Written Problem Set 1: Graphs

The first thing you should do in ps1.tex is set up your name as the author who is submitting by replacing the line, `\submitter{TODO: your name}`, with your name and UVA email id, e.g., `\submitter{Grace Hopper (gmh1a)}`.

Next, put your response for problem 1 into problem1.tex, problem 2 into problem2.tex, etc.

Before submitting , also remember to:

– List your collaborators and resources, replacing the TODO in `\collaborators{TODO: replace ...}` with your collaborators and resources. (Remember to update this before submitting if you work with more people.)

– Replace the second line in ps1.tex, `\usepackage{algo}` with `\usepackage[response]{algo}`.

Collaborators: TODO: replace this with your collaborators (if you did not have any, replace this with *None*)

Resources: TODO: replace this with your resources (if you did not have any, replace this with *None*)

Problem 1: Martian Morels

NEWSFLASH: SUBTERRANEAN HUMANOID RACE DISCOVERED ON MARS!

We've discovered Martians living in caves! Since they live underground, their cuisine heavily features of a variety of Martian mushrooms. The tastiest is the Martian Morel, said to be more delicious than any food on Earth. While many Martian Morels are safe for human consumption, one subspecies of the Martian Morel is deathly toxic to humans. To human eyes the edible and toxic varieties are indistinguishable, but the Martians may be capable of telling the difference. We're going to test this whether this is the case. Making this challenging, though, the Martian language is unlike any human languages, and the only words of their language we have learned so far are "same", "different", and "uncertain".

We have n samples of Martian morels $s_0, ..., s_{n-1}$ each of which are either edible or toxic (we don't know which for any). A Martian was given all n^2 possible (ordered) pairs of samples (s_i, s_j). The Martian then decided whether each sample's mushrooms were (a) the same (both edible or both toxic), (b) different (one edible and one toxic), or (c) that the Martian was uncertain. (All pairs were tested, but not all have "same" or "different" decisions).

At the end of the testing, the Martian decided that a total of m pairs were "same" or "different". Give an algorithm that determines whether these set of m decisions is consistent. A set of decisions is consistent if there is a way to label each sample s_i with "edible" or "toxic" such that if the Martian indicated the pair (s_i, s_j) was "same" our labels for s_i, s_j match, and if the Martian indicated the pair (s_i, s_j) was "different" our labels for s_i, s_j differ. Your algorithm should run in time $O(m + n^2)$.

Input A 2-d array with each cell containing either "same", "different" or "uncertain" representing the decision of the Martian for each pair.

Output A boolean indicating whether the given set of decisions is consistent.

Algorithm TODO: Provide a precise description of your algorithm here

Correctness TODO: Justify the correctness of your algorithm here (a formal proof is not necessary, just explain precisely why you believe your algorithm to be correct)

Running Time TODO: Give and justify a Θ bound on your algorithm's worst-case running time here (a formal proof is not necessary, just explain precisely why you believe your algorithm to have this running time).

Problem 2: Dog Walking

Nate's dog, Bucky, loves other dogs. He loves them so much that whenever he sees one he feels that he MUST meet it. The problem is that the way he expresses this is by barking hysterically, which frightens the other dogs. Nate and his neighbor both want to walk their dogs from their houses to a different park at the same time. To keep all dogs happy they want to ensure that when they are walking their dogs they do not want their dogs to see each other. Help Nate and his neighbor find the shortest paths from their homes to their parks such that the dogs will not see each other.

Your algorithm will be given an undirected, unweighted graph of the neighborhood that contains n nodes. This graph will contain two "house" nodes (call these H_{nate} and H_{neigh} for Nate's house and his neighbor's house respectively) and two "park" nodes (called P_{nate} and P_{neigh}). The remaining $n - 4$ nodes will be intersections, and the edges will represent streets. As aoutput, it should give a shortest pair of schedules (one from H_{nate} to P_{nate} and the other from H_{neigh} to P_{neigh}) that satisfies the following:

- At each step in the schedule you can either stay in the same node as the step prior, or else move to an adjacent node.
- Someone may return to a node they've already been in.
- There cannot exist a time in the schedules where nate and his neighbor are in the same node or in adjacent nodes.

For example, suppose we are given the graph shown in Figure 1.

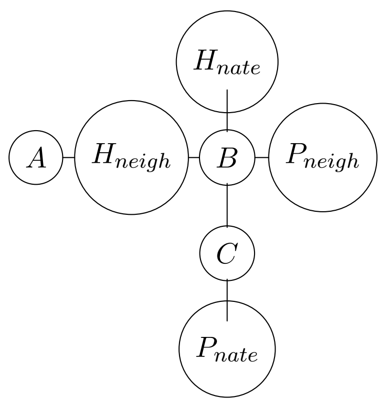

Figure 1: Nate starts at vertex H_{nate} and must get to vertex P_{nate}. Lorelai starts at vertex H_{neigh} and must get to vertex P_{neigh}.

In this case, the shortest schedule would be:

	Step 1	Step 2	Step 3	Step 4	Step 5
Nate:	H_{nate}	B	C	P_{nate}	P_{neigh}
Neighbor:	H_{neigh}	A	H_{neigh}	B	P_{neigh}

The worst case running time of your algorithm should be $O(n^4)$.

Algorithm TODO: Provide a precise description of your algorithm here

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Correctness TODO: Justify the correctness of your algorithm here (a formal proof is not necessary, just explain precisely why you believe your algorithm to be correct)

Running Time TODO: Give and justify a Θ bound on your algorithm's worst-case running time here (a formal proof is not necessary, just explain precisely why you believe your algorithm to have this running time).

Problem 3: Is it connected?

Let $G = (V, E)$ be an undirected graph such that $|V| = n$. Either prove the claim below or else find a counterexample.

Claim If every member of V has a degree of at least $\frac{n}{2}$ then G must be connected.

Proof/Counterexample TODO: Include your proof or counterexample below. Be sure to state which you are providing, and your proof strategy (if that's the direction you're answering).

Problem 4: Asymptotics

Using the definition of O, Ω, and Θ in class, demonstrate each of the following:

a. $n \in O(n^2)$

b. $n^2 \notin O(n)$

c. $\sqrt{2n} \in \Omega(2\sqrt{n})$

d. $\max(n, m) \in \Theta(n + m)$

TODO: give a proof for each item below: