Coimisiún na Scrúduithe Stáit

State Examinations Commission

Leaving Certificate Examination 2024

Applied Mathematics

Mathematical Modelling Project
Reporting Booklet

Higher and Ordinary Level

100 marks

Examination Number	142382
Date of Birth	08/01/2006

Information for Candidates, Teachers and Schools

Completion of the report

- Your report on your mathematical modelling project must be completed in this booklet, using the prescribed structure. The report must be completed and handed over to your class teacher by **02 February 2024**. The report will be submitted through the SEC student portal.
- The State Examinations Commission will provide further information in due course, in the relevant circular, in relation to the appropriate file name to use and how your school will upload the digital coursework to the school portal.
- It is your responsibility to ensure that all electronic material submitted is free from viruses etc., so that the file can be opened for assessment.
- **Examiners will only mark work that is submitted as outlined in this booklet.**
- Your report must not exceed 900 words (excluding references, equations, diagrams, graphs, etc.). Your report must not include more than 20 images. An image can be any relevant table, graph, chart, diagram or photograph. The total file size of your digital report including all embedded images must not exceed 100 MB. Videos must not be included in your report.
- You must not change the structure or format of the booklet in any way and should adhere to the following formatting guidelines:
    - The text should be in Arial, font size 12.
    - You may make use of text editing features such as italics, bullets, etc.
    - Document margins have been set and should not be changed. The text margins for each A4 page are:
        - left margin: 20 mm
        - right margin: 20 mm
        - top margin: 20 mm
        - bottom margin: 35 mm
- You are not allowed to change these settings or otherwise circumvent these restrictions, which are set to facilitate online marking. You are also not allowed to attach or include links to any other material (other than references).
- When referring in the body of the report to any specific image, the image must be properly labelled (Figure 1, Figure 2, etc.). Images should not be used as a means to include additional text. It is advisable not to use images where a person or persons in these images may be identifiable.
- Penalties may apply where the overall word count or image count or file size is exceeded.

- You must reference and acknowledge all research sources used, such as: publications – including books, professional journals and government reports; online sources and other types of media; any material generated using artificial intelligence (AI) software or applications; specialist organisations and relevant individuals. To include such material without properly referencing the source will be considered plagiarism. In addition, the copying from, or reproduction of, material from such sources may also be considered plagiarism.
- Your project must be backed up by the school on a secure encrypted storage system until the State Examinations Commission is satisfied that the projects have been received and are accessible. This will require material being retained until the examination process is complete, including until after the appeal process is completed.

Authentication

- The mathematical modelling project and report must be your own individual work – authenticated by yourself, by your teacher and by your school management authority. Authentication is an important part of how the State Examinations Commission ensures fairness to everybody in the assessment of coursework.
- While you may carry out background research relevant to the brief on your own and/or at home, all other parts of your project and your report must be completed under the supervision of your teacher in accordance with the conditions set down by the State Examinations Commission as outlined in Circulars S69/04 and S68/08, which are available at www.examinations.ie.
- Any case of suspected copying, plagiarism (which includes the use of AI software), improper assistance, or procurement of work prepared by another party will be thoroughly investigated. These actions are breaches of examination rules and attract the penalties described in the *Rules and Programme for Secondary Schools*. The penalties include: loss of the marks for the project, loss of the subject, loss of the entire examination in all subjects, or being debarred from the Certificate Examinations in subsequent years. There may be serious consequences for any person who provides you with inappropriate assistance, as this is an offence under the Education Act 1998.
- Your teacher must supervise your completion of both the project and the report. If your teacher cannot confirm that the project and report are your own work, and that you carried out the project and completed the report under your teacher's supervision, the State Examinations Commission will not accept your report for assessment. In that case, you will forfeit the marks for this component of the examination. Teachers and the authorities of schools are familiar with the detailed requirements to ensure that practical and project work is valid for examination purposes. You should comply fully with all requests that are made by the teacher and the school in order to enable authentication of your work.
- Responsibility for complying with examination requirements rests with you, the candidate. If the requirements are not followed, your teacher and school will have no choice but to bring this to the attention of the State Examinations Commission.

Report Structure and Mark Allocations

Section	**Indicative Content**	**Marks**
Introduction and Research	• Background research on brief • Identify specific problem(s) to be modelled • Research specific problem(s) • Identify relevant variables • Present relevant data • Provide citations and references	**20**
The Modelling Process	• Explain and justify model and assumptions • Compute solutions • Present solutions using appropriate mathematical and graphical representations • Analysis of solution(s) – sensitivity to changes in assumptions; comparison with other solutions or real-world data • Iterative process	**50**
Interpretation of Results	• Interpretation of solution(s) in real-world context • Conclusions and reflections	**15**
Communication and Innovation	This is not a distinct section of the report. • Innovative and creative approaches • Overall coherence	**15**

Mathematical Modelling Project

Type into the space below the title of your mathematical modelling project.

Exploration of rollercoaster physics with development of a numerical model

Word Count

Do not complete the following table until after you have finished your report.

Section	**Number of words**
Introduction and Research	200
The Modelling Process	683
Interpretation of Results	71
Total	951

Introduction and Research 20 marks

In this project, I aim to explore roller coasters with a focus on creating an enjoyable and safe ride. By utilizing the Python code, we spent weeks building in class, I will analyse various aspects of the ride to enhance its excitement while prioritizing safety. The code will assist in performing calculations and generating graphs.

Aim 1: Do a basic exam question and determine if the code functions as intended.

Aim 2: Replicate a real-life rollercoaster using the newtons laws of motion.

Aim 3: Add friction and determine how the results vary.

$mass\ of\ the\ cart\ kg$

$Initial\ position\ \theta_0$

$Initial\ velocity\ \sqrt{2gh}\ ms^{-1}$

$Radius\ of\ the\ circle\ (m)$

$Timestep\ =\ 0.0001$

We assume the G-Force is given by $G_f = \frac{a}{g_0}$

where:

1. $g_0\ is\ the\ acceleration\ due\ to\ gravity\ ms^{-1}$
2. $G_f\ is\ the\ Ge - Force\ experienced.$

Friction is given by:

$F_r = -mu * (m * g_0 * \cos(\theta) * \sin(v) + \frac{m * v^2}{R})$

Where:

1. $mu\ is\ the\ coefficient\ of\ friction. (no\ dimensions)$
2. $m\ is\ the\ mass\ of\ the\ cart\ (kg),$
3. $g_0\ is\ the\ acceleration\ due\ to\ gravity\ ms^{-1},$
4. $\theta = angle,$
5. $v = velocity (ms^{-1}),$
6. $r = radius\ (m).$

$Kinetic\ energy (KE)\ = \frac{1}{2}mv^2$

$Potential\ energy (PE)\ = mg(R - R * cos(\theta))$

$Total\ Mechanical\ Energy (TME) = KE + PE$

The Modelling Process	50 marks

Aim 1: LC question.

11. A particle of mass m kg lies on the top of a smooth fixed sphere of radius 30 cm.

The particle is slightly displaced and slides down the sphere. The particle leaves the sphere at B.

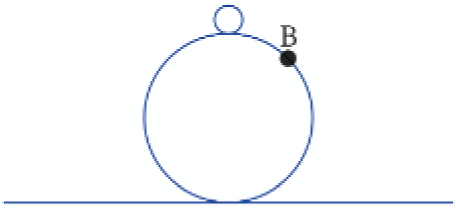


Fig. 7.39

(i) Find the speed of the particle at B.

r = 0.3

Height of initial position = 0.3 – 0.3Cos(θ) = 0.3(1 - 1Cos(θ))

Height of B above the centre = 0.3Cos(θ)

Principle of conservation of energy

$$mgh_1 + \frac{1}{2}mu^2 = mgh_2 + \frac{1}{2}mv^2$$

$$mg0.3\big(1 - 1\text{Cos}(\theta)\big) = \frac{1}{2}mv^2$$

m cancels

$$g0.3\big(1 - 1\text{Cos}(\theta)\big) = g0.3\text{Cos}(\theta) + \frac{1}{2}v^2$$

$$\big(g0.3 - g0.3\text{Cos}(\theta)\big) = g0.3\text{Cos}(\theta) + \frac{1}{2}v^2$$

÷0.3

$$\big(g - g\text{Cos}(\theta)\big) = \frac{5v^2}{3}$$

$$g - g\text{Cos}(\theta) = \frac{5v^2}{3}$$

$$g\text{Cos}(\theta) = g - \frac{5v^2}{3}$$

Centripetal force

$$mg\text{Cos}(\theta) - R = \frac{mv^2}{r}$$

R = 0 + m cancels

$$g\text{Cos}(\theta) = \frac{v^2}{0.3}$$

$$g - \frac{5v^2}{3} = \frac{v^2}{0.3}$$

$$g = \frac{v^2}{0.3} + \frac{5v^2}{3}$$

$$g = 5v^2$$

$$\sqrt{\frac{g}{5}} = v$$

$v = 1.40ms^{-1}$

Now to find angle

$$\text{gCos}(\theta) = \frac{\sqrt{\frac{g}{5}}^2}{0.3}$$

$$\text{Cos}(\theta) = \frac{\frac{\sqrt{\frac{g}{5}}^2}{0.3}}{g}$$

$$\theta = \cos^{-1}\frac{\frac{\sqrt{\frac{g}{5}}^2}{0.3}}{g}$$

$\theta = 48.19°$

Now using code to replicate this.

```python
import numpy as np

# Initial conditions and constants
initial_theta = 0.00000000000000001  # Initial angle with respect to the vertical radius from the center
initial_v = 0.0001  # Initial velocity of the sphere
radius = 0.3  # Radius of the sphere
gravity = 9.8
mass = 2
time_step = 0.0001  # Time step for the Euler iteration

# Function to update the position of the ball and the velocity
def update_position_velocity(theta, v, dt):
    # Acceleration tangent to curve due to gravity
    a_t = gravity * np.sin(theta)

    # Update velocity and angle at the next time step
    new_v = v + a_t * dt
    new_theta = theta + (v * dt / radius)

    return new_theta, new_v

# Euler method
while True:
    print(initial_theta)
    initial_theta, initial_v = update_position_velocity(initial_theta, initial_v, time_step)
    normal_force = mass * gravity * np.cos(initial_theta) - (mass * initial_v**2 / radius)
    print(normal_force)
    # If normal force is zero or smaller, exit the loop
    if normal_force <= 0:
        break

degrees = np.degrees(initial_theta)
print("Angle in degrees:", degrees)
print("Velocity:", initial_v)
```

```
0.06577972213364269
0.8395592700995251
0.04538403868162533
0.8400251549717261
0.02496918433506501
0.8404912829978937
0.004535145733004953
-0.01591809048416515
Angle in degrees: 48.21005938891341
Velocity: 1.4005742930758434
```


The code performs the following steps:

1. Read initial conditions and constants.
2. Use the **Euler method** in the update_position_velocity function to calculate the new angle and velocity of the ball at each time step.
3. Apply the Euler method in a loop until the normal force becomes zero or smaller.
4. Update the current angle and velocity within the loop using the update_position_velocity function.
5. If the normal force is zero or smaller, break the loop.
6. Print the final angle (in degrees) and velocity.

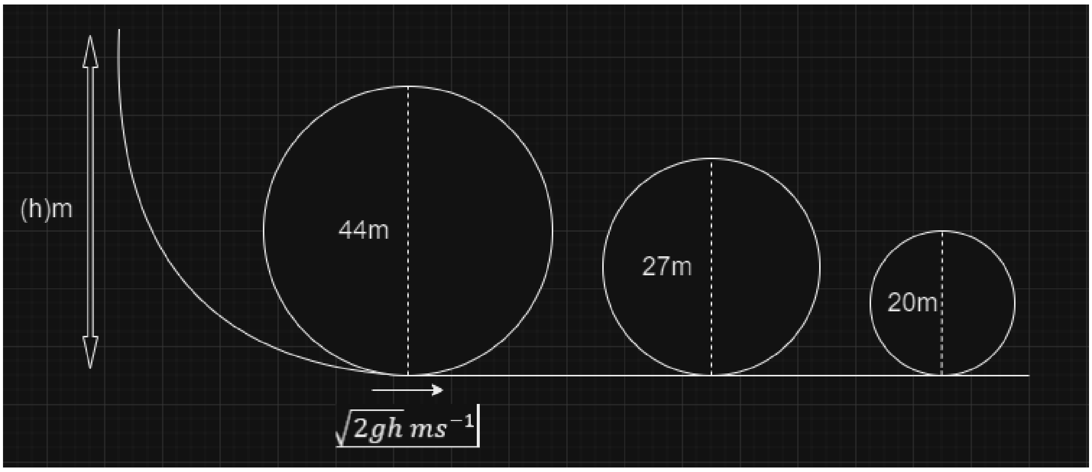
Aim 2: Replicate a real-life rollercoaster.

I will attempt to replicate the viper rollercoaster from six flags magic mountain.

