Probability 201-1-2391 ASSIGNMENT 2 - answers
The elementary probability space
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Problem 1

a) Prove the following distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

b) Prove that complementing a set is an involution: $(A^c)^c = A$. Here $A^c = X - A$ and $A \subseteq X$.
c) Prove that: $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$.

Problem 2

Which is the greatest? :
a) The probability to obtain at least one 6 in a throw of 4 dice.

$$p_a = 1 - \left(\frac{5}{6}\right)^4$$

b) The probability to obtain at least once $(6, 6)$ in 24 throws of 2 dice (in each throw).

$$p_b = 1 - \frac{35^{24}}{36^{24}}$$

so $p_b < p_a$

Problem 3

Here is a solution to Problem 2 above: When we throw one dice there are 6 possible outcomes of which only one is a "success" (getting 6). When we throw 2 (a pair) dice there are 36 possible outcomes of which only one is a "success" (getting $(6, 6)$). So for the probability of success to be equal in the two spaces the number of throws of 2 dice should be 6 times as large as that of throwing just one, i.e. $4 \times 6 = 24$. Hence the answer to problem 2 is that the 2 probabilities are equal.
Discuss this argument and express your opinion.

Problem 4

a) A person has 2 match boxes, each containing n matches. When he needs a match he pulls one randomly out of one of his 2 boxes. When he finds out that one of his boxes in empty, what is the probability that his other box contains exactly k matches? Here $0 \leq k \leq n$.

$$2 \cdot \binom{2n-k}{n} \cdot \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{n-k} \cdot \frac{1}{2} = \frac{1}{2^{2n-k}} \cdot \binom{2n-k}{n}$$

b) Prove the identity:

$$\sum_{k=0}^{n} \frac{1}{2^{2n-k}} \binom{2n-k}{n} = 1.$$

Problem 5

k married couples (pairs) arrived at a party. An even number m of the $2k$ people were chosen at random. Compute the probability that:
a) Among the chosen ones there is no married couple.

$$\left[\binom{k}{m} 2^m\right] / \binom{2k}{m}$$

b) Among the chosen ones there is exactly one married couple.

$$\left[k \binom{k-1}{m-2} 2^{m-2}\right] / \binom{2k}{m}$$

c) Among the chosen ones there are exactly 4 married couples, assuming of course that $m \geq 8$.

$$\left[\binom{k}{4} \binom{k-4}{m-8} 2^{m-8}\right] / \binom{2k}{m}$$

d) All the chosen ones are $m/2$ married couples.

$$\binom{k}{m/2} / \binom{2k}{m}$$

Problem 6

5 dice are thrown. What is the probability that at least two will show identical results?

$$1 - \frac{6!}{6^5}$$

Problem 7

What is the probability that in distributing n (distinguishable) balls in m cells, cell number 1 will contain exactly k of the balls?

$$\left[\binom{n}{k} \cdot (m-1)^{n-k}\right] / m^n$$

Problem 8

4 balls are randomly distributed into 3 cells (3^4 possibilities of equal probability). We define the events A to G below:
A - cell number 1 is empty, B - only cell number 1 is empty, C - there is exactly one empty cell, D - there is an empty cell, E - there is a non-empty cell, F - cell number 3 contains 2 balls, G - there is a cell that contains exactly 2 balls.
Compute the probabilities of these events, $p_A, p_B, p_C, p_D, p_E, p_F$ and p_G.

$$p_A = \frac{2^4}{3^4} = \frac{16}{81},\ p_B = \frac{2^4 - 2}{3^4} = \frac{14}{81},\ p_C = 3p_B = \frac{42}{81},\ p_D = \frac{5}{9},\ p_E = 1,$$

$$p_F = \left[\binom{4}{2} \cdot 2^2\right] / 3^4 = \frac{8}{27},\ p_G = \left[3\binom{4}{2} 2!\right] / 81 + \left[\binom{3}{2}\binom{4}{2}\right] / 81 = \frac{2}{3}$$

Problem 9

A jar contains 10 balls numbered: $1, 2, 3, \ldots, 10$. 5 balls are pulled out randomly. What is the probability that ball number 7 is the second in size out of the balls that were pulled out?

$$\left[3 \cdot \binom{6}{3}\right] / \binom{10}{5}$$

Problem 10

k dice are thrown. Compute the probabilities of the following two events:
A - the largest number that occurred is i,

$$\frac{i^k}{6^k} - \frac{(i-1)^k}{6^k}$$

B - the smallest number that occurred is i.

$$\frac{(6-i+1)^k}{6^k} - \frac{(6-i)^k}{6^k}$$

Problem 11

4 married couples arrived at a party. The 8 people were divided randomly into pairs.
a) What is the probability of the event in which each man found himself with his wife?

$$\frac{4!2^4}{8!}$$

b) What is the probability that each pair contains a man and a woman?

$$\frac{(4!)^2 2^4}{8!}$$

Problem 12

n married couples sit randomly around a round table. What is the probability that no woman sits besides her husband?

$$1+\sum_{i=1}^{n}(-1)^i\binom{n}{i}\frac{(2n-i-1)!}{(2n-1)!}\cdot 2^i$$

Problem 13

Out of a deck of cards we randomly choose k cards, $k \geq 13$. What is the probability that each number is represented in the sample?

$$1-\sum_{i=1}^{13}(-1)^{i-1}\binom{13}{i}\binom{52-4i}{k}\Big/\binom{52}{k}$$

The following "answer" is **wrong**

$$\left[4^{13}\binom{52-13}{k-13}\right]\Big/\binom{52}{k}$$

Problem 14

A dice is thrown 36 times. What is the probability to get each number 6 times?