Quantum Information Theory

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HT 5: Density operators

Problem 1

Consider the Hilbert space $H = \mathbb{C}^2$ and the operator

$$\rho = \begin{pmatrix} 2 & i \\ -i & 1 \end{pmatrix}$$

(a) ρ is Hermitian. Since $\mathrm{Det}(\rho) = 1$ by Sylvester's criterion it is positive definite but its trace is 3 so it is not a density operator. On the other hand, if we consider

$$\tilde{\rho} = \begin{pmatrix} \frac{3}{4} & \frac{i}{4} \\ -\frac{i}{4} & \frac{1}{4} \end{pmatrix}$$

then we have $\mathrm{Tr}(\tilde{\rho})$=1 and $\tilde{\rho}$ satisfies the sufficient properties for being a density operator.

(b)

$$\tilde{\rho}^2 = \begin{pmatrix} \frac{5}{8} & \frac{i}{4} \\ -\frac{i}{4} & \frac{1}{8} \end{pmatrix} \neq \tilde{\rho}$$

thereby $\tilde{\rho}$ does not correspond to a vector state.

Problem 2

Consider the Hermitian operator

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

(a) The eigenvalues and eigenvectors of A are

$$\begin{cases} \lambda_1 = 2 \\ \lambda_2 = 0 \end{cases}, \qquad \varphi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad \varphi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

(b) The probabilities of λ_1 and λ_2 of A for the state $\tilde{\rho}$ defined in **Problem 1** are

$$P(A = 2) = \langle \tilde{\rho}(\varphi_1), \varphi_1 \rangle = \frac{3}{4}$$

$$P(A = 0) = \langle \tilde{\rho}(\varphi_2), \varphi_2 \rangle = \frac{1}{4}$$

(c) Finally, we compute the average

$$\langle A \rangle_{\tilde{\rho}} = \mathrm{Tr}(\tilde{\rho} A) = \mathrm{Tr} \begin{pmatrix} 1 & i \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix} = \frac{3}{2}$$