Functional Analysis

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HTN9

Problem 1 (p.165 Kolmogorov, Fomin)

In a Euclidean space, the operations of addition, multiplication by numbers and the formation of scalar products are all continuous.

Proof Let $x_n \to x$, $y_n \to y$ in the sense of norm convergence and $\lambda_n \to \lambda$ in the sense of ordinary convergence. Then, given $\epsilon > 0$ there exists N_ϵ and M_ϵ such that

$$\|x_n - x\| < \frac{\epsilon}{2} \text{ and } \|y_m - y\| < \frac{\epsilon}{2}\ \forall\ n > N_\epsilon, m > M_\epsilon$$

hence,

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| < \frac{\epsilon}{2} + \frac{\epsilon}{2}\ \forall\ n > \max(N_\epsilon, M_\epsilon) \implies x_n + y_n \to x + y$$

For scalar product convergence we have that

$$(x_n, y_n) = (x_n - x + x, y_n - y + y) = (x_n - x, y_n - y) + (x_n - x, y) + (x, y_n - y) + (x, y) \underset{n\to\infty}{\longrightarrow} (0,0) + (0,y) + (x,0) + (x,y) = (x,y)$$

Finally, if $\lambda_n \to \lambda$ then $Re(\lambda_n) \to Re(\lambda)$ and $Im(\lambda_n) \to Im(\lambda)$, and in particular, $\bar{\lambda}_n \to \bar{\lambda}$. Using also the identity $\lambda\bar{\lambda} = |\lambda|^2$ we have that

$$\|\lambda_n x_n - \lambda x\|^2 = (\lambda_n x_n - \lambda x, \lambda_n x_n - \lambda x) = ... = |\lambda_n|^2\|x_n\|^2 - \lambda_n\bar{\lambda}(x_n, x) - \lambda\bar{\lambda}_n(x, x_n) + |\lambda|^2\|x\|^2 \underset{n\to\infty}{\longrightarrow} 2|\lambda|^2\|x\|^2 - 2|\lambda|^2\|x\|^2 = 0$$

$$\implies \lambda_n x_n \to \lambda x$$

Problem 5 (p.166 Kolmogorov, Fomin)

Given a Euclidean space R, let $\{\varphi_k\}$ be an orthonormal basis in R and set $f \in R$. Then the element $f - \sum_{k=1}^n a_k\varphi_k$ is orthogonal to all linear combinations of the form $\sum_{k=1}^n b_k\varphi_k$ if and only if $a_k = (f, \varphi_k)$ where $k = 1, ..., n$.

Proof

$$\begin{aligned}
(f - \sum_{k=1}^n a_k\varphi_k, \sum_{k=1}^n b_k\varphi_k) = 0 &\Leftrightarrow (f, \sum_{k=1}^n b_k\varphi_k) - (\sum_{k=1}^n a_k\varphi_k, \sum_{k=1}^n b_k\varphi_k) = 0 \\
&\Leftrightarrow \sum_{k=1}^n \bar{b}_k(f, \varphi_k) - \sum_{k=1}^n a_k(\varphi_k, \sum_{k=1}^n b_k\varphi_k) = 0 \\
&\Leftrightarrow \sum_{k=1}^n \bar{b}_k(f, \varphi_k) - \sum_{k=1}^n a_k \sum_{j=1}^n \bar{b}_j(\varphi_k, \varphi_j) = 0 \\
&\left[\{\varphi_k\} \text{ orthonormal basis} \Leftrightarrow (\varphi_k, \varphi_j) = \begin{cases} 0 & \text{if } k \neq j \\ 1 & \text{if } k = j \end{cases}\right] \\
&\Leftrightarrow \sum_{k=1}^n \bar{b}_k(f, \varphi_k) - \sum_{k=1}^n a_k\bar{b}_k = 0 \\
&\Leftrightarrow a_k = (f, \varphi_k) \text{ for } k = 1, ..., n.
\end{aligned}$$

Problem 6 (p.166 Kolmogorov, Fomin)

In elementary geometry the length of the perpendicular dropped from a point P to a line L or plane Π is smaller than the length of any other line segment joining P to L or Π. To generalize this result to arbitrary Euclidean spaces, in theorem 16.4.6 we can visualize $\|f - \sum_{k=1}^{n} a_k\varphi_k\|$ as the length between the element $f \in R$ and the element $a = \sum_{k=1}^{n} a_k\varphi_k$ that belongs to the subspace $M \subset R$ generated by $\{\varphi_k\}_{k=1}^{n}$, and by the theorem, this length achieves its minimum when $a_k = (f, \varphi_k)$ for all $k \in \{1, ..., n\}$, or equivalently, by the preceding problem, when $f - \sum_{k=1}^{n} a_k\varphi_k$ is orthogonal to M.

Problem 10 (p.167 Kolmogorov, Fomin)

Subspaces of l_2:

a) $M_1 = \{(x_1, x_2, ..., x_k, ...) \in l_2$ such that $x_1 = x_2\}$. Let $z = \alpha x + \beta y$ where $x, y \in M_1$, then we check

- Linearity: $z_1 = \alpha x_1 + \beta y_1 = \alpha x_2 + \beta y_2 = z_2 \implies z \in M_1$
- Completeness: setting $\varphi_1 = (1, 1, 0, ..., 0, ...)$ and $\varphi_k = e_{k+1} = (0, ..., \overset{(k+1)}{1}, 0, ...)$ for $k \geq 2$, it is clear that $\{\varphi_k\}$ is a complete basis for M_1.

b) $M_2 = \{(x_1, 0, x_3, 0, ..., 0, x_{2n-1}, 0, ...) \in l_2\}$. Let $z = \alpha x + \beta y$ where $x, y \in M_2$, then

- Linearity: $z_k = \left.\begin{cases} 0 & \text{if } k = 2_n \text{ for some } n \\ \alpha x_k + \beta y_k & \text{otherwise} \end{cases}\right] \implies z \in M_2$
- Completeness: $\{e_{2k-1}\}_{k\in\mathbb{N}}$ is a complete basis for M_2.

Problem 11 a) (p.167 Kolmogorov, Fomin)

Every complex euclidean space of finite dimension is isomorphic to $\mathbb{C}^n$ where $(x, y) = \sum_{k=1}^{n} x_k\bar{y}_k$ for all $x, y \in \mathbb{C}^n$.

Proof Given an n-dimensional Euclidean space R and $\{\varphi_k\}_{k=1}^{n}$ an orthonormal basis of R, consider the following mapping

$$\Phi\colon R \longrightarrow \mathbb{C}^n$$
$$f = \textstyle\sum_{k=1}^{n} a_k\varphi_k \longmapsto (a_1, ..., a_n) = f^*$$

Then,

$$f + g = \sum_{k=1}^{n} a_k\varphi_k + \sum_{k=1}^{n} b_k\varphi_k = \sum_{k=1}^{n} (a_k + b_k)\varphi_k \longmapsto (a_1 + b_1, ..., a_n + b_n) = f^* + g^*$$
$$\alpha f \longmapsto (\alpha a_1, ..., \alpha a_n) = \alpha f^*$$

and

$$(f, g) = ... = \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \bar{b}_j(\varphi_k, \varphi_j) = \sum_{k=1}^{n} a_k\bar{b}_k = (f^*, g^*)$$

Thereby Φ is an isomorphism between R and $\mathbb{C}^n$.