Probability 201-1-2391 ASSIGNMENT 7
Tow dimensional random variables
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Problem 1

A jar contains 2 white balls and 1 black ball. We pull out (and do NOT return back) 2 balls one by one. Let X be the number of white balls on the first pull out. Let Y be the number of white balls on the second pull out. What is the joint distribution of (X, Y) and what are the marginal distributions?

$p(X\,\text{black}, Y\,\text{black}) = 0$	$p(X\,\text{white}, Y\,\text{black}) = 1/3$	$p(Y\,\text{black}) = 1/3$
$p(X\,\text{black}, Y\,\text{white}) = 1/3$	$p(X\,\text{white}, Y\,\text{white}) = 1/3$	$p(Y\,\text{white}) = 2/3$
$p(X\,\text{black}) = 1/3$	$p(X\,\text{white}) = 2/3$	1

Problem 2

A balanced coin is tossed. Its faces are denoted by 0 and 1. If we get 1 then we toss again. If we get 0 we throw a dice. Let X be the result of the first toss, Y - the result of the second.
a) Find the distributions (joint and the marginals).

$p(X = 0, Y = 0) = 0$	$p(X = 1, Y = 0) = 1/4$	$p(Y = 0) = 1/4$
$p(X = 0, Y = 1) = 1/12$	$p(X = 1, Y = 1) = 1/4$	$p(Y = 1) = 1/3$
$p(X = 0, Y = 2) = 1/12$	$p(X = 1, Y = 2) = 0$	$p(Y = 2) = 1/12$
$p(X = 0, Y = 3) = 1/12$	$p(X = 1, Y = 3) = 0$	$p(Y = 3) = 1/12$
$p(X = 0, Y = 4) = 1/12$	$p(X = 1, Y = 4) = 0$	$p(Y = 4) = 1/12$
$p(X = 0, Y = 5) = 1/12$	$p(X = 1, Y = 5) = 0$	$p(Y = 5) = 1/12$
$p(X = 0, Y = 6) = 1/12$	$p(X = 1, Y = 6) = 0$	$p(Y = 6) = 1/12$
$p(X = 0) = 1/2$	$p(X = 1) = 1/2$	1

b) Are X and Y independent? No, they are NOT: $0 = p(X = 0, Y = 0) \neq p(X = 0) \cdot p(Y = 0) = 1/8$. c) compute $p(X^2 + Y^2 \leq 9)$. $p(X^2 + Y^2 \leq 9) = 3/4$.

Problem 3

We have two hats, each contains 10 notes that are numbered 1 to 10. A note is randomly pulled out of each hat. Let X be the smallest of the two numbers and let

Y be the largest of the two.
a) Compute the joint distribution of (X, Y).

$$p(X=k, Y=l) = \begin{cases} 0 & , \quad k > l \\ 1/100 & , \quad k = l \\ 2/100 & , \quad 1 \leq k < l \leq 10 \end{cases}$$

b) Are X and Y independent? No: $0 = p(X=4, Y=2) \neq p(X=4) \cdot p(Y=2) > 0$.
c) Compute the conditional distribution of X, given that $Y=2$. $p(X=k|Y=2) = p(X=k \cap Y=2)/0.03$. For $k > 1$ it is 0.

Problem 4

Dan tosses a coin that has probability p for H, 25 times. Let X be the number of H's that Dan gets. Ron tosses a coin that has probability p for H, 15 times. Y is his number of H's.
a) Find the distribution of $X+Y$. Binomial $B(40, p)$
b) Prove that given $X+Y=10$, X distributes $H(25, 15, 10)$ where $H(a, b, n)$ is the hypergeometric distribution, a is the population size being counted, b is the size of the rest of the population and n is the sample size.

$$\binom{25}{k} \cdot \binom{15}{10-k} / \binom{40}{10}$$

Problem 5

The center of a phone company receives every minute, phone calls that are distributed Poisson, $P(\lambda)$. Every call independently of the other calls is answered with probability p. Find the distribution of the answered calls per minute.

$$p(\text{answered} = y) = \frac{(\lambda \cdot p)^y}{y!} e^{-\lambda \cdot p}.$$

Problem 6

A bank branch has two entrances. The front entry receives a number of clients per minute that is distributed Poisson, $P(\lambda)$. The rear entry receives a number of clients per minute that is distributed Poisson, $P(\mu)$. The two processes are independent.

a) Prove that the total number of clients that enter the bank branch per minute distributes Poisson, $P(\lambda+\mu)$.

$$p(X+Y=k)=\frac{(\lambda+\mu)^k}{k!}e^{-(\lambda+\mu)}.$$

b) Given that the total number of clients that entered the bank branch in the last minute is 10, what is the distribution of the number of clients that entered the front entry?

$$p(X=k|X+Y=10)=\left(\begin{array}{c}10\\k\end{array}\right)\left(\frac{\lambda}{\lambda+\mu}\right)^k\left(\frac{\mu}{\lambda+\mu}\right)^{10-k}$$

Problem 7

Let X, Y be independent random variables, each distributed geometrically with the parameter p, i.e. $G(p)$. Prove that the conditional distribution of X given that $X+Y=n$ is the uniform distribution $U(1,n-1)$.

$$p(X+Y=n)=(n-1)p^2q^{n-2},\ p(X=k|X+Y=n)=\frac{p(X=k\cap X+Y=n)}{p(X+Y=n)}=$$

$$=\frac{p(X=k)p(Y=n-k)}{(n-1)p^2q^{n-2}}=\frac{(q^{k-1}p)(q^{n-k-1}p)}{(n-1)p^2q^{n-2}}=\frac{1}{n-1}.$$

Problem 8

Your TA defined X as the number of hearts in a hand of Bridge game, and Y as the number of spades. He wants to know what is the distribution of $X+Y$. So he presents the table of the joint distribution and in order to compute $p(X+Y=k)$ he sums the probabilities along diagonals of the table. The first diagonal for $k=0$. The second for $k=1$ etc... . What would you tell him?