Functional Analysis

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HTN6

Problem 1

Let X be a Banach space and $A \in \mathscr{L}(X)$ such that $A^2 = A$. Then $A^n = A$ and

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = A \cdot \sum_{n=0}^{\infty} \frac{1}{n!} = A \cdot e$$

$$\cos(A) = A \cdot \cos(1)$$

$$\sin(A) = A \cdot \sin(1)$$

Problem 2

a) Given α, $\beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$, consider the operator $A \in \mathscr{L}(\mathbb{C})$ given by

$$A = \begin{pmatrix} |\alpha|^2 & \overline{\alpha}\beta \\ \alpha\overline{\beta} & |\beta|^2 \end{pmatrix}$$

From the identity $z \cdot \overline{z} = |z|^2$ it is easy to verify that $A^2 = A$

b)

$$e^A = \begin{pmatrix} |\alpha|^2 \cdot e & \overline{\alpha}\beta \cdot e \\ \alpha\overline{\beta} \cdot e & |\beta|^2 \cdot e \end{pmatrix}$$

$$\cos(A) = \begin{pmatrix} |\alpha|^2 \cdot \cos(1) & \overline{\alpha}\beta \cdot \cos(1) \\ \alpha\overline{\beta} \cdot \cos(1) & |\beta|^2 \cdot \cos(1) \end{pmatrix}$$

$$\sin(A) = \begin{pmatrix} |\alpha|^2 \cdot \sin(1) & \overline{\alpha}\beta \cdot \sin(1) \\ \alpha\overline{\beta} \cdot \sin(1) & |\beta|^2 \cdot \sin(1) \end{pmatrix}$$

c) Consider the operator in l_2

$$A_m \colon l_2 \longrightarrow l_2$$

$$(x_1, ..., x_n, ...) \longmapsto (x_1, ..., x_m, 0, 0, ...)$$

For all $x \in l_2$, $A_m(A_m x) = A_m(x_1, ...x_m, 0, ...) = (x_1, ...x_m, 0, ...) \implies A_m^2 = A_m$ and if $x = (1, \frac{1}{2}, ..., \frac{1}{n})$, then

$$(e^{A_m})x = e \cdot A_m x = (e, \frac{e}{2}, ..., \frac{e}{m}, 0, 0...)$$

$$(\cos(A_m))x = \cos(1) \cdot A_m x = (\cos(1), \frac{\cos(1)}{2}, ..., \frac{\cos(1)}{m}, 0, 0...)$$

$$(\sin(A_m))x = \sin(1) \cdot A_m x = (\sin(1), \frac{\sin(1)}{2}, ..., \frac{\sin(1)}{m}, 0, 0...)$$

d) Consider the operator in l_2

$$A'_m : l_2 \longrightarrow l_2$$
$$(x_1, ..., x_n, ...) \longmapsto (0, 0, ..., x_{m+1}, x_{m+2})$$

It is again clear that ${A'_m}^2 = A'_m$ and by taking the same vector x as before, we get

$$(e^{A'_m})x = e \cdot A'_m x = (0, 0, ..., \frac{e}{m+1}, \frac{e}{m+2}, ...)$$
$$(\cos(A'_m))x = \cos(1) \cdot A'_m x = (0, 0, ..., \frac{\cos(1)}{m+1}, \frac{\cos(1)}{m+2}, ...)$$
$$(\sin(A'_m))x = \sin(1) \cdot A'_m x = (0, 0, ..., \frac{\sin(1)}{m+1}, \frac{\sin(1)}{m+2}, ...)$$

e) $(A_m + A'_m)x = A_m x + A'_m x = (x_1, ..., x_m, 0, ...) + (0, 0, ..., x_{m+1}, ...) = (x_1, ..., x_m, x_{m+1}, ...) = x \implies A_m + A'_m = I$

Problem 3

Let $A \in \mathscr{L}(X)$ be an idempotent operator, i.e, $A^2 = I$. Then

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I \cdot \sum_{n=0}^{\infty} \frac{1}{(2n)!} + A \cdot \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} = I \cdot \cosh(1) + A \cdot \sinh(1)$$
$$\cos(A) = I \cdot \cos(1)$$
$$\sin(A) = A \cdot \sin(1)$$

Problem 4

Example of idempotent matrix:

$$A = \begin{pmatrix} 1 & 0 \\ a & -1 \end{pmatrix}, \quad a \in \mathbb{C}$$

Problem 5

Let $A \in \mathscr{L}(X)$ be a nilpotent operator, i.e, $A^n = 0$ for some $n \in \mathbb{N}$.

a) Case $n = 2$: $e^A = I + A$; $\cos(A) = I$; $\sin(A) = A$

b) General case:

$$e^A = \sum_{n=0}^{n} \frac{A^n}{n!}$$

$$\cos(A) = \begin{cases} \sum_{j=0}^{\frac{n}{2}-1} \frac{(-1)^j \cdot A^{2j}}{(2j)!} & n \text{ is even} \\ \sum_{j=0}^{\frac{n-1}{2}} \frac{(-1)^j \cdot A^{2j}}{(2j)!} & n \text{ is odd} \end{cases}$$

$$\sin(A) = \begin{cases} \sum_{j=0}^{\frac{n}{2}-1} \frac{(-1)^j \cdot A^{2j+1}}{(2j+1)!} & n \text{ is even} \\ \sum_{j=0}^{\frac{n-1}{2}} \frac{(-1)^j \cdot A^{2j+1}}{(2j+1)!} & n \text{ is odd} \end{cases}$$

c) Example of nilpotent matrix ($n = 2$):

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Applying the identities found in a) we obtain:

$$e^A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\cos(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sin(A) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Problem 6

Given any operator $A \in \mathscr{L}(X)$ on a Banach normed space X, consider $(I - \lambda A)$ where $\lambda \in \mathbb{R}$.

Applying theorem 4 in section 23.1 (Kolmogorov & Fomin), we know that the inverse $(I - \lambda A)^{-1}$ exists and is bounded if

$$\|\lambda A\| < 1 \implies |\lambda| \|A\| < 1 \implies |\lambda| < \frac{1}{\|A\|}$$

Problem 7

If $A \in \mathscr{L}(X)$ is nilpotent and $n \in \mathbb{N}$ is the smallest natural number such that $A^n = 0$, then

$$(I - \lambda A) \cdot \sum_{k=0}^{n-1} \lambda^k A^k = \sum_{k=0}^{n-1} \lambda^k A^k \cdot (I - \lambda A) = I - \lambda^n A^n = I \implies (I - \lambda A)^{-1} = \sum_{k=0}^{n-1} \lambda^k A^k$$

In particular, $(I - \lambda A)^{-1}$ is well defined for all $\lambda \in \mathbb{R}$.