Quantum Information Theory

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HT 2: Quantum states and observables

Problem 1

Consider the 2-dimensional complex Hilbert space along with the following state and Hermitian operator:

$$\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

The eigenvalues of A, which we know are real, are computed via the characteristic polynomial:

$$|A - \lambda I| = \lambda^2 - 3\lambda + 1 \Rightarrow \begin{cases} \lambda_1 = \frac{3-\sqrt{5}}{2} \\ \lambda_2 = \frac{3+\sqrt{5}}{2} \end{cases}$$

To find valid eigenvectors, for each λ_j, we find a normalized solution e_j to the system of equations

$$A \begin{pmatrix} a+bi \\ c+di \end{pmatrix} = \lambda_j \begin{pmatrix} a+bi \\ c+di \end{pmatrix} \Leftrightarrow \begin{cases} (1-\lambda_j)a = d \\ (1-\lambda_j)b = -c \end{cases}$$

For instance,

$$e_1 = \frac{2}{\sqrt{10+2\sqrt{5}}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ i \end{pmatrix}, \qquad e_2 = \frac{2}{\sqrt{10-2\sqrt{5}}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ i \end{pmatrix}$$

Now we can compute the probabilities of observing the eigenvalues λ_1 and λ_2 given the state ψ.

$$P(A = \lambda_1, \psi) = |\langle \psi, e_1 \rangle|^2 = \frac{2}{\sqrt{10+2\sqrt{5}}} \left| \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ i \end{pmatrix} \right\rangle \right|^2 = \frac{1}{2}$$

$$P(A = \lambda_2, \psi) = 1 - P(A = \lambda_1, \psi) = \frac{1}{2}$$

Problem 2

Given the state ψ and the observable A given in **Problem 1** we can compute the mean value and dispersion as follows

$$\langle A \rangle_\psi = \langle A\psi, \psi \rangle = \frac{1}{2} \left\langle \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = \frac{3}{2}, \qquad \langle A^2 \rangle_\psi = \langle A^2\psi, \psi \rangle = \frac{7}{2}$$

$$\sigma_A(\psi) = \sqrt{\langle A^2 \rangle_\psi - \langle A \rangle_\psi^2} = \frac{\sqrt{5}}{2}$$