$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} + 2 \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} + 3 \begin{bmatrix} 7 & 8 & 9 \end{bmatrix}$$

Fundamental Law of Matrix Multiplication

$$\begin{bmatrix} * & & & \\ a_{i1} & a_{i2} & \cdots & a_{i5} \\ * & & & \\ * & & & \end{bmatrix} \begin{bmatrix} * & * & b_{ij} & * & * & * \\ & & b_{2j} & & & \\ & & \vdots & & & \\ & & b_{5j} & & & \end{bmatrix} = \begin{bmatrix} & & * & & & \\ * & * & (AB)_{ij} & * & * & * \\ & & * & & & \\ & & * & & & \end{bmatrix}$$

The second and third ways : Rows and columns

$$[\text{row } i \text{ of } A] \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = [\text{row } i \text{ of } AB]$$

Every row of A times matrix B

Every row/column of AB

is a combination of the row/column of A/B

Every column of B times Matrix A

Elimination

For $\begin{vmatrix} a_1 & \cdots \\ a_2 & \cdots \\ \vdots & \ddots \end{vmatrix}$ if we want to eliminate a_2

we need $\begin{vmatrix} a_1 & 0 & 0 & \cdots \\ -\frac{a_2}{a_1} & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & & & \ddots \end{vmatrix} \begin{vmatrix} a_1 & \cdots \\ a_2 & \cdots \\ \vdots & \ddots \end{vmatrix} = \begin{vmatrix} a_1 & \cdots \\ 0 & \cdots \\ \vdots & \ddots \end{vmatrix}$

$$\underbrace{\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right]}_{A} \cdot \left[\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right] = \left[\begin{array}{c|c} A_1B_1 + A_2B_3 & A_1B_2 + A_2B_4 \\ \hline A_3B_1 + A_4B_3 & A_3B_2 + A_4B_4 \end{array}\right]$$

Important special case

Let the blocks of A be its n columns. Let the blocks of B be its n rows. Then block multiplication AB adds up columns times rows:

Columns times rows
$$\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} - b_1 - \\ \vdots \\ - b_n - \end{bmatrix} = \begin{bmatrix} a_1b_1 + \cdots + a_nb_n \end{bmatrix}$$

e.g. $\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 5 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 2 \\ 8 & 2 \end{bmatrix}$$

Elimination by blocks

$$\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right]$$

Inverse (逆) (square matrix)

$A^{-1}A = I = AA^{-1}$ invertible

If This exists

What time doesn't a matrix be invertible?

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ isn't invertible

① because $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ are parallel, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ isn't on the line they make

② $\det \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} = 0$

③ $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

if inverse exists

$A^{-1} \cdot A \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} A^{-1}$ $\begin{bmatrix} -3 \\ 1 \end{bmatrix} = 0$ paradox

If $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} \frac{1}{d_1} & & \\ & \ddots & \\ & & \frac{1}{d_n} \end{bmatrix}$

reverse order:

$(AB)^{-1} = B^{-1}A^{-1}$

Gauss-Jordan (solve 2 eqns at once)

e.g. $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}_{A}\!\!\left|\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\right] \longrightarrow \left[\begin{matrix} 1 & 3 \\ 0 & 1 \end{matrix}\,\middle|\,\begin{matrix} 1 & 0 \\ -2 & 1 \end{matrix}\right] \longrightarrow \left[\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I}\,\middle|\,\underbrace{\begin{matrix} 7 & -3 \\ -2 & 1 \end{matrix}}_{A^{-1}}\right]$$

$$A^{-1}\,[A \;\; I] = [I \;\; A^{-1}]$$

$(R^T R)^T = R^T \cdot R^{TT} = R^T R$

conclusion $R^T \cdot R$ is symetric

rule in transpose (转置)

$(A \cdot B)^T = B^T \cdot A^T$ → easy prove by write out the items

Vector space and subspace

Every subspace contains the zero vector

A subspace containing $\vec{v}$ and $\vec{w}$ must contain all linear combinations $c\vec{v}+d\vec{w}$

The system $Ax=b$ is solvable if and only if b is in the column space of A

e.g.

$$\overset{A}{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}} x = \overset{b}{\begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}}$$

b is a subspace of R^4

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

So we can say only $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ can consitute

the subspace b and call them pivot column

Nullspace

eg.

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Nullspace

Check the solution to $Ax=0$

always give a subspace

proof: If $Av=0$ and $Aw=0$ then $A(v+w)=0$

so its a subspace

Attention $Ax=b$ $Ax=0$

not a subspace   subspace

In elimination the solution don't change, so the null space won't change

rank of A 秩

It equal to the number of pivots

the columns pivot on called pivot column
others called free column

free columns: we can assign any number freely to them

e.g.

$$\begin{array}{cccc} & \text{free} & & \text{free} \\ \end{array}$$
$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \square \\ 1 \\ \square \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \square \\ 0 \\ \square \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

we could get $X = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} . d\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

we call them special solution

* The null space contains exactly all the combination of the special solutions

columns(x) is the column in s degree of I $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

personal idea: the demension of null space equals to the number(s) of free colums(x), so we need x and set of special solutions to get the null space

---

we can also say null space $= c\begin{bmatrix} -F \\ I \end{bmatrix}$

e.g.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{R}$$

F

$$X = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = c\begin{bmatrix} -F \\ I \end{bmatrix}$$

identity matrix

$Ax = b$

To find a complete solution to $Ax=b$

① $x_{particular}$: set all free variables to zero

solve $Ax=b$ for privot variables

$+$

② $x_{nullspace}$

$$x = x_p + x_N$$

$$b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$$

e.g.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_2-b_1 \end{array}\right] \qquad x_p = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

proof: $Ax_p = b$

$Ax_N = 0$

The geometric mean of X:

$$X = \underbrace{\begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}}_{x_p} + \underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{\text{a plane in 4 dimension space}}$$

X is a plane which across $(-2, 0, \frac{3}{2}, 0)$ in 4 dimension space so X isn't a subspace m by n matrix A of rank r (know $r \le m$, $r \le n$)

Full column rank means $r=n$ No free varibles

$N(A) = \begin{Bmatrix} \text{zero} \\ \text{vector} \end{Bmatrix}$ Solution to $Ax=b$: $x = x_p$ (if solution exists)

$R \begin{bmatrix} 1 & - \\ 0 & - \\ 0 & - \\ 0 & - \end{bmatrix} \leftarrow$ e.g $\begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix} \cdot x = \begin{bmatrix} \square \\ \square \\ \square \\ \square \end{bmatrix}$

because (1,2,3,4) (5,6,7,8)'s combination can't cover all 4 dimension space

Full row rank means $r=m$

every row has a pivot

Can solve $Ax=b$ for every b

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}} x = \begin{bmatrix} \square \\ \square \end{bmatrix}$$

$$R = [\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I} \; \begin{matrix} - & - \\ - & - \end{matrix}]$$

can cover 2 dimension space

conclusion

$r=m=n$ $\quad$ $r=n<m$ $\quad$ $r=m<n$

$R=[I]$ $\quad$ $R=\begin{bmatrix} I \\ 0 \end{bmatrix}$ $\quad$ $R=[I\ F]$

1 solution $\quad$ 0 or 1 solution $\quad$ ∞ solution

$r<m,\ r<n$

$R=\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

(0 or ∞ solution)

☆: The rank tells you everything about the number of solutions

Independence

vectors $x_1, x_2, \cdots, x_n$ are independent if no combination gives you vector (except the zero combo, all $c_i = 0$)

$c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \neq 0$

rank $= n$

no free variables

when $v_1, \cdots, v_n$ are columns of A

They are independent if nullspace of A $\left\{ \begin{matrix} \text{zero} \\ \text{vector} \end{matrix} \right\}$

they are dependent if $A \cdot c = 0$ for some nonzero c

rank $< n$

Vectors $v_1, \cdots, v_\ell$ span a space means:

The space consists of all combinations of those vectors

Basis: for a space is a sequence of vectors $v_1, v_2, \cdots, v_d$ with 2 properties

基

1. they are independent
2. they span the space

vectors give basis if $n \times n$ matrix is invertiable

For a given space every basis has the same number of vectors

↳ we call it dimension

rank = pivot columns = dimension of column space

$\dim C(A) = r$

$\dim N(A) = \#$ free variables $= n - r$

Q: Given five vectors in $\mathbb{R}^7$, how do you find a basis for the space they span?

A: Put the five vectors into the columns of A. Eliminate to find the pivot columns. Those pivot columns are a basis for the column space.

4 subspace

column space $C(A)$ in R^m

null space $N(A)$ in R^n

row space $C(A^T)$ in R^n

null space of A^T $N(A^T)$ in R^m

$$\dim(C(A)) = \dim(C(A^T)) = r$$

$$\dim N(A) = n-r$$

$$\dim N(A^T) = m-r$$

e.g. How can we get $N(A^T)$

$$A=\begin{bmatrix}1&2&3&1\\1&1&2&1\\1&2&3&1\end{bmatrix}\rightarrow\begin{bmatrix}1&0&1&1\\0&1&1&0\\0&0&0&0\end{bmatrix}=R$$

Gauss-Jordan

$$\underset{m\times m}{E}\cdot[\underset{m\times n}{A}\ \underset{m\times m}{I}]\longrightarrow[\underset{m\times n}{R}\ \underset{m\times m}{E}]$$

$$E\cdot A=R$$

$$\overset{E}{\begin{bmatrix}-1&2&0\\1&-1&0\\1&0&1\end{bmatrix}}\cdot A=R$$

$\because$ the last row in R is 0

so the combination of rows in A is the last row in E

So $N(A^T)=c[1,0,1]$

Exercise: A matrix $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

1) is it a subspace if $v_1+v_2+v_3+v_4=0$?

2) what's the dimension of it?

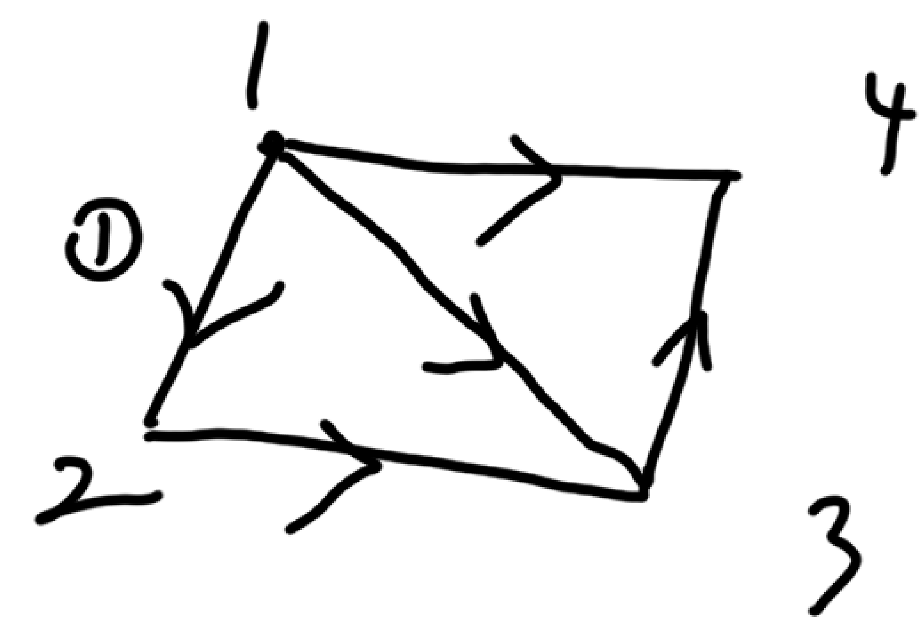
1) easy

2) actually it is the null space of

$[1, 1, 1, 1]$ the rank $=1$

so $\dim(\text{null space}) = 4-1=3$

An application of matrix



$$
\begin{array}{c|cccc|c}
\text{node} & 1 & 2 & 3 & 4 & \text{edge} \\
\hline
 & -1 & 1 & 0 & 0 & 1 \\
A = & 0 & -1 & 1 & 0 & 2 \\
 & -1 & 0 & 1 & 0 & 3 \\
 & -1 & 0 & 0 & 1 & 4 \\
 & 0 & 0 & -1 & 1 & 4
\end{array}
\qquad
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
$$

