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Probability 201-1-2391 ASSIGNMENT 1
Combinatorial Analysis
instructor: Ronen Peretz,
math dept, BGU

Problem 1

15 cubes are ordered in a row. 5 cubes are red, 7 are white and 3 are green. How many different arrangements are there?

Problem 2

In how many ways can we arrange n people in a row so that:
a) John and Jane stand next to one another?
b) John, Jane and Mary stand together?

Problem 3

How many different 4 digits numbers can we produce from the following leggo units: $8, 8, 6, 5, 3, 2$?

Problem 4

In how many ways can a group of $2n$ people be divided into n pairs?

Problem 5

A box contains 50 bulbs, 15 of which are defected. A person pulls out randomly 3 bulbs from the box. How many different ways are there in which that person will have at least one non-defected bulb?

Problem 6

A person gets 13 cards from a deck of cards. How many different possibilities he has, so that his 13 cards are: 5 hearts, 4 diamonds, 2 spades and 2 leaves?

Problem 7

In a lost ship there are 20 children. Those children do not remember their birthdays and would like to be assigned birthdays.
a) In how many ways this can be done so that exactly 2 children will get identical day, and the other 18 will get each a separate day?
b) What is the number of possibilities to assign, so that at least one day of the year will belong to at least 2 children?

Problem 8

a) 12 children want to march in a row. In how many ways this can be done so that between John and Jeff there will be exactly 5 children?
b) The same question, but this time the children want to dance in a circle?

Problem 9

a) How many different "words" could be created using all the characters of ABRACADABRA?
b) In how many of these there are no 2 A's one next to the other?
c) In how many of the "words" in a) there are no 2 identical characters one next to the other?

Problem 10

How many different solutions are there for the equation $X_1 + X_2 + \ldots + X_k = n$, where:
a) For each i, $X_i \geq 0$ is integral.
b) For each i, $X_i = 1$ or $X_i = 2$ $(k \leq n)$.
c) For each i, $X_i \geq 0$ is integral and there are exactly 4 X_i's which equal 0?

Problem 11

Prove the identity:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Problem 12

Prove the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Problem 13

Prove the identity:

$$\binom{n}{n} + \binom{n+1}{n} + \ldots + \binom{N}{n} = \binom{N+1}{n+1}, \quad (N \geq n).$$

Problem 14

Prove the identity:

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}.$$

Problem 15

Prove the identity:

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

Problem 16

n identical balls are distributed in m different cells.
a) In how many of these distributions there will be exactly k empty cells?
b) Prove the following identity:

$$\sum_{k=0}^{m-1} \binom{m}{k} \binom{n-1}{n-m+k} = \binom{n+m-1}{n}.$$

Probability 201-1-2391 ASSIGNMENT 10
Inequalities and limit laws
instructor: Ronen Peretz,
math dept, BGU

Problem 1

The number of people that arrive at the bank branch per day is distributed Poisson, $P(100)$. Give an estimate for the probability that tomorrow less than 121 people will arrive at this bank branch and more than 79. Use Cebyshev's inequality.

Problem 2

A balanced coin is tossed 280 times. Let X be the number of H's in the first 200 tosses, and let Y be the number of H's in the last 80 tosses. Prove that:

$$p(X \leq Y + 50) \leq \frac{7}{20}.$$

Problem 3

We want to prepare a sample in order to know what is the proportion of people with blue eyes in the total population. Determine using Chebyshev's inequality the size of the sample to be taken if we want that in probability greater than or equals to 0.9 the experimental proportion of the blue eyes people to be within 0.01 away (at most) from the actual proportion.

Problem 4

There are 80 stair steps in the building leading up to the roof. At each stage a man throws a dice and climbs a number of steps which equals to the number shown on the face of the dice (except maybe at the end of the stairs). He can do that at most 20 times. Give an estimate using Chebyshev's inequality to the probability that the man will make it to the roof.

Problem 5

Give an estimate, using normal approximation, to the probability that in a 1000 tosses of a coin with a probability 0.2 for an H, we will get at least 180 H's.

Problem 6

Solve problem 3 above again, using the normal approximation.

Probability 201-1-2391 ASSIGNMENT 10
Inequalities and limit laws
instructor: Ronen Peretz,
math dept, BGU

Problem 1

The number of people that arrive at the bank branch per day is distributed Poisson, $P(100)$. Give an estimate for the probability that tomorrow less than 121 people will arrive at this bank branch and more than 79. Use Cebyshev's inequality. $p(79 < X < 121) = 341/441$.

Problem 2

A balanced coin is tossed 280 times. Let X be the number of H's in the first 200 tosses, and let Y be the number of H's in the last 80 tosses. Prove that:

$$p(X \leq Y + 50) \leq \frac{7}{20}.$$

Problem 3

We want to prepare a sample in order to know what is the proportion of people with blue eyes in the total population. Determine using Chebyshev's inequality the size of the sample to be taken if we want that in probability greater than or equals to 0.9 the experimental proportion of the blue eyes people to be within 0.01 away (at most) from the actual proportion. $n \geq 25000$.

Problem 4

There are 80 stair steps in the building leading up to the roof. At each stage a man throws a dice and climbs a number of steps which equals to the number shown on the face of the dice (except maybe at the end of the stairs). He can do that at most 20 times. Give an estimate using Chebyshev's inequality to the probability that the man will make it to the roof. $P(X \geq 80) \leq 7/24$.

Problem 5

Give an estimate, using normal approximation, to the probability that in a 1000 tosses of a coin with a probability 0.2 for an H, we will get at least 180 H's. $P(Y \geq 179.5) = 1 - \Phi(-20.5/\sqrt{160}) = 0.9747$.

Problem 6

Solve problem 3 above again, using the normal approximation. $P \geq 0.95$.

Probability 201-1-2391 ASSIGNMENT 1-answers
Combinatorial Analysis
instructor: Ronen Peretz,
math dept, BGU

Problem 1

15 cubes are ordered in a row. 5 cubes are red, 7 are white and 3 are green. How many different arrangements are there?

$$\frac{15!}{5!7!3!} \text{ or } \binom{15}{5}\binom{10}{7}\binom{3}{3}$$

Problem 2

In how many ways can we arrange n people in a row so that:
a) John and Jane stand next to one another?

$$2(n-1)!$$

b) John, Jane and Mary stand together?

$$3!(n-2)!$$

Problem 3

How many different 4 digits numbers can we produce from the following leggo units: $8, 8, 6, 5, 3, 2$?

$$\binom{4}{2}\cdot 4\cdot 3+5!$$

Problem 4

In how many ways can a group of $2n$ people be divided into n pairs?

$$\frac{(2n)!}{n!2^n} \text{ or } \frac{1}{n!}\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}$$

Problem 5

A box contains 50 bulbs, 15 of which are defected. A person pulls out randomly 3 bulbs from the box. How many different ways are there in which that person will have at least one non-defected bulb?

$$\binom{50}{3} - \binom{15}{3} \text{ or } \binom{35}{3}\binom{15}{0} + \binom{35}{2}\binom{15}{1} + \binom{35}{1}\binom{15}{2}$$

Problem 6

A person gets 13 cards from a deck of cards. How many different possibilities he has, so that his 13 cards are: 5 hearts, 4 diamonds, 2 spades and 2 leaves?

$$\binom{13}{5}\binom{13}{4}\binom{13}{2}\binom{13}{2}$$

Problem 7

In a lost ship there are 20 children. Those children do not remember their birthdays and would like to be assigned birthdays.
a) In how many ways this can be done so that exactly 2 children will get identical day, and the other 18 will get each a separate day?

$$\binom{20}{2} \cdot \frac{365!}{346!}$$

b) What is the number of possibilities to assign, so that at least one day of the year will belong to at least 2 children?

$$365^{20} - \frac{365!}{345!}$$

Problem 8

a) 12 children want to march in a row. In how many ways this can be done so that between John and Jeff there will be exactly 5 children?

$$6 \cdot 2 \cdot 10!$$

b) The same question, but this time the children want to dance in a circle?

