b]}^k$ paired with the norm $\|f\|_\infty$ is not Banach since a Cauchy sequence of k-differentiable functions doesn't necessarily converge to a k-differentiable function.

3. Linear Bounded operators

- The identity operator

$$I \colon E \longrightarrow E$$

$$x \longmapsto x$$

  with norm

$$\|I\| = \sup_{x \neq 0} \frac{\|Ix\|}{\|x\|} = \sup_{x \neq 0} \frac{\|x\|}{\|x\|} = 1$$

- The zero operator

$$O \colon E \longrightarrow E_1$$

$$x \longmapsto 0$$

  with norm

$$\|O\| = \sup_{x \neq 0} \frac{\|Ox\|}{\|x\|} = \sup_{x \neq 0} \frac{\|0\|}{\|x\|} = 0$$

- The linear operator mapping between two multidimensional vector spaces paired with the norm $\|x\|_1 = \sum_{k=0}^n |x_k|$

$$A \colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$$

$$x \longmapsto Ax$$

  with norm estimate from above

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} \le \frac{\|x\|_1 \max_{1 \le j \le m} \|a_j\|_1}{\|x\|_1} = \max_{1 \le j \le m} \|a_j\|_1, \text{ where } a_j \in \mathbb{R}^n \text{ are the column vectors of A.}$$

4. Operators T_+ and T_-

The positive translation of l_∞

$$T_+^k \colon l_\infty \longrightarrow l_\infty$$
$$(x_1, ..., x_n, ...) \longmapsto (0, ..., 0^{(k)}, x_1, ..., x_n, ...)$$

satisfies $\|T_+^k x\|_\infty = \sup_{i\geq 1} x_i = \|x\|_\infty$ for all $x \in l_\infty$ and for all $k \geq 1$ which implies $\|T_+^k\| = 1 = \|T_+\| = \|T_+\|^k$.

The negative translation of l_∞

$$T_-^k \colon l_\infty \longrightarrow l_\infty$$
$$(x_1, ..., x_n, ...) \longmapsto (x_{k+1}, ..., x_n, ...)$$

satisfies $\|T_-^k\| = \sup_{\|x\|=1} \|T_-^k x\|_\infty = 1$ (choose $x \in l_\infty$ with $x_n = 1$ for some $n > k$) $\implies \|T_-^k\| = 1 = \|T_+\|^k$

5. Linear bounded operator A such that $\|A^2\| \neq \|A\|^2$

Let A be the following linear bounded operator on $\mathbb{R}^2$ together with the euclidean norm

$$A \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x_1, x_2) \longmapsto (x_1 + x_2, -x_1 - x_2)$$

It is clear that $A^2 x = 0$ for all $x \in \mathbb{R}^2$, so $\|A^2\| = 0$. Moreover,

$$\|A\| = \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} ((x_1 + x_2)^2 + (-x_1 - x_2)^2)^{\frac{1}{2}} = \sup_{\|x\|=1} (2(x_1 + x_2)^2)^{\frac{1}{2}} = \sup_{\|x\|=1} \sqrt{2}\|x\| = \sqrt{2}$$

Hence, $\|A^2\| = 0 < 2 = \|A\|^2$.

6-9. Finding operator norms

- Consider the Banach space $(C_{[0,2]}, \|\varphi\|_\infty)$ and the operator $A\varphi(x) = x\varphi(x)$. Then,

  $$\|A\varphi\|_\infty \leq \max_{0\leq x\leq 2} |x| \max_{0\leq x\leq 2} |\varphi(x)| = 2\|\varphi\|_\infty \implies \frac{\|A\varphi\|_\infty}{\|\varphi\|_\infty} \leq 2 \;\forall\; \varphi \neq 0 \implies \|A\| \leq 2$$

  But this upper bound is reached by choosing $\varphi(x) = 1$, hence, $\|A\| = 2$.

- Using the same reasoning as before, given the same space and the operator $A\varphi(x) = x^2\varphi(x)$ we find that $\|A\| = 4$
- If $A\varphi(x) = \cos x\varphi(x)$, then $\|A\| = \max_{0\leq x\leq 2} |\cos x| = 1$
- If we consider the more general space $(C_{[a,b]}, \|\varphi\|_\infty)$ and the first operator $A\varphi(x) = x\varphi(x)$, then clearly $\|A\| = \max\{|a|, |b|\}$.

  And in general, if we have the operator $A\varphi(x) = \phi(x)\varphi(x)$ for some $\phi \in C_{[a,b]}$, then $\|A\| = \|\phi\|_\infty$.