Functional Analysis

Adrian Perez Keilty

HTN5

1. Non normable topological vector spaces

1. The trivial topology on any vector space $(X, \tau = (X, \emptyset))$.

   Suppose there exists a norm $\|\cdot\|$ that generates τ. Then for all $x \in X$ and all $\varepsilon > 0$, $B_\varepsilon(x) = \{y \in X : \|x - y\| < \varepsilon\} = X$ ($B_\varepsilon(x) \neq \emptyset$ since $x \in B_\varepsilon(x)$) which would imply that $\|x\| = 0$ for all $x \in X$. Thereby, $\|\cdot\|$ would be a seminorm but not a norm.

2. The space $\mathbb{R}^\mathbb{N}$ of all sequences in $\mathbb{C}$ or $\mathbb{R}$ with the topology generated by neighborhood base at zero $\mathscr{N}_0 = \{\mathcal{U}_{k_1,...,k_r;\varepsilon} = \{x \in \mathbb{R}^\mathbb{N} \ : \ |x_{k_i}| < \varepsilon, \ i = 1, ..., r\}\}$.

   Let $x \in \mathbb{R}^\mathbb{N}$. For any neighborhood $x + \mathcal{U}_{k_1,...,k_r;\varepsilon}$, by taking $\mathcal{U}_{k;\varepsilon} \in \mathscr{N}_0$ such that $k \notin \{k_1, ..., k_r\}$, it is clear that $x + \mathcal{U}_{k_1,...,k_r;\varepsilon} \not\subset \alpha\mathcal{U}_{k;\varepsilon}$ for all $\alpha > 0 \implies \mathbb{R}^\mathbb{N}$ is not locally bounded $\implies \mathbb{R}^\mathbb{N}$ not normable.

3. The space $K_{[a,b]}$ of all infinitely differentiable functions on $[a, b]$ with the topology generated by the neighborhood base $\mathscr{N}_0 = \{\mathcal{U}_{r;\varepsilon} = \{\phi \in K_{[a,b]} \ : \ |\phi^{(k)}| < \varepsilon, \ k = 0, ..., r\}\}$.

   Let $\phi \in_{[a,b]}$. For any neighborhood $\phi + \mathcal{U}_{r;\varepsilon}$, by taking $\mathcal{U}_{s;\varepsilon} \in \mathscr{N}_0$ such that $s > r$, it is clear that $\phi + \mathcal{U}_{r;\varepsilon} \not\subset \alpha\mathcal{U}_{s;\varepsilon}$ for all $\alpha > 0 \implies K_{[a,b]}$ is not locally bounded $\implies K_{[a,b]}$ not normable.

2. Continuity of an operator on a topological vector space induced by a family of seminorms

Consider the space $\mathscr{C}(\mathbb{R})$ of all continuous functions, the family of seminorms $P_N(\phi) = \max_{t\in[-N,N]} |\phi(t)|$, $N \in \mathbb{N}$ and the operator

$$\begin{aligned} A \colon \mathscr{C}(\mathbb{R}) &\longrightarrow \mathscr{C}(\mathbb{R}) \\ \phi(t) &\longmapsto t\phi(t) \end{aligned}$$

In particular, paired with the topology induced by the seminorms P_N (coarsest topology on $\mathscr{C}(\mathbb{R})$ that makes P_N continuous for all $N \in \mathbb{N}$), the space $\mathscr{C}(\mathbb{R})$ satisfies the first axiom of countability.
Furthermore, $P_N(A\phi) = \max_{[-N,N]} |t \cdot \phi(t)| = N \cdot \max_{[-N,N]} |\phi(t)| = N \cdot P_N(\phi) \implies P_N(\phi) = \frac{1}{N} \cdot P_N(A\phi) \implies A$ is bounded $\implies A$ continuous.