10!

2

Problem 9

a) How many different "words" could be created using all the characters of ABRACADABRA?

$$\frac{11!}{5!2!2!} \text{ or } \binom{11}{5}\binom{6}{2}\binom{4}{2}\binom{2}{1}\binom{1}{1}$$

b) In how many of these there are no 2 A's one next to the other?

$$\frac{6!}{2!2!} \cdot \binom{7}{5}$$

c) In how many of the "words" in a) there are no 2 identical characters one next to the other?

$$\frac{6!}{2!2!} \cdot \binom{7}{5} - \frac{5!}{2!}\binom{6}{5} - \frac{5!}{2!}\binom{6}{5} + 4! \cdot \binom{5}{5}$$

Problem 10

How many different solutions are there for the equation $X_1 + X_2 + \ldots + X_k = n$, where:

a) For each i, $X_i \geq 0$ is integral.

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}$$

b) For each i, $X_i = 1$ or $X_i = 2$ ($k \leq n$).

$$\binom{k}{n-k}$$

c) For each i, $X_i \geq 0$ is integral and there are exactly 4 X_i's which equal 0?

$$\binom{k}{4}\binom{n-1}{n-k+4}$$

Problem 11

Prove the identity:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Problem 12

Prove the identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Problem 13

Prove the identity:

$$\binom{n}{n} + \binom{n+1}{n} + \ldots + \binom{N}{n} = \binom{N+1}{n+1}, \quad (N \geq n).$$

Problem 14

Prove the identity:

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}.$$

Proof. The left hand side is the total number of ways to choose a committee of k out of a group of $n+m$ people. This is also the number on the right hand side, for if we split our group of people into two sub-groups, one of size n and a second of size m then

$$\binom{n}{i} \binom{m}{k-i}$$

is the number of committees of size k, which have exactly i members from the first sub-group.

Problem 15

Prove the identity:

$$\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \ldots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

Problem 16

n identical balls are distributed in m different cells.
a) In how many of these distributions there will be exactly k empty cells?

$$\binom{m}{k} \cdot \binom{n-1}{n-m+k}$$

b) Prove the following identity:

$$\sum_{k=0}^{m-1} \binom{m}{k} \binom{n-1}{n-m+k} = \binom{n+m-1}{n}.$$

Probability 201-1-2391 ASSIGNMENT 2
The elementary probability space
instructor: Ronen Peretz,
math dept, BGU

Problem 1

a) Prove the following distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

b) Prove that complementing a set is an involution: $(A^c)^c = A$. Here $A^c = X - A$ and $A \subseteq X$.
c) Prove that: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Problem 2

Which is the greatest? :
a) The probability to obtain at least one 6 in a throw of 4 dice.
b) The probability to obtain at least once $(6, 6)$ in 24 throws of 2 dice (in each throw).

Problem 3

Here is a solution to Problem 2 above: When we throw one dice there are 6 possible outcomes of which only one is a "success" (getting 6). When we throw 2 (a pair) dice there are 36 possible outcomes of which only one is a "success" (getting $(6, 6)$). So for the probability of success to be equal in the two spaces the number of throws of 2 dice should be 6 times as large as that of throwing just one, i.e. $4 \times 6 = 24$. Hence the answer to problem 2 is that the 2 probabilities are equal.
Discuss this argument and express your opinion.

Problem 4

a) A person has 2 match boxes, each containing n matches. When he needs a match he pulls one randomly out of one of his 2 boxes. When he finds out that one of his boxes in empty, what is the probability that his other box contains exactly k matches? Here $0 \leq k \leq n$.
b) Prove the identity:

$$\sum_{k=0}^{n} \frac{1}{2^{2n-k}} \binom{2n-k}{n} = 1.$$

Problem 5

k married couples (pairs) arrived at a party. An even number m of the $2k$ people were chosen at random. Compute the probability that:
a) Among the chosen ones there is no married couple.
b) Among the chosen ones there is exactly one married couple.
c) Among the chosen ones there are exactly 4 married couples, assuming of course that $m \geq 8$.
d) All the chosen ones are $m/2$ married couples.

Problem 6

5 dice are thrown. What is the probability that at least two will show identical results?

Problem 7

What is the probability that in distributing n (distinguishable) balls in m cells, cell number 1 will contain exactly k of the balls?

Problem 8

4 balls are randomly distributed into 3 cells (3^4 possibilities of equal probability). We define the events A to G below:
A - cell number 1 is empty, B - only cell number 1 is empty, C - there is exactly one empty cell, D - there is an empty cell, E - there is a non-empty cell, F - cell number 3 contains 2 balls, G - there is a cell that contains exactly 2 balls.
Compute the probabilities of these events, $p_A, p_B, p_C, p_D, p_E, p_F$ and p_G.

Problem 9

A jar contains 10 balls numbered: $1, 2, 3, \ldots, 10$. 5 balls are pulled out randomly. What is the probability that ball number 7 is the second in size out of the balls that were pulled out?

Problem 10

k dice are thrown. Compute the probabilities of the following two events:
A - the largest number that occurred is i,
B - the smallest number that occurred is i.

Problem 11

4 married couples arrived at a party. The 8 people were divided randomly into pairs.
a) What is the probability of the event in which each man found himself with his wife?
b) What is the probability that each pair contains a man and a woman?

Problem 12

n married couples sit randomly around a round table. What is the probability that no woman sits besides her husband?

Problem 13

Out of a deck of cards we randomly choose k cards, $k \geq 13$. What is the probability that each number is represented in the sample?

Problem 14

A dice is thrown 36 times. What is the probability to get each number 6 times?

Probability 201-1-2391 ASSIGNMENT 2 - answers
The elementary probability space
instructor: Ronen Peretz,
math dept, BGU

Problem 1

a) Prove the following distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

b) Prove that complementing a set is an involution: $(A^c)^c = A$. Here $A^c = X - A$ and $A \subseteq X$.
c) Prove that: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Problem 2

Which is the greatest? :
a) The probability to obtain at least one 6 in a throw of 4 dice.

$$p_a = 1 - \left(\frac{5}{6}\right)^4$$

b) The probability to obtain at least once $(6, 6)$ in 24 throws of 2 dice (in each throw).

$$p_b = 1 - \frac{35^{24}}{36^{24}}$$

so $p_b < p_a$

Problem 3

Here is a solution to Problem 2 above: When we throw one dice there are 6 possible outcomes of which only one is a "success" (getting 6). When we throw 2 (a pair) dice there are 36 possible outcomes of which only one is a "success" (getting $(6, 6)$). So for the probability of success to be equal in the two spaces the number of throws of 2 dice should be 6 times as large as that of throwing just one, i.e. $4 \times 6 = 24$. Hence the answer to problem 2 is that the 2 probabilities are equal.
Discuss this argument and express your opinion.

Problem 4

a) A person has 2 match boxes, each containing n matches. When he needs a match he pulls one randomly out of one of his 2 boxes. When he finds out that one of his boxes in empty, what is the probability that his other box contains exactly k matches? Here $0 \leq k \leq n$.

$$2 \cdot \binom{2n-k}{n} \cdot \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{n-k} \cdot \frac{1}{2} = \frac{1}{2^{2n-k}} \cdot \binom{2n-k}{n}$$

b) Prove the identity:

$$\sum_{k=0}^{n} \frac{1}{2^{2n-k}} \binom{2n-k}{n} = 1.$$

Problem 5

k married couples (pairs) arrived at a party. An even number m of the $2k$ people were chosen at random. Compute the probability that:
a) Among the chosen ones there is no married couple.

$$\left[\binom{k}{m} 2^m\right] / \binom{2k}{m}$$

b) Among the chosen ones there is exactly one married couple.

$$\left[k \binom{k-1}{m-2} 2^{m-2}\right] / \binom{2k}{m}$$

c) Among the chosen ones there are exactly 4 married couples, assuming of course that $m \geq 8$.

$$\left[\binom{k}{4} \binom{k-4}{m-8} 2^{m-8}\right] / \binom{2k}{m}$$

d) All the chosen ones are $m/2$ married couples.

$$\binom{k}{m/2} / \binom{2k}{m}$$

Problem 6

5 dice are thrown. What is the probability that at least two will show identical results?

$$1 - \frac{6!}{6^5}$$

Problem 7

What is the probability that in distributing n (distinguishable) balls in m cells, cell number 1 will contain exactly k of the balls?

$$\left[\binom{n}{k} \cdot (m-1)^{n-k}\right] / m^n$$

Problem 8

4 balls are randomly distributed into 3 cells (3^4 possibilities of equal probability). We define the events A to G below:
A - cell number 1 is empty, B - only cell number 1 is empty, C - there is exactly one empty cell, D - there is an empty cell, E - there is a non-empty cell, F - cell number 3 contains 2 balls, G - there is a cell that contains exactly 2 balls.
Compute the probabilities of these events, $p_A, p_B, p_C, p_D, p_E, p_F$ and p_G.

$$p_A = \frac{2^4}{3^4} = \frac{16}{81},\ p_B = \frac{2^4-2}{3^4} = \frac{14}{81},\ p_C = 3p_B = \frac{42}{81},\ p_D = \frac{5}{9},\ p_E = 1,$$

$$p_F = \left[\binom{4}{2} \cdot 2^2\right] / 3^4 = \frac{8}{27},\ p_G = \left[3\binom{4}{2} 2!\right] / 81 + \left[\binom{3}{2}\binom{4}{2}\right] / 81 = \frac{2}{3}$$

Problem 9

A jar contains 10 balls numbered: $1, 2, 3, \ldots, 10$. 5 balls are pulled out randomly. What is the probability that ball number 7 is the second in size out of the balls that were pulled out?

$$\left[3 \cdot \binom{6}{3}\right] / \binom{10}{5}$$

Problem 10

k dice are thrown. Compute the probabilities of the following two events:
A - the largest number that occurred is i,

$$\frac{i^k}{6^k} - \frac{(i-1)^k}{6^k}$$

B - the smallest number that occurred is i.

$$\frac{(6-i+1)^k}{6^k} - \frac{(6-i)^k}{6^k}$$

Problem 11

4 married couples arrived at a party. The 8 people were divided randomly into pairs.
a) What is the probability of the event in which each man found himself with his wife?

$$\frac{4!2^4}{8!}$$

b) What is the probability that each pair contains a man and a woman?

$$\frac{(4!)^2 2^4}{8!}$$

Problem 12

n married couples sit randomly around a round table. What is the probability that no woman sits besides her husband?

$$1 + \sum_{i=1}^{n} (-1)^i \binom{n}{i} \frac{(2n-i-1)!}{(2n-1)!} \cdot 2^i$$

Problem 13

Out of a deck of cards we randomly choose k cards, $k \geq 13$. What is the probability that each number is represented in the sample?

$$1 - \sum_{i=1}^{13} (-1)^{i-1} \binom{13}{i} \binom{52-4i}{k} / \binom{52}{k}$$

The following "answer" is **wrong**

$$\left[4^{13} \binom{52-13}{k-13} \right] / \binom{52}{k}$$

Problem 14

A dice is thrown 36 times. What is the probability to get each number 6 times?

Probability 201-1-2391 ASSIGNMENT 3
The Borel system
instructor: Ronen Peretz,
math dept, BGU

Problem 1

We defined a field of events β as a family $\beta \subseteq 2^{\Omega}$ that satisfies the following 3 axioms:
I. $\Omega \subseteq \beta$.
II. If $A, B \in \beta$ then $A - B \in \beta$.
III. If $A_1, A_2, \ldots \in \beta$ is countable then the union $\bigcup_{n=1}^{\infty} A_n \in \beta$

Let us state the following axiom:
II'. If $A \in \beta$, then $A^c \in \beta$ where A^c is the complimentary set of A.
Prove that the three axioms I,II,III are equivalent to I,II',III.

Problem 2

Using the definition we gave to a Borel system (Ω, β, p), prove (using the axioms):
1. $p(\emptyset) = 0$.
2. If $A, B \in \beta$ satisfy $A \subseteq B$ then $p(A) \leq p(B)$.

Problem 3

Give an example of a probability space (Ω, β, p) such there are $A, B \in \beta$, $A \neq \emptyset$ and $B \neq \Omega$, but $p(A) = 0$ and $p(B) = 1$.

Problem 4

Give an example of a probability space (Ω, β, p) such that Ω is an infinite sample space but $|\beta| < \infty$, or prove that no such a probability space can exist.

Problem 5

Give an example of a probability space (Ω, β, p) such that $|\beta| = 3$ or prove that no such a probability space can exist.

Problem 6

Give an example of a probability space (Ω, β, p) such that $|\beta| = 4$ or prove that no such a probability space can exist.

Problem 7

Give an example of a probability space (Ω, β, p) such that $|\beta|$ is an odd integer or prove that no such a probability space can exist.

Probability 201-1-2391 ASSIGNMENT 3-answers
The Borel system
instructor: Ronen Peretz,
math dept, BGU

Problem 1

We defined a field of events β as a family $\beta \subseteq 2^{\Omega}$ that satisfies the following 3 axioms:
I. $\Omega \subseteq \beta$.
II. If $A, B \in \beta$ then $A - B \in \beta$.
III. If $A_1, A_2, \ldots \in \beta$ is countable then the union $\bigcup_{n=1}^{\infty} A_n \in \beta$

Let us state the following axiom:
II'. If $A \in \beta$, then $A^c \in \beta$ where A^c is the complimentary set of A.
Prove that the three axioms I,II,III are equivalent to I,II',III.

Axioms I,II,III imply axiom II': Let $A \in \beta$. Take $B = \Omega$. By I $\Omega \in \beta$. By II $B - A \in \beta$. But $B - A = A^c$.
Axioms I,II',III imply axiom II: Let $A, B \in \beta$. Then by II' $A^c \in \beta$. So by III $A^c \cup B \in \beta$. So $(A^c \cup B)^c \in \beta$. But $A - B = (A^c \cup B)^c$ and we finished.

Problem 2

Using the definition we gave to a Borel system (Ω, β, p), prove (using the axioms):
1. $p(\emptyset) = 0$.

By axiom III of p we have $p(\emptyset \cup \Omega) = p(\emptyset) + p(\Omega)$. So $p(\Omega) = p(\emptyset) + p(\Omega)$ and hence $p(\emptyset) = 0$.
2. If $A, B \in \beta$ satisfy $A \subseteq B$ then $p(A) \leq p(B)$.

By axiom III of p we have $p(B) = p(B - A) + P(A)$ because $A \subseteq B$. By axiom II $p(B - A) \geq 0$ and so $p(B) \geq p(A)$.

Problem 3

Give an example of a probability space (Ω, β, p) such there are $A, B \in \beta$, $A \neq \emptyset$ and $B \neq \Omega$, but $p(A) = 0$ and $p(B) = 1$.

Take $\Omega = [0, 1]$ the real numbers between 0 and 1, and β the field of events generated by the closed sub-intervals in $[0, 1]$. Also $p([a, b]) = b - a$. Then we can take $A = \{0\}$ and $B = \Omega - A$.

Problem 4

Give an example of a probability space (Ω, β, p) such that Ω is an infinite sample space but $|\beta| < \infty$, <u>or</u> prove that no such a probability space can exist.

Take Ω any infinite set and $\beta = \{\Omega, \emptyset\}$, with $p(\Omega) = 1$ and $p(\emptyset) = 0$.

Problem 5

Give an example of a probability space (Ω, β, p) such that $|\beta| = 3$ <u>or</u> prove that no such a probability space can exist.

No such a space exists. Because if $\beta = \{\Omega, \emptyset, A\}$ then also $A^c \in \beta$ and of course $A^c \notin \beta$.

Problem 6

Give an example of a probability space (Ω, β, p) such that $|\beta| = 4$ <u>or</u> prove that no such a probability space can exist.

Take $\Omega = \{1, 2\}$ the elementary probability space.

Problem 7

Give an example of a probability space (Ω, β, p) such that $|\beta|$ is an odd integer <u>or</u> prove that no such a probability space can exist.

Probability 201-1-2391 ASSIGNMENT 4
Conditional probability, independence
instructor: Ronen Peretz,
math dept, BGU

Problem 1

Give an example of two independent events.

Problem 2

Give an example of:
a) Three events that are pairwise independent but NOT independent.
b) Three events that satisfy $p(A \cap B \cap C) = p(A) \cdot p(B) \cdot p(C)$, but are NOT independent.

Problem 3

In a certain country there are equal numbers of Men and of women. 20 percent of the men are color blind and 2 percent of the women are color blind.
a) What is the probability that a person chosen at random in this country is color blind?
b) A person was randomly chosen and was found to be color blind. What is the probability that the person is a man?

Problem 4

We randomly distribute 7 balls into 7 cells.
a) What is the probability that there are exactly 2 empty cells and there is a cell containing exactly 3 balls?
b) What is the probability that there are 2 empty cells?
c) What is the probability that there is a cell containing 3 balls given that there are 2 empty cells?

Problem 5

We toss 5 times a balanced coin. Each time: if we get H we put into a jar a white ball, otherwise we put a black ball.
a) What is the probability that the jar contains only white balls?
b) Answer the question in a, given that we pulled out of the jar 5 balls (returning the pulled out ball each time), and found out that only white balls came.

Problem 6

A basketball player tries to shoot the ball 5 times from a long distance. The probability for a hit is $1/5$. Assuming independence of events:
a) What is the probability to hit at least twice?
b) The same question, given that the player already hit once.

Problem 7

There are N children in a family. The probability for a child to be a son is $1/2$. We define A - the family has boys and girls. B - the family has at most one girl.For which values of N the events A and B are independent?

Problem 8

A person is about to try his luck by gambling 3 times. At the front of him there are 2 machines. The probability to win on machine A is $p_A = 0.6$ and that on machine B is $p_B = 0.3$. The first time he chooses a machine randomly. At every stage (after the first) if he wins, he will try the same machine again. Otherwise he switches to the other machine.
a) What is the probability that on the second time he will use machine A?
b) The same question, given that on the third time machine A was used?

Problem 9

A jar contains 3 red balls, 4 white balls and 2 black balls. Balls are being pulled out and returned back to the jar forever. What is the probability that the first red ball that was pulled out came out before the first white ball that was pulled out?

Problem 10

There are 3 types of pine trees in the forest, A, B and C. Because of a disease 40 percent of the trees in the forest were damaged. 1/3 of type A, 1/2 of type B, 1/4 of type C. 1/4 of the damaged trees are pines of type A. What is the percentage of pines of type B in the forest?

Problem 11

Two jars contain (each) black and white balls.The probability to pull out a white ball from jar A is p, and the probability to pull out a white ball from jar B is q. Balls are pulled with placing them back into the jars A ans B. At each stage, if the ball pulled out is white, the next ball will be pulled out of the same jar. Otherwise we switch to the other jar. The first ball is pulled out of jar A.

a) Let $\alpha(n)$ be the probability that the n'th ball is pulled out of jar A. Show that for $n \geq 2$ we have:

$$\alpha(n) = \alpha(n-1) \cdot (p+q-1) + (1-q).$$

b) Prove that for $n \geq 1$ we have:

$$\alpha(n) = \frac{1-q}{2-p-q} + \frac{1-p}{2-p-q} \cdot (p+q-1)^{n-1}.$$

Problem 12

Two candidates A and B raced each other in an election campaign. Candidate A accumulated a votes and candidate B accumulated b votes. Assuming $a > b$ what is the probability that candidate A was leading all the way?

Problem 13

Prove that for any two events A, B we have: $R(A, B) = -R(A^c, B) = R(A^c, B^c)$.

Problem 14

A jar contains $a$vwhite balls and b black balls. Two players pull out the balls in turn (one by one). The first who pulls out a white ball wins the game. What is the probability that the first player wins the game?

Probability 201-1-2391 ASSIGNMENT 4-answers
Conditional probability, independence
instructor: Ronen Peretz,
math dept, BGU

Problem 1

Give an example of two independent events.

From a deck of cards we pull out a card. Define A-the card is a heart. B- the card is a King or a Queen or an Ace.

Problem 2

Give an example of:
a) Three events that are pairwise independent but NOT independent.
b) Three events that satisfy $p(A \cap B \cap C) = p(A) \cdot p(B) \cdot p(C)$, but are NOT independent.

In throwing two dice let A-the first one gives a number ≤ 3, B-the first one shows one of $3, 4$ or 5, C-the sum of the two is 9.

Problem 3

In a certain country there are equal numbers of Men and of women. 20 percent of the men are color blind and 2 percent of the women are color blind.
a) What is the probability that a person chosen at random in this country is color blind?

$0.2 \cdot 1/2 + 0.02 \cdot 1/2 = 0.11$
b) A person was randomly chosen and was found to be color blind. What is the probability that the person is a man?

$$\frac{0.2 \cdot 1/2}{0.11} = \frac{10}{11}.$$

Problem 4

We randomly distribute 7 balls into 7 cells.
a) What is the probability that there are exactly 2 empty cells and there is a cell containing exactly 3 balls?

$$\frac{7!}{2!1!4!}\binom{7}{3}4!/7^7.$$

b) What is the probability that there are 2 empty cells?

$$\left\{\frac{7!}{2!1!4!}\binom{7}{3}4!+\frac{7!}{2!2!3!1!1!}\binom{7}{2}\binom{5}{2}3!\right\}/7^7.$$

c) What is the probability that there is a cell containing 3 balls given that there are 2 empty cells? 1/4.

Problem 5

We toss 5 times a balanced coin. Each time: if we get H we put into a jar a white ball, otherwise we put a black ball.
a) What is the probability that the jar contains only white balls? $(1/2)^5$.
b) Answer the question in a, given that we pulled out of the jar 5 balls (returning the pulled out ball each time), and found out that only white balls came.

$$(1/32)/\left\{\sum_{i=0}^{5}\left(\frac{i}{5}\right)^5\binom{5}{i}\left(\frac{1}{2}\right)^5\right\}=\frac{3125}{11000}.$$

Problem 6

A basketball player tries to shoot the ball 3 times from a long distance. The probability for a hit is 1/5. Assuming independence of events:
a) What is the probability to hit at least twice?

$$\binom{3}{2}\left(\frac{1}{5}\right)^2\frac{4}{5}+\left(\frac{1}{5}\right)^3=\frac{13}{125}.$$

b) The same question, given that the player already hit once.

$$\frac{13/125}{1-(4/5)^3}=\frac{13}{16}.$$

Problem 7

There are N children in a family. The probability for a child to be a son is 1/2. We define A - the family has boys and girls. B - the family has at most one girl.For which values of N the events A and B are independent?

A and B are independent for $N = 0, 1, 3$ and are not independent for all other N.

Problem 8

A person is about to try his luck by gambling 3 times. At the front of him there are 2 machines. The probability to win on machine A is $p_A = 0.6$ and that on machine

B is $p_B = 0.3$. The first time he chooses a machine randomly. At every stage (after the first) if he wins, he will try the same machine again. Otherwise he switches to the other machine.
a) What is the probability that on the second time he will use machine A?

$$\frac{1}{2} \cdot 0.6 + \frac{1}{2}(1 - 0.3) = 0.65.$$

b) The same question, given that on the third time machine A was used?

$$\frac{0.6 \cdot 0.65}{0.635} = \frac{39}{635},$$

where we computed before $0.6 \cdot 0.65 + (1 - 0.3)(1 - 0.65) = 0.635$.

Problem 9

A jar contains 3 red balls, 4 white balls and 2 black balls. Balls are being pulled out and returned back to the jar forever. What is the probability that the first red ball that was pulled out came out before the first white ball that was pulled out?

$$\frac{3}{7}.$$

Problem 10

There are 3 types of pine trees in the forest, A, B and C. Because of a disease 40 percent of the trees in the forest were damaged. 1/3 of type A, 1/2 of type B, 1/4 of type C. 1/4 of the damaged trees are pines of type A. What is the percentage of pines of type B in the forest?

50 percents are of type B.

Problem 11

Two jars contain (each) black and white balls.The probability to pull out a white ball from jar A is p, and the probability to pull out a white ball from jar B is q. Balls are pulled with placing them back into the jars A ans B. At each stage, if the ball pulled out is white, the next ball will be pulled out of the same jar. Otherwise we switch to the other jar. The first ball is pulled out of jar A.

a) Let $\alpha(n)$ be the probability that the n'th ball is pulled out of jar A. Show that for $n \geq 2$ we have:

$$\alpha(n) = \alpha(n-1) \cdot (p+q-1) + (1-q).$$

b) Prove that for $n \geq 1$ we have:

$$\alpha(n) = \frac{1-q}{2-p-q} + \frac{1-p}{2-p-q} \cdot (p+q-1)^{n-1}.$$

Problem 12

Two candidates A and B raced each other in an election campaign. Candidate A accumulated a votes and candidate B accumulated b votes. Assuming $a > b$ what is the probability that candidate A was leading all the way?

$$\frac{a-b}{a+b}.$$

Problem 13

Prove that for any two events A, B we have: $R(A,B) = -R(A^c,B) = R(A^c,B^c)$.

Problem 14

A jar contains $a$vwhite balls and b black balls. Two players pull out the balls in turn (one by one). The first who pulls out a white ball wins the game. What is the probability that the first player wins the game?

$$\frac{a}{a+b} + \left(\frac{b}{a+b}\right)\left(\frac{b-1}{a+b-1}\right)\left(\frac{a}{a+b-2}\right) +$$

$$+\left(\frac{b}{a+b}\right)\left(\frac{b-1}{a+b-1}\right)\left(\frac{b-2}{a+b-2}\right)\left(\frac{b-3}{a+b-3}\right)\left(\frac{a}{a+b-4}\right) + \ldots =$$

$$= \left(\frac{a}{a+b}\right)\left(1 + \sum_{k=2}^{b} \binom{b}{k} \Big/ \binom{a+b-1}{k}\right).$$

Probability 201-1-2391 ASSIGNMENT 5-answers
Asymptotic laws: The theorems of DeMoivre, Laplace, Bernoulli and Poisson
instructor: Ronen Peretz,
math dept, BGU

Problem 1

Give an estimate, using normal approximation, to the probability that in a 1000 tosses of a coin with a probability 0.2 for an H, we will get at least 180 H's.

$$p(N(0,1) \geq \frac{179.5 - 200}{\sqrt{160}}) = 1 - \Phi(\frac{-20.5}{\sqrt{160}}) = 0.9747.$$

Problem 2

A balanced coin is tossed 300 times. Compute the probability that it will show 140 times 'H':
(a) Using the Stirling asymptotic formula, and

$$p = \binom{300}{140}\left(\frac{1}{2}\right)^{300} \approx \frac{300^{300}}{140^{140} \cdot 160^{160}}\sqrt{\frac{300}{2\pi \cdot 140 \cdot 160}}\left(\frac{1}{2}\right)^{300}$$

(b) Assuming the normal distribution.
What is the relative deviation between the two methods?

$$\text{By DeMoivre } p \approx \frac{1}{\sqrt{75}}\phi\left(\frac{140 - 150}{\sqrt{75}}\right) = \frac{1}{\sqrt{75}\sqrt{2\pi}}e^{-(-10/\sqrt{75})^2/2}.$$

Problem 3

We do 15000 independent trials with the probability 1/3 for a success. Compute the probability that the number of successes is between 4950 and 5050.

$$n = 15000,\ p = \frac{1}{3},\ q = \frac{2}{3},\ \mu = np = 5000,\ \sigma = \sqrt{npq} = \frac{100}{\sqrt{3}}$$

$$a = 4950,\ b = 5050,\ X_a = \frac{4950 - 5000}{100/\sqrt{3}},\ X_b = \frac{5050 - 5000}{100/\sqrt{3}}$$

So, answer (using Laplace Theorem): $p(4950 \leq m \leq 5050) \approx \Phi(X_b) - \Phi(X_a)$.

Problem 4

A balanced die is thrown 24000 times one after the other. The face 5 appeared in m of these throws. Estimate the probability of the event $3900 \leq m \leq 4050$.
Use Laplace as in Problem 3.

Problem 5

Find the smallest natural number x that satisfies the condition: If in k out of 10000 deliveries male babies were born then

$$P(5000 - x \leq k \leq 5000 + x) \geq 0.95.$$

$$n = 10000,\ p = q = \frac{1}{2},\ \mu = np = 5000,\ \sigma = \sqrt{npq} = 50.$$

$$\text{By Laplace}: p(5000 - x \leq k \leq 5000 + x) \approx \Phi(X_{5000+x}) - \Phi(X_{5000-x}).$$

$$\text{We have}: X_{5000-x} = \frac{(5000 - x) - 5000}{50} = -\frac{x}{50},\ X_{5000+x} = \frac{x}{50}.$$

$$\text{SO}: 2\Phi\left(\frac{x}{50}\right) - 1 = 0.95,\ \text{hence}: x = 50 \times \Phi^{-1}\left(\frac{1.95}{2}\right)$$

Problem 6

Find the probability that in a group of 600 people there will be m birthdays on the New Years day.

Problem 7

In a certain typing process the probability for a typo is 0.001. Each line contains 50 characters. Compute the probability for two typos to occur
(a) Using the exact distribution.
(b) Using Poisson's approximation.
What is the relative deviation between the two results?

Problem 8

There is 1 percent of sick people in a population. Estimate the probability that among 200 of the people we will find at least 4 sick.

Problem 9

A certain population has 0.5 percents color blinds. We choose out of the population n people in a blind sampling (a person might be chosen more than once). Find the minimal value of n if the probability to find a color blind man in this sample is at least 0.95.

Probability 201-1-2391 ASSIGNMENT 5
Asymptotic laws: The theorems of DeMoivre, Laplace, Bernoulli and Poisson
instructor: Ronen Peretz,
math dept, BGU

Problem 1

Give an estimate, using normal approximation, to the probability that in a 1000 tosses of a coin with a probability 0.2 for an H, we will get at least 180 H's.

Problem 2

A balanced coin is tossed 300 times. Compute the probability that it will show 140 times 'H':
(a) Using the Stirling asymptotic formula, and
(b) Assuming the normal distribution.
What is the relative deviation between the two methods?

Problem 3

We do 15000 independent trials with the probability 1/3 for a success. Compute the probability that the number of successes is between 4950 and 5050.

Problem 4

A balanced die is thrown 24000 times one after the other. The face 5 appeared in m of these throws. Estimate the probability of the event $3900 \leq m \leq 4050$.

Problem 5

Find the smallest natural number x that satisfies the condition: If in k out of 10000 deliveries male babies were born then

$$P(5000 - x \leq k \leq 5000 + x) \geq 0.95.$$

Problem 6

Find the probability that in a group of 600 people there will be m birthdays on the New Years day.

Problem 7

In a certain typing process the probability for a typo is 0.001. Each line contains 50 characters. Compute the probability for two typos to occur
(a) Using the exact distribution.
(b) Using Poisson's approximation.
What is the relative deviation between the two results?

Problem 8

There is 1 percent of sick people in a population. Estimate the probability that among 200 of the people we will find at least 4 sick.

Problem 9

A certain population has 0.5 percents color blinds. We choose out of the population n people in a blind sampling (a person might be chosen more than once). Find the minimal value of n if the probability to find a color blind man in this sample is at least 0.95.

Probability 201-1-2391 ASSIGNMENT 6
One dimensional distributions and random variables
instructor: Ronen Peretz,
math dept, BGU

Problem 1

A balanced dice is thrown twice. Let X denote the maximal of the two results. Find the distribution function of X.

Problem 2

A balanced dice is thrown 5 times. Let X be the number of times that a number smaller than 3 had shown up.
a) Find the distribution function of X.
b) Find the distribution function of $Y = 2X - 1$.

Problem 3

An athlete gets 5 opportunities to jump over the bar. If he succeeds on a certain jump he does not try again. If he fails all the 5 chances then he is out. A certain jumper tries to jump over 2.15 meters and it is known that the probability that he passes the bar is r. Assuming that the jumps are independent, find the distribution of X - the number of jumps that he will do.

Problem 4

A jar contains 15 black balls and 30 white balls. The balls are randomly being pulled out of the jar till the first white ball is pulled out. Let X be the number of the balls pulled out. Compute the distribution of X.

Problem 5

Same as in Problem 4 above, but balls are being pulled out till the second white ball is pulled out.

Problem 6

Let X be a random variable that assumes all the non-negative integral values. We are given that for any positive integer: $k \cdot p(X = k) = 10 \cdot p(X = k - 1)$. Find the distribution of X.

Problem 7

A jar contains a white balls and b black balls. Two opponents pull out (with returning the balls back) balls one by one. The first who pulls out a white ball, wins.
a) What is the probability that the first player wins?
b) What is the distribution of the number of pull outs till the determination of the winner (inclusive)?

Problem 8

A robot is located at point 0 on the real axis. Then it forms N steps where in each step (which is independent of the others) it moves rightwards once with probability p, and leftwards once with probability $q = 1 - p$. Let X be the number on the real axis at which the robot arrives after N steps. What is the distribution of X?

Problem 9

There are 20 notes in a hat that are numbered with the numbers $1, 2, 3, \ldots, 20$. One number is written on each note.You pull out the notes one by one till you reach 13. Let X be the number of notes pulled out. What is the distribution of X?

Probability 201-1-2391 ASSIGNMENT 6 -answers
One dimensional distributions and random variables
instructor: Ronen Peretz,
math dept, BGU

Problem 1

A balanced dice is thrown twice. Let X denote the maximal of the two results. Find the distribution function of X.

$$F(k) = \begin{cases} 0 & , \quad k < 1 \\ (k/6)^2 & , \quad 1 \le k \le 6 \\ 1 & , \quad 6 < k \end{cases}$$

Problem 2

A balanced dice is thrown 5 times. Let X be the number of times that a number smaller than 3 had shown up.
a) Find the distribution function of X.

$$p_5(m) = \binom{5}{m} \left(\frac{1}{3}\right)^m \left(\frac{2}{3}\right)^{5-m}, \quad 0 \le m \le 5.$$

b) Find the distribution function of $Y = 2X - 1$.

$$p(Y = k) = \binom{5}{(k+1)/2} \left(\frac{1}{3}\right)^{(k+1)/2} \left(\frac{2}{3}\right)^{(9-k)/2}$$

Problem 3

An athlete gets 5 opportunities to jump over the bar. If he succeeds on a certain jump he does not try again. If he fails all the 5 chances then he is out. A certain jumper tries to jump over 2.15 meters and it is known that the probability that he passes the bar is r. Assuming that the jumps are independent, find the distribution of X - the number of jumps that he will do.

$p(\text{succeeds in } j) = (1-r)^{j-1} \cdot r, \;\; j = 1, 2, 3, 4. \; p(\text{don't care about } 5) = (1-r)^4.$Else 0.

Problem 4

A jar contains 15 black balls and 30 white balls. The balls are randomly being pulled out of the jar till the first white ball is pulled out. Let X be the number of the balls pulled out. Compute the distribution of X.

$$p(X = k) = \frac{15!(45-k)! \cdot 30}{45!(16-k)!}.$$

Problem 5

Same as in Problem 4 above, but balls are being pulled out till the second white ball is pulled out.

$$p(X = k) = \left\{ \left(\begin{array}{c} 30 \\ 1 \end{array} \right) \left(\begin{array}{c} 15 \\ k-2 \end{array} \right) \cdot 29 \right\} / \left\{ \left(\begin{array}{c} 45 \\ k-1 \end{array} \right) \cdot (46-k) \right\}.$$

Problem 6

Let X be a random variable that assumes all the non-negative integral values. We are given that for any positive integer: $k \cdot p(X = k) = 10 \cdot p(X = k-1)$. Find the distribution of X.

$$p(X = k) = \frac{10^k}{k!} e^{-10}.$$

Problem 7

A jar contains a white balls and b black balls. Two opponents pull out (with returning the balls back) balls one by one. The first who pulls out a white ball, wins.
a) What is the probability that the first player wins?

$$p(\text{The first player wins}) = \frac{a+b}{a+2b}.$$

b) What is the distribution of the number of pull outs till the determination of the winner (inclusive)?

$$p(X = k) = \left(\frac{b}{a+b}\right)^{k-1} \left(\frac{a}{a+b}\right).$$

Problem 8

A robot is located at point 0 on the real axis. Then it forms N steps where in each step (which is independent of the others) it moves rightwards once with probability p, and leftwards once with probability $q = 1 - p$. Let X be the number on the real axis at which the robot arrives after N steps. What is the distribution of X?

$$p(X = k) = \binom{N}{(k+N)/2} p^{(k+N)/2} q^{(N-k)/2}.$$

Problem 9

There are 20 notes in a hat that are numbered with the numbers $1, 2, 3, \ldots, 20$. One number is written on each note.You pull out the notes one by one till you reach 13. Let X be the number of notes pulled out. What is the distribution of X?

$$p(X = k) = \frac{1}{20}.$$

Probability 201-1-2391 ASSIGNMENT 7
Tow dimensional random variables
instructor: Ronen Peretz,
math dept, BGU

Problem 1

A jar contains 2 white balls and 1 black ball. We pull out (and do NOT return back) 2 balls one by one. Let X be the number of white balls on the first pull out. Let Y be the number of white balls on the second pull out. What is the joint distribution of (X, Y) and what are the marginal distributions?

Problem 2

A balanced coin is tossed. Its faces are denoted by 0 and 1. If we get 1 then we toss again. If we get 0 we throw a dice. Let X be the result of the first toss, Y - the result of the second.
a) Find the distributions (joint and the marginals).
b) Are X and Y independent?
c) compute $p(X^2 + Y^2 \leq 9)$.

Problem 3

We have two hats, each contains 10 notes that are numbered 1 to 10. A note is randomly pulled out of each hat. Let X be the smallest of the two numbers and let Y be the largest of the two.
a) Compute the joint distribution of (X, Y).
b) Are X and Y independent?
c) Compute the conditional distribution of X, given that $Y = 2$.

Problem 4

Dan tosses a coin that has probability p for H, 25 times. Let X be the number of H's that Dan gets. Ron tosses a coin that has probability p for H, 15 times. Y is his number of H's.
a) Find the distribution of $X + Y$.
b) Prove that given $X + Y = 10$, X distributes $H(25, 15, 10)$ where $H(a, b, n)$ is the hypergeometric distribution, a is the population size being counted, b is the size of the rest of the population and n is the sample size.

Problem 5

The center of a phone company receives every minute, phone calls that are distributed Poisson, $P(\lambda)$. Every call independently of the other calls is answered with probability p. Find the distribution of the answered calls per minute.

Problem 6

A bank branch has two entrances. The front entry receives a number of clients per minute that is distributed Poisson, $P(\lambda)$. The rear entry receives a number of clients per minute that is distributed Poisson, $P(\mu)$. The two processes are independent.
a) Prove that the total number of clients that enter the bank branch per minute distributes Poisson, $P(\lambda + \mu)$.
b) Given that the total number of clients that entered the bank branch in the last minute is 10, what is the distribution of the number of clients that entered the front entry?

Problem 7

Let X, Y be independent random variables, each distributed geometrically with the parameter p, i.e. $G(p)$. Prove that the conditional distribution of X given that $X + Y = n$ is the uniform distribution $U(1, n-1)$.

Problem 8

Your TA defined X as the number of hearts in a hand of Bridge game, and Y as the number of spades. He wants to know what is the distribution of $X + Y$. So he presents the table of the joint distribution and in order to compute $p(X + Y = k)$ he sums the probabilities along diagonals of the table. The first diagonal for $k = 0$. The second for $k = 1$ etc... . What would you tell him?

Probability 201-1-2391 ASSIGNMENT 7
Tow dimensional random variables
instructor: Ronen Peretz,
math dept, BGU

Problem 1

A jar contains 2 white balls and 1 black ball. We pull out (and do NOT return back) 2 balls one by one. Let X be the number of white balls on the first pull out. Let Y be the number of white balls on the second pull out. What is the joint distribution of (X, Y) and what are the marginal distributions?

$$\begin{array}{lll} p(X\,\text{black}, Y\,\text{black}) = 0 & p(X\,\text{white}, Y\,\text{black}) = 1/3 & p(Y\,\text{black}) = 1/3 \\ p(X\,\text{black}, Y\,\text{white}) = 1/3 & p(X\,\text{white}, Y\,\text{white}) = 1/3 & p(Y\,\text{white}) = 2/3 \\ p(X\,\text{black}) = 1/3 & p(X\,\text{white}) = 2/3 & 1 \end{array}$$

Problem 2

A balanced coin is tossed. Its faces are denoted by 0 and 1. If we get 1 then we toss again. If we get 0 we throw a dice. Let X be the result of the first toss, Y - the result of the second.
a) Find the distributions (joint and the marginals).

$$\begin{array}{lll} p(X=0, Y=0) = 0 & p(X=1, Y=0) = 1/4 & p(Y=0) = 1/4 \\ p(X=0, Y=1) = 1/12 & p(X=1, Y=1) = 1/4 & p(Y=1) = 1/3 \\ p(X=0, Y=2) = 1/12 & p(X=1, Y=2) = 0 & p(Y=2) = 1/12 \\ p(X=0, Y=3) = 1/12 & p(X=1, Y=3) = 0 & p(Y=3) = 1/12 \\ p(X=0, Y=4) = 1/12 & p(X=1, Y=4) = 0 & p(Y=4) = 1/12 \\ p(X=0, Y=5) = 1/12 & p(X=1, Y=5) = 0 & p(Y=5) = 1/12 \\ p(X=0, Y=6) = 1/12 & p(X=1, Y=6) = 0 & p(Y=6) = 1/12 \\ p(X=0) = 1/2 & p(X=1) = 1/2 & 1 \end{array}$$

b) Are X and Y independent? No, they are NOT: $0 = p(X = 0, Y = 0) \neq p(X = 0) \cdot p(Y = 0) = 1/8$. c) compute $p(X^2 + Y^2 \leq 9)$. $p(X^2 + Y^2 \leq 9) = 3/4$.

Problem 3

We have two hats, each contains 10 notes that are numbered 1 to 10. A note is randomly pulled out of each hat. Let X be the smallest of the two numbers and let

Y be the largest of the two.
a) Compute the joint distribution of (X, Y).

$$p(X = k, Y = l) = \begin{cases} 0 & , \quad k > l \\ 1/100 & , \quad k = l \\ 2/100 & , \quad 1 \leq k < l \leq 10 \end{cases}$$

b) Are X and Y independent? No: $0 = p(X = 4, Y = 2) \neq p(X = 4) \cdot p(Y = 2) > 0$.
c) Compute the conditional distribution of X, given that $Y = 2$. $p(X = k | Y = 2) = p(X = k \cap Y = 2)/0.03$. For $k > 1$ it is 0.

Problem 4

Dan tosses a coin that has probability p for H, 25 times. Let X be the number of H's that Dan gets. Ron tosses a coin that has probability p for H, 15 times. Y is his number of H's.
a) Find the distribution of $X + Y$. Binomial $B(40, p)$
b) Prove that given $X + Y = 10$, X distributes $H(25, 15, 10)$ where $H(a, b, n)$ is the hypergeometric distribution, a is the population size being counted, b is the size of the rest of the population and n is the sample size.

$$\binom{25}{k} \cdot \binom{15}{10 - k} / \binom{40}{10}$$

Problem 5

The center of a phone company receives every minute, phone calls that are distributed Poisson, $P(\lambda)$. Every call independently of the other calls is answered with probability p. Find the distribution of the answered calls per minute.

$$p(\text{answered} = y) = \frac{(\lambda \cdot p)^y}{y!} e^{-\lambda \cdot p}.$$

Problem 6

A bank branch has two entrances. The front entry receives a number of clients per minute that is distributed Poisson, $P(\lambda)$. The rear entry receives a number of clients per minute that is distributed Poisson, $P(\mu)$. The two processes are independent.

a) Prove that the total number of clients that enter the bank branch per minute distributes Poisson, $P(\lambda + \mu)$.

$$p(X+Y=k) = \frac{(\lambda+\mu)^k}{k!}e^{-(\lambda+\mu)}.$$

b) Given that the total number of clients that entered the bank branch in the last minute is 10, what is the distribution of the number of clients that entered the front entry?

$$p(X=k|X+Y=10) = \left(\begin{array}{c} 10 \\ k \end{array} \right) \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{10-k}$$

Problem 7

Let X, Y be independent random variables, each distributed geometrically with the parameter p, i.e. $G(p)$. Prove that the conditional distribution of X given that $X+Y=n$ is the uniform distribution $U(1, n-1)$.

$$p(X+Y=n) = (n-1)p^2q^{n-2}, \; p(X=k|X+Y=n) = \frac{p(X=k \cap X+Y=n)}{p(X+Y=n)} =$$

$$= \frac{p(X=k)p(Y=n-k)}{(n-1)p^2q^{n-2}} = \frac{(q^{k-1}p)(q^{n-k-1}p)}{(n-1)p^2q^{n-2}} = \frac{1}{n-1}.$$

Problem 8

Your TA defined X as the number of hearts in a hand of Bridge game, and Y as the number of spades. He wants to know what is the distribution of $X+Y$. So he presents the table of the joint distribution and in order to compute $p(X+Y=k)$ he sums the probabilities along diagonals of the table. The first diagonal for $k=0$. The second for $k=1$ etc... . What would you tell him?

Probability 201-1-2391 ASSIGNMENT 8
Expectation, variance and covariance
instructor: Ronen Peretz,
math dept, BGU

Problem 1

There are 6 lottery cards in a hat. Three are marked with 0, two are marked with 20 and one is marked with 40. A gambler invests 20 Dollars in order to participate. He pulls out of the hat two cards and gets an amount that equals the average of his cards. Compute the distribution of the amount that the gambler will get, the expectation of the gain and its variance.

Problem 2

Two dice are thrown 1000 times. Let X be the number of throws in which the sum is at least 11. Compute the expectation and the variance of X.

Problem 3

In a TV game, handles of three machines can be pulled down. In machine 1 the probability to win 500 Dollars is 1/2, in machine 2 the probability to win 1000 Dollars is 0.6, and in machine 3 the probability to win 1500 Dollars is 0.7. Let X be the total win. Find $E(X)$.

Problem 4

Let X, Y be two random variables that assume (each) the values $0, 1$. Write down formulas for $E(X)$, $Var(X)$ and $E(X \cdot Y)$.

Problem 5

Out of 10 people 3 are randomly chosen, returning back (each time). Let X be the number of different people chosen in a sample. Find the expectation and the variance of X.

Problem 6

n numbers are randomly chosen from $1, 2, \ldots, N$ (without returning). Let X be the largest number chosen. For $n < N$ prove that:

$$E(X) = \frac{n}{n+1} \cdot (N+1).$$

Problem 7

n balls that are numbered $1, 2, \ldots, n$ are randomly distributed into n cells that are also numbered $1, 2, \ldots, n$. Every cell has a room for exactly 1 ball. Let X be the number of balls that were put into cells with identical numbers (i.e. the ball and it's cell have identical numbers). Compute $E(X)$ and $Var(X)$.

