Functional Analysis

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HTN8

Problem 5 (p.238 Kolmogorov, Fomin)

Let $A \in \mathcal{L}(H)$ where H is a Hilbert space and let $M \subset H$ be an invariant subspace under A. Then the ortogonal complement $M' = H \ominus M$ is invariant under the adjoint operator A^*.

Proof Let $x_0 \in M'$. Then $(x_0, x) = 0 \; \forall x \in M$ and since $Ax \in M$,

$$(A^* x_0, x) = \overline{(x, A^* x_0)} = \overline{(Ax, x_0)} = (x_0, Ax) = 0 \; \forall x \in M$$

which implies that M' is invariant under A^*.

Problem 6 (p.238 Kolmogorov, Fomin)

By the adjoint operator of A in a Hilbert space, we mean the operator defined by $A^* = \tilde{A}^* = \tau^{-1} \circ A^* \circ \tau$ where τ assigns the linear functional $(x, y_0) = \tau^{-1}(y_0)$ to every $y_0 \in H$ and A^* is the usual adjoint of a mapping defined on an arbitrary Banach space. Now, let $A, B \in \mathcal{L}(H)$ where H is a complex Hilbert space. Then

a) $(\alpha A + \beta B)^* = \bar{\alpha} A^* + \bar{\beta} B^*$

Proof

$$\begin{aligned}(\alpha A + \beta B)^*(y_0) = \tau^{-1}(\alpha A + \beta B)^*(x, y) = \tau^{-1}(((\alpha A + \beta B)(x), y_0)) &= \tau^{-1}(\alpha(Ax, y_0) + \beta(Bx, y_0)) \\ &= \tau^{-1}(\alpha(x, A^* y_0) + \beta(x, B^* y_0)) \\ &= \tau^{-1}((x, \bar{\alpha} A^* y_0 + \bar{\beta} B^* y_0)) \\ &= (\bar{\alpha} A^* + \bar{\beta} B^*)(y_0)\end{aligned}$$

b) $(AB)^* = B^* A^*$

Proof

$$(AB)^*(y_0) = \tau^{-1}(AB)^* \tau(y_0) = \tau^{-1}((A(Bx), y_0)) = \tau^{-1}((Bx, A^* y_0)) = \tau^{-1}((x, B^* A^* y_0)) = B^* A^*(y_0)$$

<u>Lemma</u>: If H is a Hilbert space and $(x, y_1) = (x, y_2) \; \forall x \in H$ then $y_1 = y_2$ (let $\{e_i\}$ be an orthonormal basis in H, then using the Fourier coefficients representation, $y_1 = \sum_{i=1}^{\infty} (e_i, y_1) \cdot e_i = \sum_{i=1}^{\infty} (e_i, y_2) \cdot e_i = y_2$).

c) $(A^*)^* = A$

Proof $\forall x, y \in H, (x, Ay) = \overline{(Ay, x)} = \overline{(y, A^* x)} = (A^* x, y) = (x, (A^*)^* y) \implies (A^*)^* = A$

d) $I^* = I$

Proof $\forall x, y \in H \; (x, y) = (Ix, y) = (x, I^* y) \implies I^* = I$

Problem 9 (p.239 Kolmogorov, Fomin)

Let $R_\lambda = (A-\lambda I)^{-1}$ and $R_\mu = (A-\mu I)^{-1}$ be the resolvents corresponding to the points $\lambda, \mu \in \mathbb{C}$ where A is a linear operator on a topological linear space E. Then

$$\begin{aligned}
R_\lambda R_\mu(x_0) &= (A-\lambda I)^{-1} \circ (A-\mu I)^{-1}(x_0) \\
&= (A-\lambda I)^{-1}(\{x \in E \;:\; A(x) - \mu x = x_0\}) \\
&= \{x \in E \;:\; A(A(x)-\lambda x) - \mu(A(x) - \lambda x) = x_0\} \\
&= \{A^2 x - (\lambda+\mu)Ax + \mu\lambda x = x_0\} \qquad \text{(symmetric expression } \to \lambda \text{ and } \mu \text{ interchangeable)} \\
&= R_\mu R_\lambda(x_0)
\end{aligned}$$

which implies that $(A-\lambda I)(A-\mu I) = (A-\mu I)(A-\lambda I)$ and in particular, $R_\mu - R_\lambda = (\mu-\lambda)R_\mu R_\lambda$ since

$$\begin{aligned}
R_\mu - R_\lambda = (\mu-\lambda)R_\mu R_\lambda &\Leftrightarrow (A-\mu I)^{-1} - (A-\lambda I)^{-1} = (\mu-\lambda)(A-\mu I)^{-1}(A-\lambda I)^{-1} \\
&\Leftrightarrow (A-\lambda I) - (A-\mu I) = \mu - \lambda \qquad \text{(multiply both sides by } (A-\lambda I)(A-\mu I)) \\
&\Leftrightarrow (\mu-\lambda)I = (\mu-\lambda)
\end{aligned}$$

Problem 10 (p.239 Kolmogorov, Fomin)

Let H be a complex Hilbert space and $A \in \mathcal{H}$ be self-adjoint. Then the spectrum of A is a closed bounded subset of $\mathbb{R}$.

Proof Using the fact that a Hilbert space is a Banach space (H is isomorphic to l_2) and by combining theorems 23.4.6, 23.4.7 and 23.4.8, we have that the spectrum of A (the complement of the set Δ of all regular points) is closed and is contained in the complex disc of radius $\|A\|$ centered at the origin (spectrum is bounded), and additionally, that all eigenvalues of A are real. Finally, by the Banach theorem on the inverse operator, if $(A-\lambda I)^{-1}$ exists then it is automatically continuous which implies that A does not possess a continuous spectrum so the whole spectrum of A is then real, closed and bounded.