To replicate the Viper roller coaster accurately, we need to calculate the weight of the cart. According to (source), each car can carry 4 people. Assuming a steel car weighs around 1000kg without passengers, and estimating an average adult weight of 100kg, the cart's weight would be approximately 1400kg. Considering there are 7 cars in total, the entire train would weigh around 9800kg or almost 10 tons.

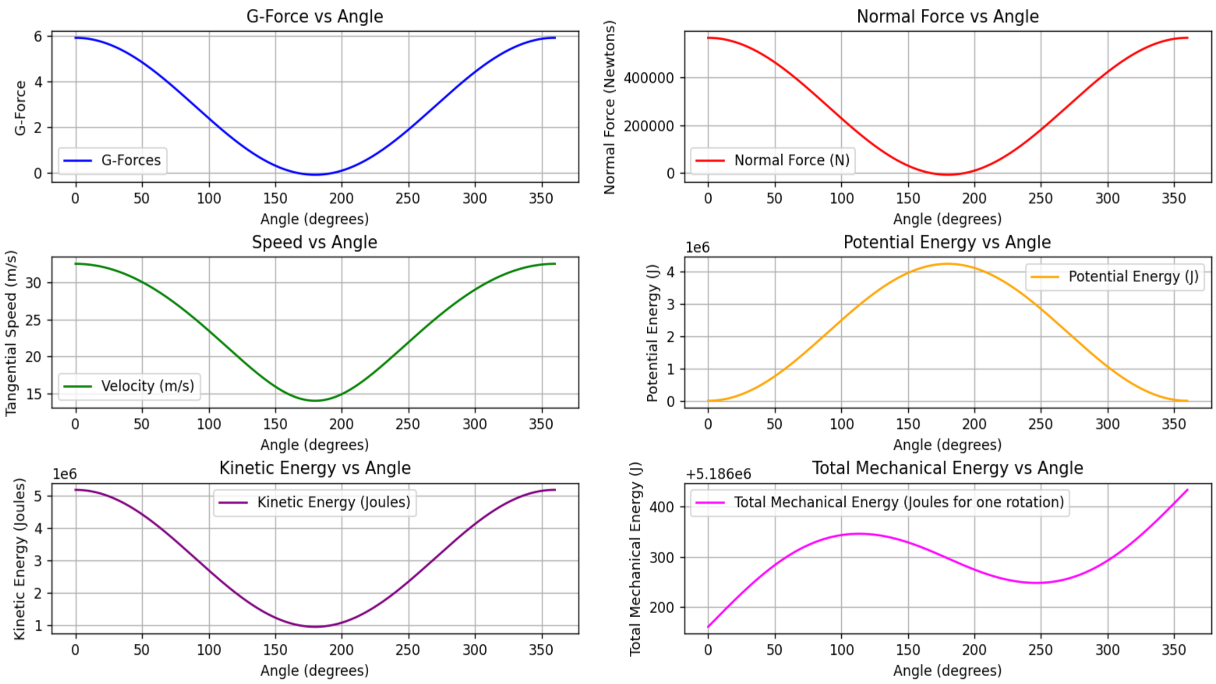
The coaster has 3 consecutive loops with 3 heights, 144ft (44 meters), 90ft (27 meters) and 62 ft (20 meters), before it experiences a 171ft drop (52 meters) which I will adjust with (h) (source)

It will look like this.



Starting with only **gravitational forces**,

The rollercoaster enters the loop at $\sqrt{2gh}\ ms^{-1}$ and setting the h as 52 as it is on the real viper these are the results that the code gives us, I am going to be dealing with the first loop only to start with.



