Functional Analysis

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HTN7

Problem 1

a) Continuous linear functionals in the space $\mathcal{C}_{[a,b]}$ equipped with the norm $\|x\|_\infty = \max_{t\in[a,b]}|x(t)|$.

- $$\phi_1(x) = \int_a^b x(t)\,dt \tag{1}$$

  ϕ_1 is linear and $|\phi_1(x)| \leq (b-a)\cdot\|x\|_\infty$ where equality holds when x is a constant. ϕ_1 is therefore bounded (and hence continuous) and $\|\phi_1\| = b-a$.

- Let $y(t) \in \mathcal{C}_{[a,b]}$. Then we can produce the functional

  $$\phi_2(x) = \int_a^b x(t)y(t)\,dt \tag{2}$$

  where linearity is clear and $|\phi_2(x)| \leq \|x\|_\infty \int_a^b |y(t)|\,dt$ where equality is reached again when x is constant. ϕ_2 is then bounded with norm $\|\phi_2\| = \int_a^b |y(t)|\,dt$.

- Given $t_0 \in [a,b]$, let

  $$\phi_3(x) = x(t_0) \tag{3}$$

  linear and $|\phi_3| \leq \|x\|_\infty$ where equality holds if $x(t)$ happens to reach its maximum at t_0, thereby $\|\phi_3\| = 1$.

b) Continuous linear functionals in the space $\mathcal{C}_{[a,b]}$ equipped with the norm $\|x\| = \int_a^b |x(t)|\,dt$.

- $$\phi_1(x) = \int_a^b x(t)\,dt \tag{4}$$

  linear and $|\phi_1(x)| \leq \|x\|$ where equality holds when $x(t) = |x(t)| \implies \|\phi_1\| = 1$.

- Let $k \in \mathbb{R}$. Then

  $$\phi_2(x) = \int_a^b k\cdot x(t)\,dt \tag{5}$$

  is linear and $|\phi_2(x)| \leq k\cdot\|x\|$ where equality is reached when $x = \frac{1}{b-a} \implies \|\phi_2\| = k$.

- More generally, let $y \in \mathcal{C}_{[a,b]}$. Then

  $$\phi_3(x) = \int_a^b x(t)y(t)\,dt \tag{6}$$

  is linear and $|\phi_3(x)| \leq \|x\|\cdot\|y\|_\infty$, so

  $$\|\phi_3\| \begin{cases} = \|y\|_\infty & \text{if y is constant} \\ \leq \|y\|_\infty & \text{otherwise} \end{cases} \tag{7}$$

c) Continuous linear functionals in the space of sequences l_2 where $\|x\| = \sum_{j=1}^{\infty} x_j^2$.

- Let $k \in \mathbb{N}$. Then
$$\phi_1(x) = x_k \tag{8}$$
is linear and
$$\|\phi_1\| = \sup_{\|x\|=1} |x_k| = 1, \text{ where the supremum is reached when } x = e_k$$

- Let $n \in \mathbb{N}$. Then
$$\phi_2(x) = \sum_{j=1}^{n} x_j \tag{9}$$
is linear and
$$\|\phi_2\| = \sup_{\|x\|=1} |\sum_{j=1}^{n} x_j| = \frac{n}{\sqrt{n}}, \text{ where the supremum is reached when } x = (\frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}^{(n)}, 0, 0, ...)$$

- Let $y \in l_1$. Then
$$\phi_3(x) = \sum_{j=1}^{\infty} x_j y_j \tag{10}$$
is linear and an estimate from above is given by
$$\|\phi_3(x)\| = \sup_{\|x\|=1} |\sum_{j=1}^{\infty} x_j y_j| \leq \|x\|_\infty \cdot \|y\|_1$$

d) Continuous linear functionals in the space of convergent sequences to zero, c_0 where $\|x\|_\infty = \sup_j |x_j|$.

- ϕ_1 from (8) is also valid in this space and with the same reasoning its norm is again 1.
- ϕ_2 shown in (9) where
$$\|\phi_2\| = \sup_{\|x\|=1} |\sum_{j=1}^{n} x_j| = n, \text{ and supremum reached when } x = (1, 1, ..., 1^{(n)}, 0, 0, ...)$$

- Let $y \in l_1$. Then
$$\phi_3(x) = \sum_{j=1}^{\infty} x_j y_j \tag{11}$$
is linear and
$$|\phi_3(x)| \leq \|x\|_\infty \cdot \|y\|_1 \implies \|\phi_3\| \leq \|y\|_1$$

Now, by choosing $x^{(n)} \in c_0$ where
$$x_j^{(n)} = \begin{cases} \frac{y_j}{|y_j|} & \text{if } 1 \leq j \leq n \text{ and } y_j \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$
since $\|x^{(n)}\|_\infty \leq 1$, we obtain
$$\|\phi_3\| = \sup_{\|x\|\leq 1} |\sum_{j=1}^{\infty} x_j y_j| \geq \sum_{j=1}^{n} x^{(n)} \cdot y_j = \sum_{j=1}^{n} \text{sign}\,(y_j) \cdot y_j = \sum_{j=1}^{n} |y_j| \xrightarrow[n\to\infty]{} \sum_{j=1}^{\infty} |y_j| = \|y\|_1$$
thus,
$$\|\phi_3\| = \|y\|_1$$

Problem 2

Regardless of the completeness of a normed space E, its conjugate space $E^* = \mathcal{L}(E, \mathbb{R})$ is always complete, (Banach) since the codomain R is itself complete. This is based on the general result for linear bounded operators between normed spaces.

