Quantum Information Theory

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HT 4: Schödinger's equation in two dimensions

In this home task we express the solution of Schrödinger's equation using the eigenvalues and eigenvectors of our own Hamiltonian operator in the finite dimensional Hilbert space $H = \mathbb{C}^2$.

Consider the following initial state and Hamiltonian operator:

$$\psi_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

The eigenvalues and eigenvectors of $\mathcal{H}$ were found in HT 2:

$$\begin{cases} \lambda_1 = \frac{3-\sqrt{5}}{2} \\ \lambda_2 = \frac{3+\sqrt{5}}{2} \end{cases}, \qquad \varphi_1 = \frac{2}{\sqrt{10+2\sqrt{5}}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ i \end{pmatrix}, \qquad \varphi_2 = -\frac{2}{\sqrt{10-2\sqrt{5}}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} i \\ 1 \end{pmatrix}$$

$\mathcal{H}$ is Hermitian and since $\lambda_1, \lambda_2 > 0$ we have that $\mathcal{H}$ is also positive definite. Now consider the Schrödinger equation

$$\begin{cases} i\dot{\psi}(t) = \mathcal{H}\psi(t) \\ \psi(0) = \psi_0 \end{cases} \tag{1}$$

Then, since $\{\varphi_1, \varphi_2\}$ is an orthonormal basis of H, Ψ_0 can then be expressed as

$$\psi_0 = c_1(0)\varphi_1 + c_2(0)\varphi_2$$

for some $c_1(0), c_2(0) \in \mathbb{C}$. If we denote

$$\Lambda = \begin{pmatrix} \varphi_1(1) & \varphi_2(1) \\ \varphi_1(2) & \varphi_2(2) \end{pmatrix}$$

then by orthogonality, $\Lambda = \overline{\Lambda^{\mathrm{T}}}$ and

$$\psi_0 = \Lambda \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} \iff \begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix} = \overline{\Lambda^{\mathrm{T}}}\psi_0 = ... = \frac{(1+\sqrt{5}-2i)}{2\sqrt{5+\sqrt{5}}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the solution to (1) is given by

$$\psi(t) = c_1(0)e^{-i\lambda_1 t}\varphi_1 + c_2(0)e^{-i\lambda_2 t}\varphi_2$$