Probability 201-1-2391 ASSIGNMENT 3-answers
The Borel system
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Problem 1

We defined a field of events β as a family $\beta \subseteq 2^{\Omega}$ that satisfies the following 3 axioms:
I. $\Omega \subseteq \beta$.
II. If $A, B \in \beta$ then $A - B \in \beta$.
III. If $A_1, A_2, \ldots \in \beta$ is countable then the union $\bigcup_{n=1}^{\infty} A_n \in \beta$

Let us state the following axiom:
II'. If $A \in \beta$, then $A^c \in \beta$ where A^c is the complimentary set of A.
<u>Prove</u> that the three axioms I,II,III are equivalent to I,II',III.

Axioms I,II,III imply axiom II': Let $A \in \beta$. Take $B = \Omega$. By I $\Omega \in \beta$. By II $B - A \in \beta$. But $B - A = A^c$.
Axioms I,II',III imply axiom II: Let $A, B \in \beta$. Then by II' $A^c \in \beta$. So by III $A^c \cup B \in \beta$. So $(A^c \cup B)^c \in \beta$. But $A - B = (A^c \cup B)^c$ and we finished.

Problem 2

Using the definition we gave to a Borel system (Ω, β, p), prove (using the axioms):
1. $p(\emptyset) = 0$.

By axiom III of p we have $p(\emptyset \cup \Omega) = p(\emptyset) + p(\Omega)$. So $p(\Omega) = p(\emptyset) + p(\Omega)$ and hence $p(\emptyset) = 0$.
2. If $A, B \in \beta$ satisfy $A \subseteq B$ then $p(A) \leq p(B)$.

By axiom III of p we have $p(B) = p(B - A) + P(A)$ because $A \subseteq B$. By axiom II $p(B - A) \geq 0$ and so $p(B) \geq p(A)$.

Problem 3

Give an example of a probability space (Ω, β, p) such there are $A, B \in \beta$, $A \neq \emptyset$ and $B \neq \Omega$, <u>but</u> $p(A) = 0$ and $p(B) = 1$.

Take $\Omega = [0,1]$ the real numbers between 0 and 1, and β the field of events generated by the closed sub-intervals in $[0,1]$. Also $p([a,b]) = b-a$. Then we can take $A = \{0\}$ and $B = \Omega - A$.

Problem 4

Give an example of a probability space (Ω, β, p) such that Ω is an infinite sample space but $|\beta| < \infty$, or prove that no such a probability space can exist.

Take Ω any infinite set and $\beta = \{\Omega, \emptyset\}$, with $p(\Omega) = 1$ and $p(\emptyset) = 0$.

Problem 5

Give an example of a probability space (Ω, β, p) such that $|\beta| = 3$ or prove that no such a probability space can exist.

No such a space exists. Because if $\beta = \{\Omega, \emptyset, A\}$ then also $A^c \in \beta$ and of course $A^c \notin \beta$.

Problem 6

Give an example of a probability space (Ω, β, p) such that $|\beta| = 4$ or prove that no such a probability space can exist.

Take $\Omega = \{1, 2\}$ the elementary probability space.

Problem 7

Give an example of a probability space (Ω, β, p) such that $|\beta|$ is an odd integer or prove that no such a probability space can exist.