Problem 3

$\boxed{c_0^* \cong l_1}$

Proof To each $y \in l_1$ we associate the functional described in (11), i.e., we construct the correspondence

$$\begin{aligned} \Phi\colon l_1 &\longrightarrow c_0^* \\ y &\longmapsto f(x) = \textstyle\sum_{j=1}^{\infty} x_j y_j \end{aligned}$$

where $\{x_j\}_j \in c_0$. It preserves linear operations:

If

$$y \longleftrightarrow f, \quad \tilde{y} \longleftrightarrow \tilde{f}$$

then

$$\alpha y + \beta \tilde{y} \longleftrightarrow \sum_{j=1}^{\infty} x_j(\alpha y_j + \beta \tilde{y_j}) = \alpha \cdot \sum_{j=1}^{\infty} x_j y_j + \beta \cdot \sum_{j=1}^{\infty} x_j \tilde{y_j} = \alpha f + \beta \tilde{f}$$

and as shown before, in the third example of Problem 1.d) the correspondence is also norm-preserving,

$$\|y\| = \|f\|$$

It remains to show that Φ is bijective. Let $f \in c_0^*$, $x \in c_0$ and e_j denote the canonical vector of coordinate j. Then, x can be represented as the sum of its canonical coordinates $x_j e_j$ and applying the continuity of f we have

$$f(x) = f(\sum_{j=1}^{\infty} x_j e_j) = f(\lim_{n\to\infty} \sum_{j=1}^{n} x_j e_j) = \lim_{n\to\infty} f(\sum_{j=1}^{n} x_j e_j) = \lim_{n\to\infty} \sum_{j=1}^{n} x_j f(e_j) = \sum_{j=1}^{\infty} x_j f(e_j)$$

To see that $y = \{f(e_j)\}_j \in l_1$ set $x^{(n)} \in c_0$ where

$$x_j^{(n)} = \begin{cases} \frac{f(e_j)}{|f(e_j)|} & \text{if } 1 \le j \le n \text{ and } f(e_j) \ne 0 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

Then, $\|x^{(n)}\| \le 1$ and

$$\infty > \|f\| = \sup_{\|x\| \le 1} |f(x)| \ge f(x^{(n)}) = \sum_{j=1}^{n} \frac{f(e_j)}{|f(e_j)|} \cdot f(e_j) = \sum_{j=1}^{n} |f(e_j)| \xrightarrow[n\to\infty]{} \|y\| \implies y \in l_1$$

Therefore each $f \in c_0^*$ is uniquely determined by an element $y \in l_1$ and the spaces are then isomorphic.

Problem 4

$\boxed{l_1^* \cong l_\infty}$

Proof To each $y \in l_\infty$ we associate the same functional as before

$$\begin{aligned} \Phi\colon l_\infty &\longrightarrow l_1^* \\ y &\longmapsto f(x) = \textstyle\sum_{j=1}^{\infty} x_j y_j \end{aligned}$$

where $\{x_j\}_j \in l_1$. As shown before, it preserves linear operations. To see that it is also norm-preserving, set $x^{(n)} = e_k$ where k is determined such that $|y_k| = \sup\limits_{1 \le j \le n} |y_j|$.

Then $\|x^{(n)}\| = 1$, and we obtain the following estimates:

$$|f(x)| \leq \sup_{1\leq j<\infty} |y_j| \cdot \sum_{j=1}^{\infty} |x_j| = \|y\|_\infty \cdot \|x\|_1 < \infty \implies \|f\| \leq \|y\|_\infty$$

$$\|f\| = \sup_{\|x\|=1} |\sum_{j=1}^{\infty} x_j y_j| \geq |f(x^{(n)})| = \sup_{1\leq j\leq n} |y_j| \xrightarrow[n\to\infty]{} \|y\|_\infty \implies \|f\| \geq \|y\|_\infty$$

Thus,

$$\|f\| = \|y\|_\infty$$

Finally, to show that Φ is bijective, we set $x^{(n)} = e_k$ where k is determined such that $|f(e_k)| = \sup_{1\leq j\leq n} |f(e_j)|$ and take $f \in l_1^*$, $x \in l_1$. Then, using the same reasoning as before

$$f(x) = \sum_{j=1}^{\infty} x_j f(e_j)$$

and since $\|x^{(n)}\| \leq 1$ we have

$$\infty > \|f\| = \sup_{\|x\|=1} |\sum_{j=1}^{\infty} x_j f(e_j)| \geq |\sum_{j=1}^{\infty} x_j^{(n)} f(e_j)| = |f(x^{(n)})| = \sup_{1\leq j\leq n} |f(e_k)| \xrightarrow[n\to\infty]{} \|\{f(e_j)\}_j\|_\infty$$

hence, $y = \{f(e_j)\}_j \in l_\infty$ and is uniquely determined by f from which we obtain that $l_1^* \cong l_\infty$.