Problem 8

A dice is thrown n times. Let X be the number of different faces that came up. What is $E(X)$?

Problem 9

A dice is thrown again and again till all the faces show up. Let Y be the number of throws. What is $E(Y)$?

Problem 10

There are n students in the class. Out of all the

$$\binom{n}{2}$$

pairs of students we let X be the number of pairs in which the two students were born in the same day of the year. Compute $E(X)$ and $Var(X)$.

Problem 11

A balanced coin is tossed N times. Let X be the number of sequences of three adjacent H's. Compute $E(X)$ and $Var(X)$.

Problem 12

X is distributed $B(n, p)$. $Y|X$ is distributed $U(X, m)$. What is $E(Y)$?

Problem 13

A dice is thrown N times. Let X be the number of times 6 showed. Let Y be the number of times 5 showed. Compute $Cov(X, Y)$.

Problem 14

A red dice is thrown, and a blue dice is thrown. Let X be the result shown on the red dice, and let Y be the maximal result. Compute $Cov(X, Y)$.

Probability 201-1-2391 ASSIGNMENT 8
Expectation, variance and covariance
instructor: Ronen Peretz,
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Problem 1

There are 6 lottery cards in a hat. Three are marked with 0, two are marked with 20 and one is marked with 40. A gambler invests 20 Dollars in order to participate. He pulls out of the hat two cards and gets an amount that equals the average of his cards. Compute the distribution of the amount that the gambler will get, the expectation of the gain and its variance.
We define: X is the amount that the gambler gets, Y is the gain of the gambler. So $Y = -20 + X$. Then:

$$p(X = 0) = \frac{3}{15},\ P(X = 10) = \frac{6}{15},\ P(X = 20) = \frac{4}{15},\ P(X = 30) = \frac{2}{15}.$$

So:

$$E(X) = \frac{4}{3},\ E(Y) = -\frac{2}{3},\ E(X^2) = \frac{4000}{15},\ \sigma^2(Y) = \frac{800}{9}.$$

Problem 2

Two dice are thrown 1000 times. Let X be the number of throws in which the sum is at least 11. Compute the expectation and the variance of X.
$E(X) = 83\frac{1}{3}$, $\sigma^2(X) = 76\frac{7}{18}$.

Problem 3

In a TV game, handles of three machines can be pulled down. In machine 1 the probability to win 500 Dollars is 1/2, in machine 2 the probability to win 1000 Dollars is 0.6, and in machine 3 the probability to win 1500 Dollars is 0.7. Let X be the total win. Find $E(X)$.
$E(X) = 1900$.

Problem 4

Let X, Y be two random variables that assume (each) the values $0, 1$. Write down formulas for $E(X)$, $Var(X)$ and $E(X \cdot Y)$.
$E(X) = P(X = 1)$, $\sigma^2(X) = P(X = 1) \cdot P(X = 0)$, $E(X \cdot Y) = P(X = 1 \cap Y = 1)$.

Problem 5

Out of 10 people 3 are randomly chosen, returning back (each time). Let X be the number of different people chosen in a sample. Find the expectation and the variance of X.
$E(X) = 2.71$, $\sigma^2(X) = 0.2259$.

Problem 6

n numbers are randomly chosen from $1, 2, \ldots, N$ (without returning). Let X be the largest number chosen. For $n < N$ prove that:

$$E(X) = \frac{n}{n+1} \cdot (N + 1).$$

Problem 7

n balls that are numbered $1, 2, \ldots, n$ are randomly distributed into n cells that are also numbered $1, 2, \ldots, n$. Every cell has a room for exactly 1 ball. Let X be the number of balls that were put into cells with identical numbers (i.e. the ball and it's cell have identical numbers). Compute $E(X)$ and $Var(X)$.
$E(X) = 1$, $\sigma^2(X) = 1$.

Problem 8

A dice is thrown n times. Let X be the number of different faces that came up. What is $E(X)$?

$$E(X) = 6 - \frac{5^n}{6^{n-1}}.$$

Problem 9

A dice is thrown again and again till all the faces show up. Let Y be the number of throws. What is $E(Y)$?
$E(Y) = 14.7$.

Problem 10

There are n students in the class. Out of all the

$$\binom{n}{2}$$

pairs of students we let X be the number of pairs in which the two students were born in the same day of the year. Compute $E(X)$ and $Var(X)$.

$$E(X) = \binom{n}{2} \cdot \frac{1}{365}, \ \sigma^2(X) = \binom{n}{2} \cdot \frac{364}{365^2}.$$

Problem 11

A balanced coin is tossed N times. Let X be the number of sequences of three adjacent H's. Compute $E(X)$ and $Var(X)$.

$$E(X) = \frac{N-2}{8}, \ \sigma^2(X) = \frac{15N-40}{64}.$$

Problem 12

X is distributed $B(n,p)$. $Y|X$ is distributed $U(X,m)$. What is $E(Y)$?

$$E(Y) = \frac{np+m}{2}.$$

Problem 13

A dice is thrown N times. Let X be the number of times 6 showed. Let Y be the number of times 5 showed. Compute $Cov(X,Y)$.

$$Cov(X,Y) = -\frac{N}{36}.$$

Problem 14

A red dice is thrown, and a blue dice is thrown. Let X be the result shown on the red dice, and let Y be the maximal result. Compute $Cov(X, Y)$.

$$Cov(X, Y) = \frac{35}{24}.$$

Probability 201-1-2391 ASSIGNMENT 9
The correlation coefficient
instructor: Ronen Peretz,
math dept, BGU

Problem 1

Three balls are distributed into three cells. Compute $R(X, Y)$ where X is the number of balls in cell 1 and Y is the number of balls in cell 2.

Problem 2

We throw a red dice and a green dice. Let X be the result in the red dice, and Y the maximum result. compute $R(X, Y)$.

Problem 3

Prove that if $Y = n - X$, then $R(X, Y) = -1$.

Problem 4

Let X be the number of sons in a family that has N children. Let Y be the number of daughters in that family. We define $T = X - Y$. Compute $Var(T)$.

Problem 5

A jar contains 5 balls that are numbered $1, 2, 3, 4, 5$. Three balls are randomly pulled out (they are NOT returned back). Let X be the minimal number pulled out, and Y be the maximal number pulled out.
a) Construct the table of the joint distribution of (X, Y) and compute the marginal distributions of X and of Y.
b) Compute $R(X, Y)$.

Probability 201-1-2391 ASSIGNMENT 9
The correlation coefficient
instructor: Ronen Peretz,
math dept, BGU

Problem 1

Three balls are distributed into three cells. Compute $R(X, Y)$ where X is the number of balls in cell 1 and Y is the number of balls in cell 2. $R(X, Y) = -1/2$

Problem 2

We throw a red dice and a green dice. Let X be the result in the red dice, and Y the maximum result. compute $R(X, Y)$.
$R(X, Y) = 0.6$.

Problem 3

Prove that if $Y = n - X$, then $R(X, Y) = -1$.

Problem 4

Let X be the number of sons in a family that has N children. Let Y be the number of daughters in that family. We define $T = X - Y$. Compute $Var(T)$. $Var(T) = N$.

Problem 5

A jar contains 5 balls that are numbered $1, 2, 3, 4, 5$. Three balls are randomly pulled out (they are NOT returned back). Let X be the minimal number pulled out, and Y be the maximal number pulled out.
a) Construct the table of the joint distribution of (X, Y) and compute the marginal distributions of X and of Y.

.	3	4	5	.
1	0.1	0.2	0.3	0.6
2	0	0.1	0.2	0.3
3	0	0	0.1	0.1
.	0.1	0.3	0.6	.

b) Compute $R(X, Y)$. $R(X, Y) = 1/3$.