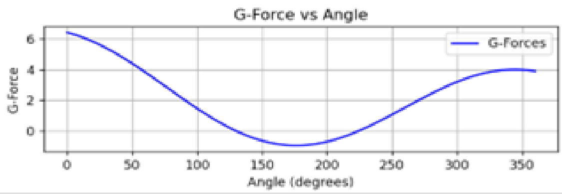
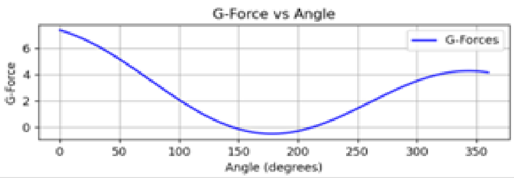
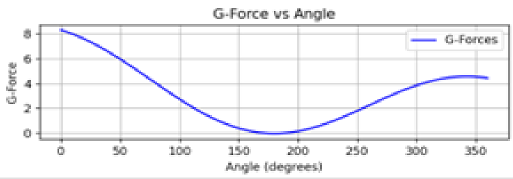
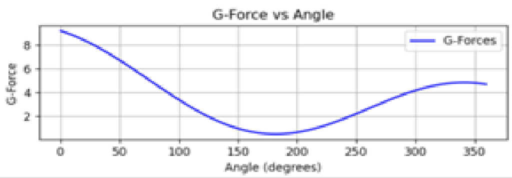
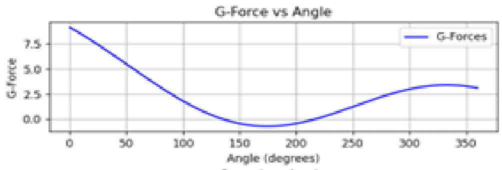
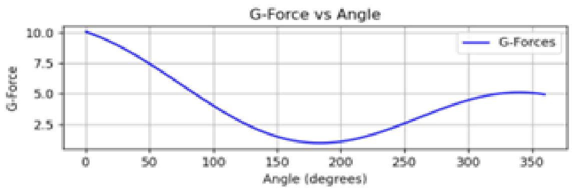
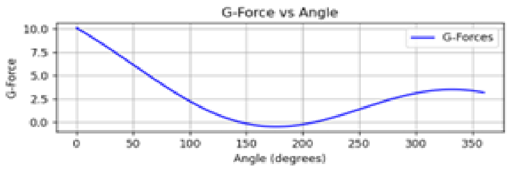
A **G-Force** of almost 6G's is considered on the extreme end and unsafe for some![(source)] I am hoping adding the forces of **friction** will put that down to around 4-4.5 G's to at least to keep it thrilling but safe.

Aim 3: Incorporating friction.

- The code simulates the motion of a rollercoaster using a ring on a wire.
- It calculates the position, velocity, forces, and energies of the rollercoaster at each time step.
- Factors like friction and gravitational forces are considered in the simulation.
- **The Euler method** is used to update the position and velocity of the rollercoaster.
- The simulation continues until the rollercoaster completes a full rotation.
- Normal force, g-force, potential energy, kinetic energy, and total mechanical energy are calculated during each iteration.
- The calculated values put into graphs.

```python
import numpy as np
import matplotlib.pyplot as plt

# Function to calculate derivatives including friction and drag
def derivatives(theta, v):
    # Calculate tangential acceleration
    a_tangential = -g * np.sin(theta)

    # Calculate friction force
    F_friction = -mu * (mass * g * np.cos(theta) * np.sign(v) + mass * v ** 2 / R)


    # Calculate net force
    F_net = -mass * g * np.sin(theta) + F_friction

    # Calculate angular velocity (derivative of theta)
    dtheta_dt = v / R

    # Calculate acceleration (derivative of velocity)
    dv_dt = F_net / mass

    return dtheta_dt, dv_dt

# Euler Method to update the position (in radians) and velocity
def euler_step(theta, v, dt):
    # Calculate derivatives
    dtheta_dt, dv_dt = derivatives(theta, v)

    # Update angle and velocity using Euler method
    theta = theta + dtheta_dt * dt
    v = v + dv_dt * dt

    return theta, v

# Lists to store theta, reaction force, g-force values, and mechanical energy
theta_values = []
normal_forces = []
g_forces = []
velocity = []
Kinetic_En = []
Potential_En = []
Total_Energy = []

# Set up parameters for the simulation
h = 90
R = 22  # radius of the sphere
g = 9.8
mass = 9800
dt = 0.0001  # time step for the Euler iteration
v = np.sqrt(2 * g * h)

# Friction and Drag
mu = 0.2
A = np.pi * (1) ** 2  # cross-sectional area of the ring, of diameter 1m

theta = 0

Kinetic = 0.5 * mass * v ** 2
Potential = 0

# Simulation Loop
while theta <= 2 * np.pi:
    # Update angle and velocity using Euler method
    theta, v = euler_step(theta, v, dt)

    # Calculate normal force and g-force
    normal_force = mass * g * np.cos(theta) + mass * v ** 2 / R
    g_force = normal_force / (mass * g)

    # Calculate potential and kinetic energy
    Potential = mass * g * (R - R * np.cos(theta))
    Kinetic = 0.5 * mass * v ** 2

    # Calculate total mechanical energy
    Total = Potential + Kinetic

    # Append values to respective lists
    theta_values.append(np.degrees(theta))
    velocity.append(v)
    normal_forces.append(normal_force)
    g_forces.append(g_force)
    Kinetic_En.append(Kinetic)
    Potential_En.append(Potential)
    Total_Energy.append(Total)
```

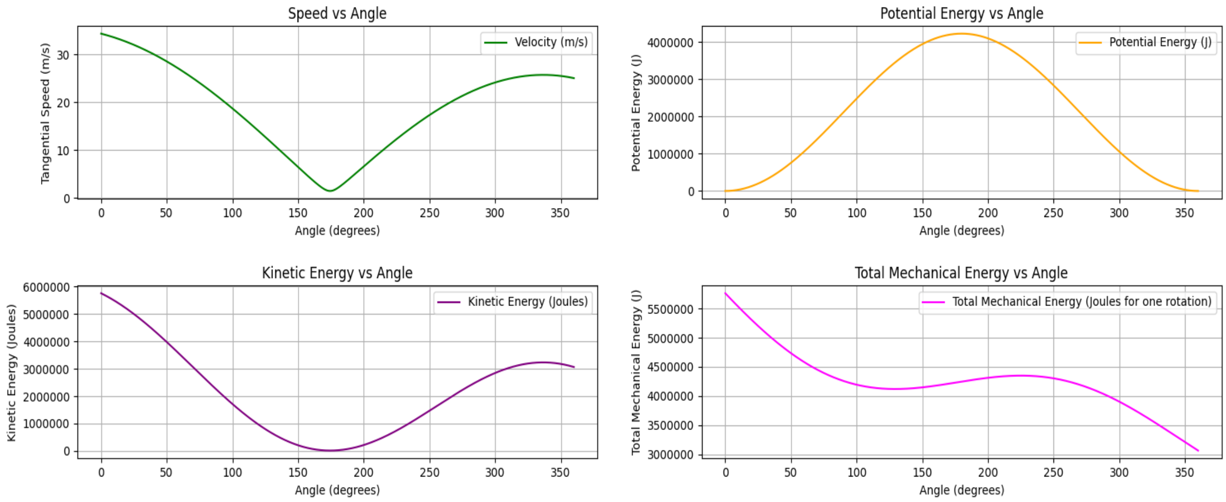
According to (source), advancements in technology have significantly reduced the **friction** on rollercoaster tracks. Considering this, I will analyse various heights for the rollercoaster drop to ensure the **G-forces** remain within **safe limits**. I won't explore heights exceeding 100 meters, as they seem impractical. To present data I will use Microsoft Excel.

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr>
<tr><td>1</td><td></td><td>mu</td><td>mu</td><td></td></tr>
<tr><td>2</td><td>drop height</td><td>0.1</td><td>0.2</td><td></td></tr>
<tr><td>3</td><td>60m</td><td>
</td><td>doesn't reach</td><td></td></tr>
<tr><td>4</td><td>70m</td><td>
</td><td>doesn't reach</td><td></td></tr>
<tr><td>5</td><td>80m</td><td>
</td><td>doesn't reach</td><td></td></tr>
<tr><td>6</td><td>90m</td><td>
</td><td>
</td><td></td></tr>
<tr><td>7</td><td>100m</td><td>
</td><td>
</td><td></td></tr>
</table>

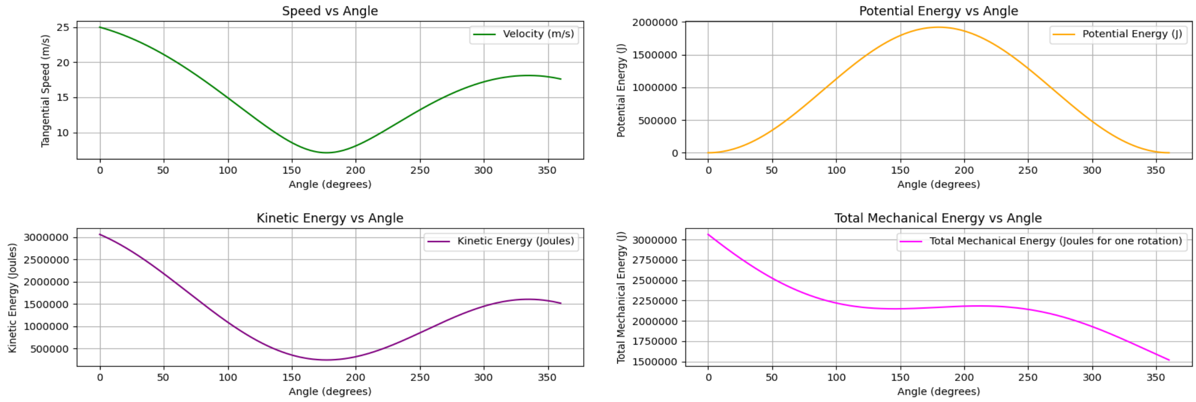
The rollercoaster appears to have been skilfully engineered to achieve a balance of high yet secure **G-forces** at approximately 60 meters. Incorporating drag into our calculations, it is likely that the G-force of 6 could have been slightly reduced, resulting in a safer experience. It is evident that the initial **G-force** is exceptionally intense, but as the loop progresses, the coaster gradually slows down because of **friction**.

Now that we know a drop of 60 meters and 0.1 mu is the **safest** while being the most **exhilarating**, we will go more in depth about the **energy conversions** taking place.

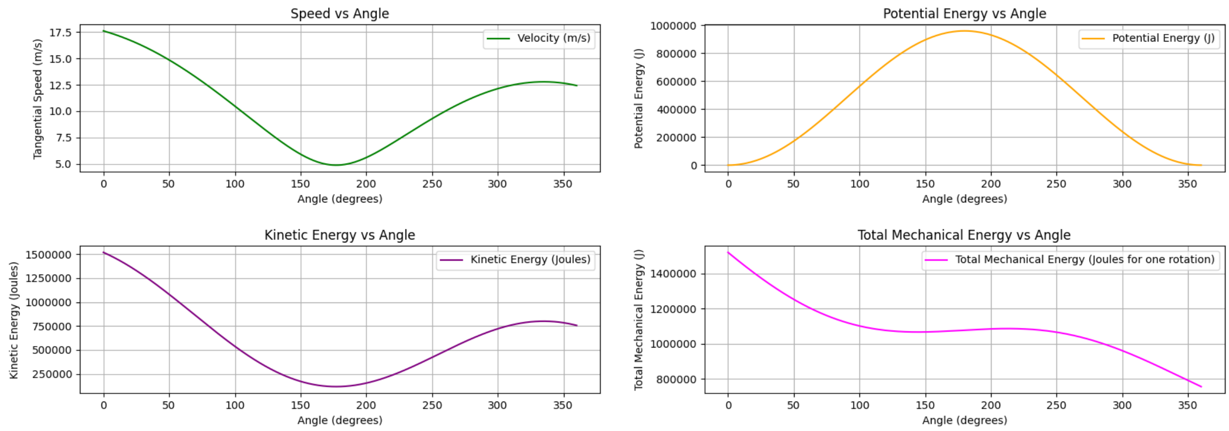
$$Loop\ one\ radius\ (22m)\ Initial\ velocity: \sqrt{2g(60)}\ ms^{-1}$$



$$Loop\ two\ radius\ (13.5m):\ Initial\ velocity: 25.00833073341539ms^{-1}$$



$$Loop\ 3\ radius\ (10m): Initial\ velocity:\ 17.610992407296333ms^{-1}$$



From this, we can understand that:

• Friction during the rollercoaster's journey through loops leads to energy losses. The potential energy decreases, and the kinetic energy increases, but the total mechanical energy of the system is not conserved. At the top of the loop, the potential energy remains maximum, but due to friction, the kinetic energy at the bottom is lower.

• While the principle of energy conservation still holds, the rollercoaster's mechanical energy is not fully preserved due to friction.

• Friction converts some of the mechanical energy into other forms, such as heat and sound, causing a gradual decrease in the total mechanical energy of the rollercoaster as it moves through the loops.

Interpretation of Results **15 marks**

- Aim 1: Successfully replicated LCH question using my model.
- Aim 2: Explored gravitational forces and gained insights into their impact on rollercoasters.
- Aim 3: Focused on friction's influence on rollercoasters and explored the energy conversions taking place.

Results indicate that the rollercoasters analysed were designed safely.
Exploring drag would have increased accuracy by accounting for speed reduction and decreased G-forces however due to word count limitations I couldn't.
Clothoid loops were not included due to being beyond the syllabus.
Euler method used is first order inhomogeneous.