Project 2: Maneuver Reconstruction

Tyson Warner and Jack Pence

I. PROJECT STATEMENT: MANEUVER RECONSTRUCTION

Maneuver reconstruction is an important part of the astrodynamics suite of applications. Specially in sparse data environments, or in the event that teh spacecraft is performing multiple maneuvers, identifying how much delta-v was expended is an important metric in characterizing the capabilities of a satelltie. In this project, we will idealize a simplified maneuver reconstruction experiment.

Consider a spacecraft that is continuously being tracked. The ground station looses tracking when a maneuver is performed, but has regained its custody at a later time. (Maneuver detection and tracking is another important and difficult problem in astrodynamic. For this project, let us assume that this problem is solved). A schematic of one such maneuver reconstruction is shown as follows:

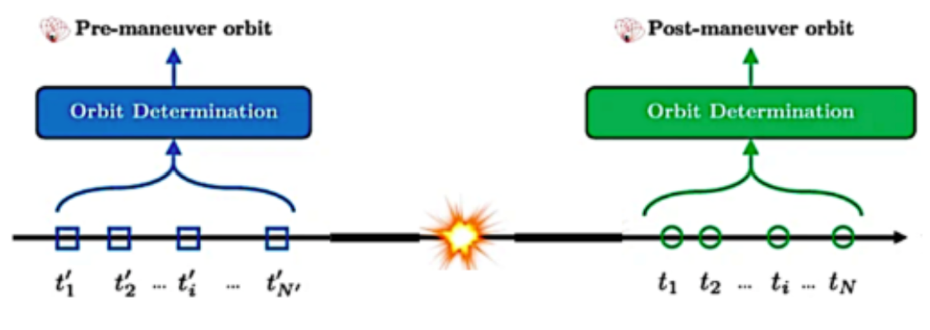

Figure 1. Maneuver Reconstruction Schematic.

We will utilize Gauss' initial orbit determination to identify the orbit that the spacecraft occupies before and after the maneuver and calculate how much delta-v was used to transfer between them.

II. ANALYTICAL PART: COMPUTING THE MANEUVER

Given the pre-maneuver and post-maneuver orbit parameters, compute the delta-v used to transfer from one to the other. Let us assume a coplanar case for this project, i.e. the inclination and RAAN of the pre and post-maneuver orbit are the same. However, the argument of perigee may be different.

In such a case, let us suppose that we adopt the perifocal reference frame for the first ellipse. In this reference frame, then, the argument of periapsis of the second ellipse is rotated counter-clockwise from the x-axis by an angle $\Delta\omega$ - where $\Delta\omega$ is the difference in the argument of periapsis of the two co-planar orbits.

So, the radial distance of a satellite on the first orbit can be thought of as:

$$r_1 = \frac{p_1}{1 + e_1 cos(f)}$$

and that on the second orbit is:

$$r_2 = \frac{p_2}{1 + e_2 cos(f - \Delta\omega)}$$

If the two orbits intersect, the two radial distances must be equal:

$$r_1 = r_2 = r$$

Our aim in this analytical part of the project is to find the value of the radius where the two orbits intersect. This leads to a quadratic equation in the free variable r. Let's go about this step-by-step.

A. Derivation of radius of intersecting orbits

1. Using the first relation:

$$r_1 = \frac{p_1}{1 + e_1 cos(f)}$$

rearrange to identify an expression for $cos(f)$. Also, using trigonometric identities, identify an expression for $r^2 sin^2(f)$. We will keep these quantities aside, as we will be using them at a later step.

SOLUTION:

Using $r_1 = r$ and solving for $cos(f)$:

$$cos(f) = \frac{1}{e_1}\left(\frac{p_1}{r} - 1\right)$$

Solving for $r^2 sin^2(f)$ using Pythagoras' theorem $(sin^2(f) = 1 - cos^2(f))$ and squaring the previous, rearranged expression:

$$r^2 sin^2(f) = r^2(1 - cos^2(f))$$

$$r^2 sin^2(f) = r^2 \left(1 - \left(\frac{1}{e_1}\left(\frac{p_1}{r} - 1\right)\right)^2\right)$$