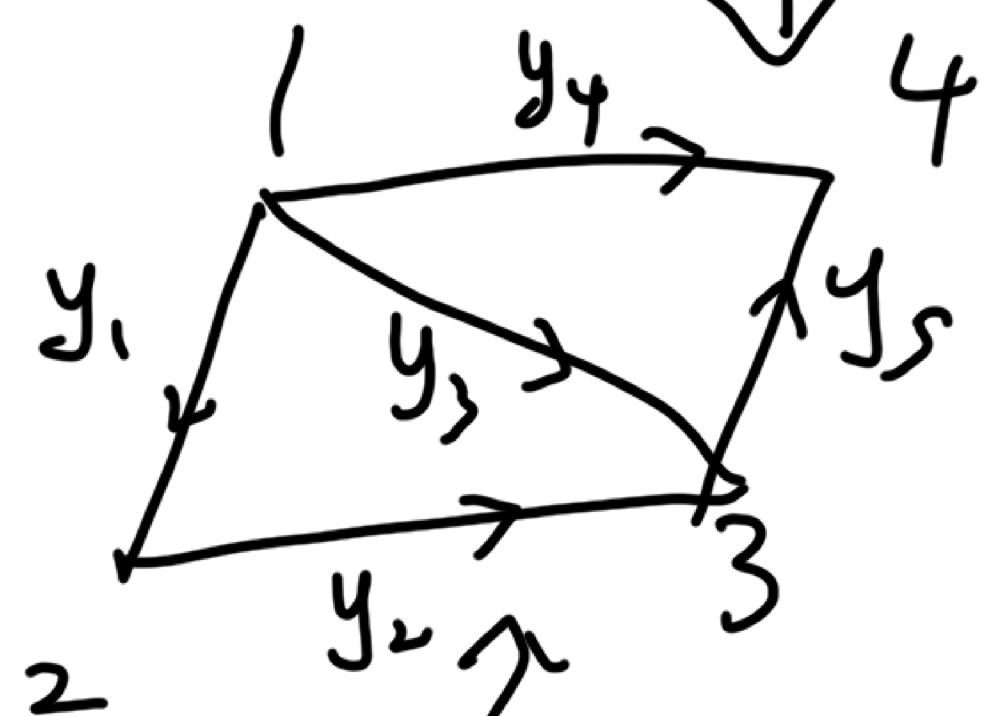
$$
Ax = 0 = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\therefore \text{Null space} = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}
$$

$A^T y = 0$

$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Loop ②



$-y_1 - y_3 - y_4 = 0$

$y_1 - y_2 = 0$

$y_2 + y_3 - y_5 = 0$

$y_4 + y_5 = 0$

Loop ①

Basis for $N(A^T)$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

So Basis for $N(A^T) = \#$ loop (independent)

*: the pivot columns of A^T y_1 y_2 y_4, have 4 nodes but 3 edge. so it's not loop, we call it tree

we can convert the formula :

$$\dim N(A^T) = m - r$$

$$\# \text{Loops} = \# \text{edge} - (\# \text{nodes} - 1)$$

$m = \# \text{edge}$

$n = \# \text{node}$

$\dim N(A) = 1$

$\therefore r = n - 1$

Eular's Formula

$$\# \text{nodes} - \# \text{edges} + \# \text{loops} = 1$$

Exercise: $Ax = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$ $\quad x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

A is a 3×3 matrix

and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is the null vector

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

Orthogonal

orthogonal space:

def: two vectors and of and inner product space are called orthogonal if the inner product of and is 0.

row space is orthogonal to nullspace

$$\begin{bmatrix} \text{row } 1 \text{ of } A \\ \text{row } 2 \text{ of } A \\ \cdots \\ \text{row } m \text{ of } A \end{bmatrix} \begin{bmatrix} \vdots \\ x \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and column space is orthogonal to nullspace of A^T

$N(A^TA) = N(A)$

rank of A^TA = rank of A

If A has independent columns then A^TA is invertible.

proof:

Suppose $A^TAx = 0$

$$x^TA^TA^Tx = 0$$

$$(Ax)^T(Ax) = 0$$

$$Ax = 0$$

$$x = 0$$

$\therefore$ A^TA is invertible

Projection

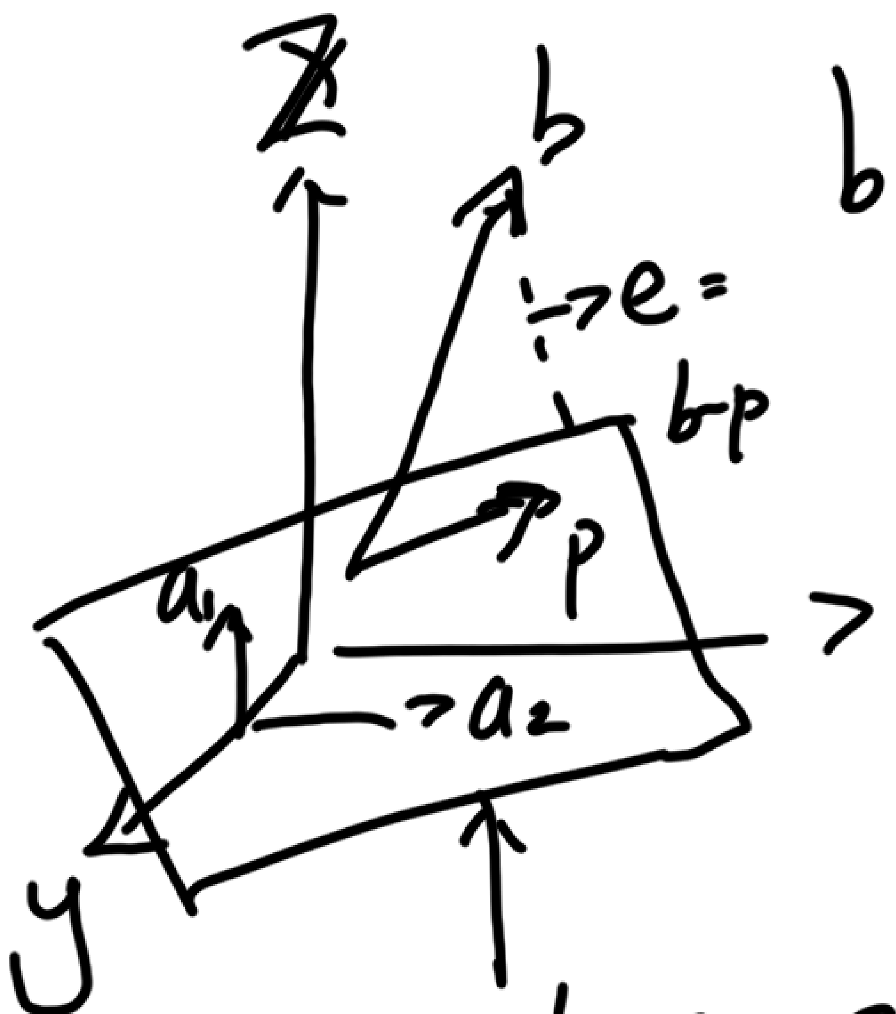
Because $Ax=b$ may have no solution

we solve $A\hat{x}=p$ instead

↳ projection of b onto column space

e.g

e is prep to plane



b is out of the plane

p is in

plane of a_1, a_2

$P = A\hat{x}$ Find $\hat{x}$

Key: $b-A\hat{x}$ is prep. to plane

column space of $A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$

$a_1^T(b-A\hat{x})=0 \qquad a_2^T(b-A\hat{x})=0$

$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix}(b-A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A^T\underbrace{(b-A\hat{x})}_{e} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

e in $N(A^T)$

$e \perp C(A)$

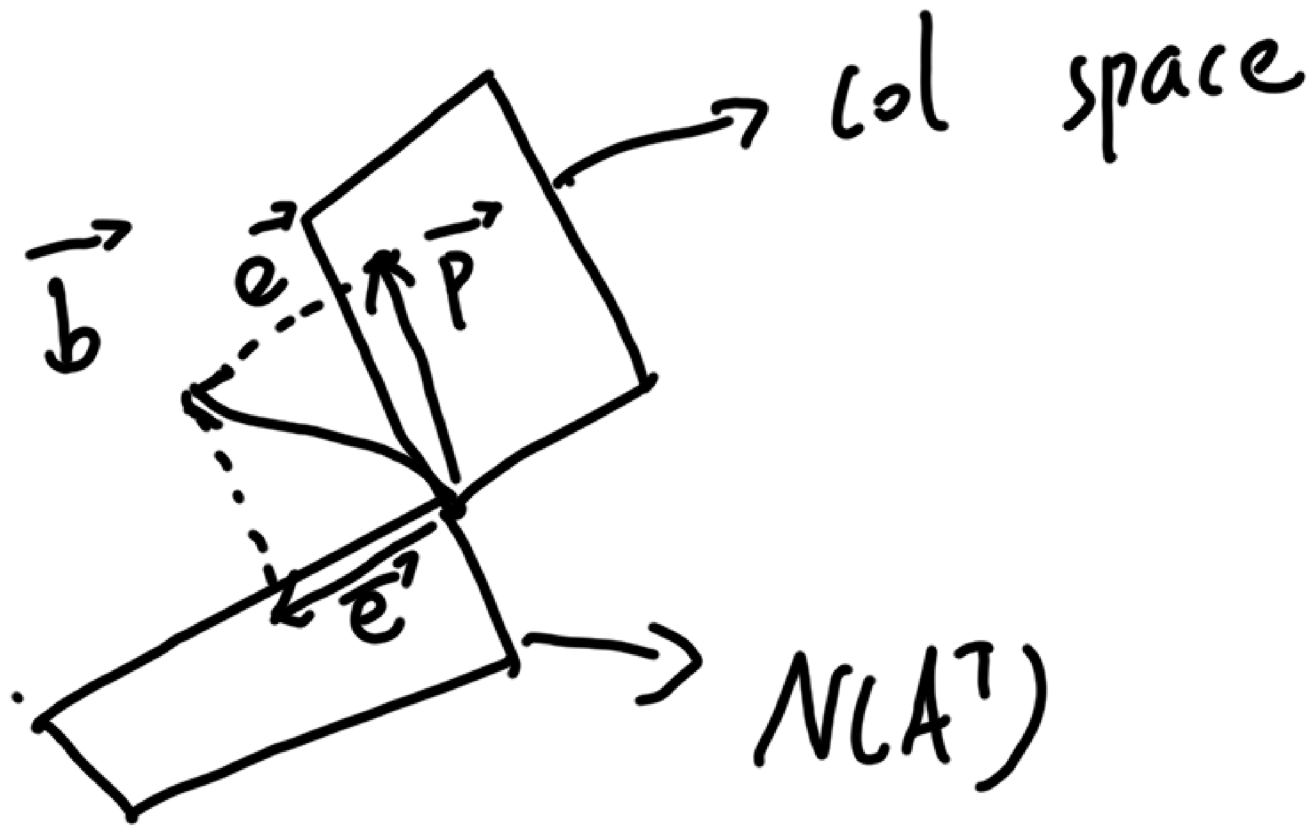
$A^TA\hat{x} = A^Tb$

$\hat{x} = (A^TA)^{-1}A^Tb$

$p = A\hat{x} = A(A^TA)^{-1}A^Tb$

$p = Pb$

$P^T = P \qquad P^2 = P$



$$\vec{p} + \vec{e} = \vec{b}$$

$$P\vec{b} \qquad (I-P)\vec{b}$$

Key. $A^TAx = A^Tb$

property of projection matrix P: $P^2 = P$

$$\left(\frac{AA^T}{A^TA}\right)^2 = \frac{AA^T}{A^TA}$$

Excercise:

(1,1) (2,2) (3,2)

$$y = Ax + b$$

$$A + b = 1$$

$$2A + b = 2$$

$$3A + b = 2$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$A^T A x = A^T b$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 14 & 6 \end{bmatrix} \begin{bmatrix} A \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$$A = \frac{1}{2}$$

$$b = \frac{2}{3}$$

$$A^TA = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & \cdots & 1 \\ t_1 & \cdots & t_m \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$$

$$\begin{bmatrix} m & \sum t_i \\ \sum t_i & \sum t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \sum b_i \\ \sum t_i b_i \end{bmatrix}$$

Problem: Fit height $b_1, \cdots, b_m$ at times $t_1, \cdots, t_m$ by a parabola $C+Dt+Et^2$

Solution: $C+Dt_1+Et_1^2=b_1$

$$\vdots$$

$$C+Dt_m+Et_m^2=b_m$$

is $Ax=b$ with m by 3 matrix

$$\begin{bmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_m & t_m^2 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

The closest parabola $C+Dt+Et^2$ chooses $\hat{x}=(C,D,E)$ to satisfy the three normal equations $A^TA\hat{x}=A^Tb$

Orthonormal Bases and Gram-Schmit

Def:

The vectors $q_1, \cdots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \qquad \|q_i\| = 1 \end{cases}$$

A matrix with orthonormal columns satisfies $Q^T Q = I$

when matrix is square: $Q^T = Q^{-1}$

- Every permutation matrix is an orthogonal matrix

Reflection: If u is any unit vector, set $Q = I - 2uu^T$. Notice that uu^T is a matrix while $u^T u$ is a number $\|u\|^2 = 1$

$Q^T = I - 2uu^T = Q \qquad Q^T Q = I - 4uu^T + 4uu^T uu^T = I \qquad Q^T = Q^{-1}$

e.g. $u = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \qquad Q = I - 2\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$

When (x, y) goes to (y, x) a vector like $(3,3)$ doesn't move it's on the mirror line

If Q has orthonormal columns $(Q^TQ=I)$, it leaves lengths unchanged:

$$\|Qx\| = \|x\| \qquad (Qx)^T(Qy) = x^TQ^TQy = x^Ty$$

property of orthogonal matrix: $|\lambda|=1$

$$Ax = \lambda x$$

$$\|A\|\,\|x\| = |\lambda|\,\|x\|$$

$$|\lambda| = 1$$

$$AA^T = A^TA = E$$

$$Ax = \lambda x$$

$$x^TA^T = \lambda x^T$$

$$x^TA^TAx = \lambda^2 x^Tx$$

$$\lambda^2 = 1$$

Gram- Schmidt

① Begin by choosing $A = a$

② start with b and subtract its projection along A

$$B = b - A\frac{A^Tb}{A^TA}$$

A and B are orthogonal

$$C = c - A\frac{A^Tc}{A^TA} - B\frac{B^Tc}{B^TB}$$

Example of Gram- Schmidt

$A = QR$:

first step: $q_1 = \frac{a}{\|a\|}$

$$\begin{bmatrix} | & | & | \\ a & b & c \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ & q_2^T b & q_2^T c \\ & & q_3^T c \end{bmatrix}$$

$$\therefore \quad b = \|b\| q_2 + q_1 \frac{q_1^T b}{q_1^T q_1} \qquad c = \|c\| q_3 + q_1 \frac{q_1^T c}{q_1^T q_1} + q_2 \frac{q_2^T c}{q_2^T q_2}$$

$$A^T A \hat{x} = A^T b$$

$$R^T R \hat{x} = R^T Q^T b$$

$$R \hat{x} = Q^T b$$

$$A^{-1} = \frac{1}{\det A} C$$

$$A = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -3 \\ 0 & -2 & 3 \end{bmatrix} \end{array} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{18} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{bmatrix} = QR$$

$$q_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

$$q_2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \frac{a^T b}{a^T a} a$$

$$= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$= \sqrt{6} \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \frac{a^T c}{a^T a} a - \frac{b^T c}{b^T b} b$$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Eigenvalues - Eigenvectors

$$Ax = \lambda x$$

Eigenvalues   Eigenvectors

$$(A - \lambda I)x = 0$$

$$\therefore \det(A - \lambda I) = 0$$

Trace: $\lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$

$$\begin{cases} Ax = \lambda x \\ (A+3I)x = (\lambda+3)x \end{cases}$$

Every matrix $C = B^{-1}AB$ has the same eigenvalue as A

C's are "similar" to A

proof

Suppose $Cx = \lambda x$ $\quad BCB^{-1}$ has the same eigenvalue λ with the new eigenvector Bx:

$$(BCB^{-1})(Bx) = BCx = B\lambda x = \lambda(Bx)$$

eigenvalues might be complex number

e.g.

$Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ trace = 0

det = 1

$$\det(Q - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\therefore \lambda_1 = i$

$\lambda_2 = -i$

Suppose n independent eigenvector of A

put them in columns of S

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & 0 \\ 0 & \ddots & 0 \\ \vdots & & \vdots \\ 0 & & \lambda_n \end{bmatrix} = S\Lambda$$

$$S^{-1}AS = \Lambda \qquad A = S\Lambda S^{-1}$$

If $Ax = \lambda x$

$$A^2 x = \lambda A x = \lambda^2 x \qquad A^2 = S\Lambda^2 S^{-1}$$

A^2 eigenvector is same eigenvalue is λ^2

$$A^k = S\Lambda^k S^{-1}$$

Theorem

$A^k \to 0$ as $k \to \infty$

if all $|\lambda_i| < 1$

A is sure to have n independent eigenvectors (and be diagonaliable) if all the λ are different

Reapeated eigenvalues may or may not have n independent eigenvector

e.g. I $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

proof:

Suppose $c_1x_1+c_2x_2=0$

Multipy by A: $c_1\lambda_1x_1+c_2\lambda_2x_2=0$

Multipy by λ_2: $c_1\lambda_2x_1+c_2\lambda_2x_2=0$

$\therefore (\lambda_1-\lambda_2)c_1x_1=0$

$\therefore c_1=0$

Similarily $c_2=0$

$\therefore c_1=c_2=0$ $\qquad c_1x_1+c_2x_2=0$

$\therefore x_1$ x_2 must be independent

This proof extends directly to j eigenvectors.

If x_1, x_2 is symetric matrix A's two eigenvectors : $Ax_1 = \lambda_1 x_1$ $Ax_2 = \lambda_2 x_2$ and $\lambda_1 \neq \lambda_2$

we can have : $x_1^T x_2 = 0 \Longrightarrow x_1\ x_2$ are orthorgonal

When things are evolving in time by a first-order system $u_{k+1} = Au_k$

$$u_k = A^k u_0$$

$$u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n = Sc$$

$$A^{100}u_0 = c_1\lambda_1^{100}x_1 + c_2\lambda_2^{100}x_2 + \cdots + c_n\lambda_n^{100}x_n = \Lambda^{100}Sc$$

e.g. Fibonacci sequence $0,1,1,2,3,5\cdots$ $F_{100} = ?$

$F_{k+2} = F_{k+1} + F_k$

$F_{k+1} = F_{k+1}$

$$u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} \quad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}} u_k$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_k \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$c_1x_1 + c_2x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2}$$

$$\lambda_2 = \frac{1-\sqrt{5}}{2}$$

e.g. $\frac{du_1}{dt} = -u_1 + 2u_2$

$\frac{du_2}{dt} = u_1 - 2u_2$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\downarrow$

$A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$

$\lambda_1 = 0 \qquad \lambda_2 = -3$

$x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$u(t) = c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2$$

$$= c_1 \cdot 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Differential equations 微分方程

$$\frac{du}{dt} = Au$$

eigenvector matrix

set $u = Sv$

$$\frac{du}{dt} = Au$$

$\downarrow$

$$\frac{dv}{dt} = S^{-1}ASv = \Lambda v$$

$$\frac{dv_1}{dt} = \lambda_1 v_1$$

$\vdots$

If $f(x) = Ce^{\lambda x}$

$f'(x) = C\lambda e^{\lambda x} = \lambda(Ce^{\lambda x})$

$f(x) = f(0) \cdot e^{\lambda x}$

$\star$

$\therefore$ $v(t) = e^{\Lambda t} v(0)$

$u(t) = Se^{\Lambda t}S^{-1}u(0)$

$$e^{At} = Se^{\Lambda t}S^{-1}$$

Matrix exponential

$$e^{At} = \underline{I} + At + \frac{(At)^2}{2} + \frac{(At)^2}{6} + \cdots + \frac{(At)^n}{n!} + \cdots$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$$= S S^{-1} + S\Lambda S^{-1} t + \frac{S\Lambda^2 S^{-1}}{2} t^2 + \cdots$$

$$= S e^{\Lambda t} S^{-1}$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_2 t} \end{bmatrix}$$

$$u(t) = e^{At} u(0) = S e^{\Lambda t} S^{-1} u(0)$$

$$= c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$$

$$= \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

if $Re\ \lambda_i < 0$

the exponential goes to 0

$u(t)$ is steady

$u(t)$ approaches zero (stability) if every λ has negative real part: All $e^{\lambda t} \to 0$

$$\because |e^{(-3+6i)t}| = e^{-3t} \qquad |e^{6it}| = 1$$

$$y'' + by' + ky = 0$$

$$y' = y'$$

$$u = \begin{bmatrix} y' \\ y \end{bmatrix} \qquad u' = \begin{bmatrix} -b & -k \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \end{bmatrix}$$

if we have a matrix goes from

a n order, we can put it to

first order

Markov matrix

Two properties:

1. every entry is greater equal zero
2. all columns add to 1

so the powers of Markov matrix are all Markov

the matrix has an eigenvalue of 1

1. $\lambda = 1$ is an eigenvalue
2. All other $|\lambda_i| < 1$

$$U_k = A^k u_0 = c_1 \lambda_1^k x_1 + c_2 \underbrace{\lambda_2^k}_{\to 0} x_2 \ldots$$

$$= c_1 \lambda_1^k x_1$$

$$= c_1 x_1$$

proof:

$$A-1I = \begin{bmatrix} -0.9 & 0.01 & 0.3 \\ 0.2 & -0.01 & 0.3 \\ 0.7 & 0 & -0.6 \end{bmatrix}$$

because $(1,1,1)$ is in $N(A^T)$

$\therefore \det(A-I)=0$

eigenvalue of A = eigenvalue of A^T

e.g. Application of Markov matrix

$$\begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_{t=k+1} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_k \qquad \begin{bmatrix} u_{cal} \\ u_{mass} \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 1000 \end{bmatrix}$$

$$u_k = c_1 1^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 (0.7)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad = \frac{1000}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$= \frac{1000}{3}$ $\qquad \frac{2000}{3}$

Fourier Series

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

dot product of vectors / functions

vector

$$v^T w = v_1 w_1 + \cdots + v_n w_n$$

function

$$f^T g = \int_0^{2\pi} f(x)\, g(x)\, dx$$

$$\int_0^{2\pi} f(x) \cos x \, dx = a_1 \int_0^{2\pi} \cos^2 x = a_1 \pi$$

$$a_1 = \frac{1}{\pi} \int_0^{\pi} f(x) \cos x$$

eigenvalues of A^{-1} are $\frac{1}{\lambda_1}$ $\frac{1}{\lambda_2}$ $\cdots$

$|A-\lambda I|=0 \Rightarrow |A^{-1}|\,|A-\lambda I|=0 \Rightarrow |A^{-1}A-\lambda A^{-1}|=0$

$|I-\lambda A^{-1}|=0 \Rightarrow |\frac{1}{\lambda}I-A^{-1}|=0$

Symmetric matrix

$A = A^T$

① The eigenvalues are real

② The eigenvectors are perpendicular

usual

$A = S\Lambda S^{-1}$

symmetric:

$A = Q\Lambda Q^{-1}$

Why eigenvalues real?

(A is real)

$Ax = \lambda x \xRightarrow{\text{always}} A\bar{x} = \bar{\lambda}\bar{x} \Rightarrow \bar{x}^T A = \bar{x}^T \bar{\lambda}$

$\bar{x}^T A x = \bar{x}^T \lambda x$

$\bar{x}^T A x = \lambda \bar{x}^T x$

$\therefore \lambda = \bar{\lambda}$

GOOD MATRIX:

Real λ's

Perpendicular x's

$A = \bar{A}^T$

if A is complex: $A = \bar{A}^T \Rightarrow \lambda = \bar{\lambda}$

Eigenvectors of a real symmetric matrix (when they correspond to different λ's) are perpendicular

Proof:

Suppose $Sx = \lambda_1 x$ and $Sy = \lambda_2 y$ $\quad \lambda_1 \neq \lambda_2$

$$(\lambda_1 x)^T y = (Sx)^T y = x^T S^T y = x^T S y = x^T \lambda_2 y$$

$$x^T \lambda_1 y = x^T \lambda_2 y$$

$$\therefore x^T y = 0$$

If $A^{(n\times n)}$ is symmetric, $P = T^{-1}AT$ is diagonal

T is orthogonal matrix

proof: mathematical induction 数学归纳法

$n=1$ easy

when $n=t-1$ suppose it's right

when $n=t$

suppose λ_1 is an eignvalue of A, $Ax_1 = \lambda_1 x_1$ let $\|x_1\|=1$

span x_1 into a orthnomal basis in R^t $\{x_1, \cdots, x_t\}$

$T_0 = (x_1, x_2 \cdots x_n)$ T_0 is orthogonal matrix $T_0^{-1} = T_0^T$

$$T_0^{-1}AT_0 = \begin{pmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_t^T \end{pmatrix} A (x_1, x_2 \cdots x_t) = \begin{pmatrix} x_1^TAx_1 & x_1^TAx_2 & \cdots & x_1^TAx_t \\ x_2^TAx_1 & x_2^TAx_2 & \cdots & x_2^TAx_t \\ \vdots & \vdots & & \vdots \\ x_t^TAx_1 & x_t^TAx_2 & \cdots & x_t^TAx_t \end{pmatrix}$$

$$= \left(\begin{array}{c|ccc} \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & A_1 & \\ 0 & & & \end{array}\right)$$

$\because \exists$ orthogonal matrix T_1

s.t. $T_1^{-1}A_1T_1 = P_1$ P_1 is diagonal

$$= \begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix} \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix}^{-1}$$

A_1 is symmetric

$T = T_0 \begin{pmatrix} 1 & 0 \\ 0 & T_1 \end{pmatrix}$ $T^TT = I$ $\therefore T$ is or...

Schur's Theorem

Every square A factor into QTQ^{-1} where T is upper triangular and $\overline{Q}^T = Q^{-1}$. If A has real eigenvalues then Q and T can be chosen real: $Q^T Q = I$

Proof: mathematical induction

$A = A^T$

$A = Q\Lambda Q^T$

$$= \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$= \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots$$

Symmetric matrix:

signs of pivots same as signs of λ's

pivots = # positive λ's
(positive)

If $AA^T = A^TA$ the eigenvectors of A are orthogonal

A: symmetric

anti-symmetric

orthonormal

$Qx = \lambda_1 x$ ①

$Qy = \lambda_2 y$

$y^T Q^T = \lambda_2 y^T$ ②

② × ①

$y^T x = \lambda_1 \lambda_2 y^T x$

$\because |\lambda| = 1$

$\therefore y^T x = 0$

Complex matrix

inner product :

$$\overline{z}^T \cdot z \qquad z^H z$$

Symmetric

$$A^H = A \qquad e.g: \begin{bmatrix} 2 & 3+i \\ 3-i & 5 \end{bmatrix}$$

Orthnormal

$q_1, q_2 \cdots q_n$

$$\overline{q_i}^T \cdot q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Fouier

$$F_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w & w^2 & \cdots & w^{n-1} \\ 1 & w^2 & w^4 & \cdots & w^{2(n-1)} \\ \vdots & & & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \cdots & w^{(n-1)^2} \end{bmatrix} \qquad (F_n)_{ij} = w^{ij}$$

$$i, j = 0, \cdots, n-1$$

$$w^n = 1 \qquad w = e^{\frac{2\pi}{n}}$$

$$F_4 : \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

$$F_4^H \cdot F_4 = I$$

Fast Fouier Transform

$$\begin{bmatrix} F_{64} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{32} & 0 \\ 0 & F_{32} \end{bmatrix} \overset{P_{64}}{\begin{bmatrix} & 1 & & & \\ & & & 1 & \\ 1 & & 1 & & \\ & & & & 1 \end{bmatrix}}$$

takes the even (0, 2, 4 ...) numbered components first and then the odds

$$D = \begin{bmatrix} 1 & & & & \\ & w & & & \\ & & w^2 & & \\ & & & \ddots & \\ & & & & w^{31} \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} \begin{matrix} I & D \\ I & -D \end{matrix} & 0 \\ 0 & \begin{matrix} I & D \\ I & -D \end{matrix} \end{bmatrix} \begin{bmatrix} F_{16} & & & \\ & F_{16} & & \\ & & F_{16} & \\ & & & F_{16} \end{bmatrix} \begin{bmatrix} P_{32} & \\ & P_{32} \end{bmatrix}$$

only need $\frac{n}{2} \log_2 n$ multiply

Positive Definite Matrix

① $\lambda_1 > 0 \quad \lambda_2 > 0 \ \ldots$

② det > 0

③ pivot > 0

④ $x^T A x > 0$

If A, B are pos def $\quad A+B$ is pos def

$A^T A$ is pos def

Similar Matrix

A and B are similar

means: for some M

$B = M^{-1}AM$

A, B have same eigenvalues

eigenvector of $A = M^{-1}$(evector of A)

ATTENTION: If A, B have same eigenvalues

the probably not similar

The Singular Value Decomposition (SVD)

$$A = U \Sigma V^T$$

$v_1, v_2 \cdots$ in row space

$\sigma_1 u_1 = A v_1$ orthonormal

$u_1, u_2 \cdots$ in col space

Idea: We want to find an orthonormal basis in row space and A multiply it can get a orthogonal basis in col space

If there's some null space, then we want to stick in a basis for that

$$\underbrace{v_1, v_2, \cdots v_r}_{\text{row}}, \underbrace{v_{r+1} \cdots v_n}_{\text{null}} \qquad \underbrace{u_1, u_2, \cdots, u_r}_{\text{col}}, \underbrace{u_{r+1}, \cdots u_n}_{\text{null of } A^T}$$

$$A^TA = V \Sigma^T \Sigma V^T \qquad AA^T = U \Sigma \Sigma^T U^T$$

$$\underbrace{= V \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_3^2 \end{bmatrix} V^T}_{\downarrow}$$

$Q \Lambda Q^T$ of A^TA

e.g. $\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \qquad 32$$

$$x_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} \qquad 18$$

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}}_{V^T}$$

$$U = \begin{bmatrix} \frac{1}{\sqrt{32}} & 0 \\ 0 & \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{32}} & 0 \\ 0 & \frac{1}{\sqrt{18}} \end{bmatrix} \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & -3\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

e.g. $\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

$\lambda_1 = 125 \qquad \lambda_2 = 0$

$x_1 = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0.6 \\ -0.8 \end{bmatrix}$

$$\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

Linear Transform

$$T(cv+dw) = cT(v) + dT(w)$$

T: with respect to $v_1 \cdots v_n$ it has matrix A

with respect to $w_1, \cdots, w_n$ has matrix B

$$B = M^{-1}AM$$

similar

2-sided inverse

$AA^{-1} = I = A^{-1}A$ condition: full rank $r=m=n$

left inverse full column rank $r=n$

nullspace $=\{0\}$ A^TA is full rank

$\underbrace{(A^TA)^{-1}A^T}_{\text{left inverse}}$ $(A^TA)^{-1}A^TA = I$

right inverse full row rank $r=m$

nullspace of $A^T = \{0\}$ AA^T is full rank

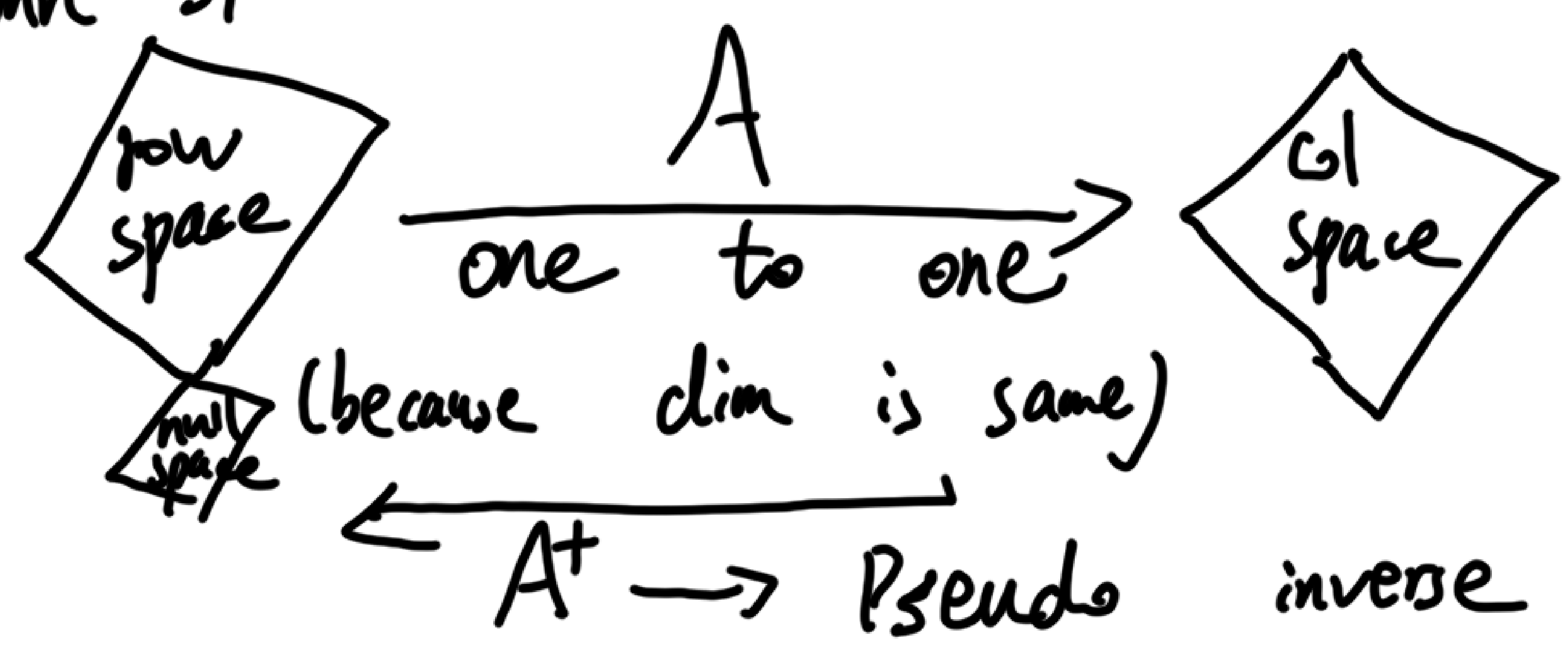
$\underbrace{A^T(AA^T)^{-1}}_{\text{right inverse}}$ $AA^T(AA^T)^{-1} = I$

* if $A \cdot$ (left inverse), (right inverse) A

they are projection matrix

Pseudo inverse

key idea: $Ax=b$ b is a combination of column vector, b is in column space



If $x \neq y$ both in row space the col space

$Ax \neq Ay$

Proof

Suppose $Ax = Ay$

$A(x-y) = 0$ $\to$ in nullspace

$\because$ $(x-y)$ is in rowspace

$\because$ rowspace and nullspace are orthogonal

$\therefore$ contradict

The way to find Pseudo inverse A^+

① SVD

$$A = U \Sigma V^T \qquad \Sigma = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ & \sigma_n & \\ 0 & & 0 \end{bmatrix}_{n \times m}$$

$$\Sigma^+ = \begin{bmatrix} \sigma_1^{-1} & & 0 \\ & \sigma_2^{-1} & \\ 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

idea: $\Sigma x = b$

$\Sigma^+ b = x$

because U, V^T are orthonormal

$$A^+ = V \Sigma^+ U^T$$