Expanding this expression (this will be helpful for step 5):

$$r^2 sin^2(f) = \left(\frac{e_1^2 - 1}{e_1^2}\right) r^2 + \left(\frac{2p_1}{e_1^2}\right) r + \left(-\frac{p_1^2}{e_1^2}\right)$$

2. Using the relation

$$r_2 = \frac{p_2}{1 + e_2 cos(f - \Delta\omega)}$$

rearrange to identify an expression for $cos(f - \Delta\omega)$

SOLUTION:

Repeat the first part of step 1 but replacing subscripts and substituting $cos(f - \Delta\omega)$ for $cos(f)$:

$$cos(f - \Delta\omega) = \frac{1}{e_2}\left(\frac{p_2}{r} - 1\right)$$

3. Expand the above expression using the trigonometric identity for $cos(A - B)$ and rearrange to find an expression for $rsin(f)$. After this step, you may also substitute the expression for $cos(f)$ from step 1.

On performing this step, you should find the resulting expressions contain the terms:

$$\alpha = e_2 cos(\Delta\omega) - e_1$$

$$\beta = e_1 p_2 - e_2 p_1 cos(\Delta\omega)$$

$$\gamma = e_1 e_2 sin(\Delta\omega)$$

you may use these placeholder variables (α, β, γ) to simplify your math process.

SOLUTION:

Expanding the resulting expression from step 2 and solving for $sin(f)$:

$$cos(f - \Delta\omega) = cos(f)cos(\Delta\omega) - sin(f)sin(\Delta\omega)$$

$$sin(f) = \frac{cos(f - \Delta\omega) - cos(f)cos(\Delta\omega)}{sin(\Delta\omega)}$$

Multiplying r to both sides:

$$rsin(f) = r\left(\frac{cos(f - \Delta\omega) - cos(f)cos(\Delta\omega)}{sin(\Delta\omega)}\right)$$

Substituting the results from steps 1 and 2:

$$rsin(f) = r\left(\frac{\frac{1}{e_2}\left(\frac{p_2}{r} - 1\right) - \frac{1}{e_1}\left(\frac{p_1}{r} - 1\right)cos(\Delta\omega)}{sin(\Delta\omega)}\right)$$

Simplifying by factoring out $\frac{1}{r}$ in the numberator and multiplying the numberator and denominator by e_1 and e_2:

$$rsin(f) = \left(\frac{e_1(p_2 - r) - e_2(p_1 - r)cos(\Delta\omega)}{e_1 e_2 sin(\Delta\omega)}\right)$$

$$rsin(f) = \left(\frac{(e_2 cos(\Delta\omega) - e_1)r + e_1 p_2 - e_2 p_1 cos(\Delta\omega)}{e_1 e_2 sin(\Delta\omega)}\right)$$

Substituting for the placeholder variables mentioned earlier in this step:

$$rsin(f) = \frac{\alpha r + \beta}{\gamma}$$

4. Now, you may square the resulting expression on either side to obtain another expression for $r^2 sin^2(f)$ i.e., you will obtain an expression as:

$$r^2 sin^2(f) =$$

SOLUTION:

Squaring both sides of the resulting expression from step 3:

$$r^2 sin^2(f) = \frac{(\alpha r + \beta)^2}{\gamma^2}$$

Expanding this expression (this will be helpful for the next step):

$$r^2 sin^2(f) = \frac{\alpha^2}{\gamma^2} r^2 + 2\frac{\alpha\beta}{\gamma^2} r + \frac{\beta^2}{\gamma^2}$$

5. Plug in your relation for $r^2 sin^2(f)$ from step 1. At this stage, we have eliminated true anomaly from the equations.

SOLUTION:

Equating the results for $r^2 sin^2(f)$ from steps 1 and 4:

$$\left(r^2 sin^2(f)\right)_{step1} = \left(r^2 sin^2(f)\right)_{step4}$$
$$\left(\frac{e_1^2 - 1}{e_1^2}\right) r^2 + \left(\frac{2p_1}{e_1^2}\right) r + \left(-\frac{p_1^2}{e_1^2}\right) = \left(\frac{\alpha^2}{\gamma^2}\right) r^2 + \left(\frac{2\alpha\beta}{\gamma^2}\right) r + \left(\frac{\beta^2}{\gamma^2}\right)$$

6. You will notice that you have a quadratic expression in r.

$$ar^2 + br + c = 0$$

Collect the terms that multiply r^2, r, and the constant term, i.e. a, b, c. These are the coefficients you will use to compute the roots of a quadratic equation:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As the final step, you should obtain something of the form:

$$a = \frac{e_1^2 - 1}{e_1^2} - \frac{\alpha^2}{\gamma^2}$$

$$b = \frac{2p_1}{e_1^2} - \frac{2\alpha\beta}{\gamma^2}$$

$$c = -\left(\frac{p_1^2}{e_1^2} + \frac{\beta^2}{\gamma^2}\right)$$

SOLUTION:

Collecting terms from the result of the previous step:

$$\left(\frac{e_1^2-1}{e_1^2}\right)r^2+\left(\frac{2p_1}{e_1^2}\right)r+\left(-\frac{p_1^2}{e_1^2}\right)=\left(\frac{\alpha^2}{\gamma^2}\right)r^2+\left(\frac{2\alpha\beta}{\gamma^2}\right)r+\left(\frac{\beta^2}{\gamma^2}\right)$$

$$\left(\frac{e_1^2-1}{e_1^2}-\frac{\alpha^2}{\gamma^2}\right)r^2+\left(\frac{2p_1}{e_1^2}-\frac{2\alpha\beta}{\gamma^2}\right)r+\left(-\left(\frac{p_1^2}{e_1^2}+\frac{\beta^2}{\gamma^2}\right)\right)=0$$

Equating the previous result to a quadratic expression with constants a, b, and c ($a^2r+br+c=0$):

$$a=\frac{e_1^2-1}{e_1^2}-\frac{\alpha^2}{\gamma^2}$$

$$b=\frac{2p_1}{e_1^2}-\frac{2\alpha\beta}{\gamma^2}$$

$$c=-\left(\frac{p_1^2}{e_1^2}+\frac{\beta^2}{\gamma^2}\right)$$

B. Testing the algorithm

Now, consider two sets of orbital elements of two spacecraft:

	Orbit 1	Orbit 2
semimajor axis (km)	13000	7226.58
eccentricity	0.3	0.444819
inclination (deg)	20	20
RAAN (deg)	30	30
AOP (deg)	50	301.901

1. Identify the radius at the intersecting point.

   SOLUTION:

   Calculating semi-latus rectum (in km) using given eccentricities and semimajor axes for orbits 1 and 2 (p_1 and p_2 respectively):

$$p_n=a_n\left(1-e_n^2\right)$$

$$p_1=13000\left(1-(0.3)^2\right)=11830$$

$$p_2=7226.58\left(1-(0.444819)^2\right)=5796.7$$

   Calculating the difference in the argument of periapsis ($\Delta\omega$) between the two orbits:

$$\Delta\omega=\omega_2-\omega_1=301.901^\circ-50^\circ=251.901^\circ$$

   Calculating the constants from part A, step 3 (α, β, γ):

$$\alpha=e_2cos(\Delta\omega)-e_1=-0.4382$$

$$\beta = e_1 p_2 - e_2 p_1 cos(\Delta\omega) = 3373.767$$

$$\gamma = e_1 e_2 sin(\Delta\omega) = -0.1268$$

Calculating the constants from part A, step 6 (a, b, and c):

$$a = \frac{e_1^2 - 1}{e_1^2} - \frac{\alpha^2}{\gamma^2} = -22.045$$

$$b = \frac{2p_1}{e_1^2} - \frac{2\alpha\beta}{\gamma^2} = 446657.9$$

$$c = -\left(\frac{p_1^2}{e_1^2} + \frac{\beta^2}{\gamma^2}\right) = -2262440535.1$$

Calculating the radius using the relation obtained from part A, step 6:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the two possible solutions are $r = 10129.35$ km and $r = 10131.71$ km. Both values are valid solutions of where the orbits intersect, so we chose to continue using $r = 10129.35$ km.

2. Compute the position and velocity vectors at these points on the two orbits. Note, that position vectors will be the same, but the velocity vectors may be different.

   SOLUTION:

3. Compute the delta-v at the intersection point to transfer from one orbit to another.

   SOLUTION: