MEE5114 Advanced Control for Robotics

Lecture 8: Rigid Body Dynamics

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Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Acceleration

- Given a rigid body with spatial velocity $\mathcal{V} = (\omega, v_o)$, its spatial acceleration is

$$\mathcal{A} = \dot{\mathcal{V}} = \begin{bmatrix} \dot{\omega} \\ \dot{v}_o \end{bmatrix}$$

$$A = \lim_{\delta \to 0} \frac{\mathcal{V}(t+\delta) - \mathcal{V}(t)}{\delta}$$

  - Recall that: v_o is the velocity of the body-fixed particle coincident with frame origin o at the current time t.
  - Note: $\dot{\omega}$ is the angular acceleration of the body

$V_o = \dot{q}(t)$. for some body-fixed point

  - $\dot{v}_o$ is not the acceleration of any body-fixed point! but $\dot{v}_o \neq \ddot{q}(t)$
  - In fact, $\dot{v}_o$ gives the rate of change in stream velocity of body-fixed particles passing through o

O is fixed

particle is changing

Spatial vs. Conventional Accel. (1/2)

- Why "$\dot{v}_o$ is not the acceleration of any body-fixed point"?

  $q(t_0) = 0$

- Suppose $q(t)$ is the body fixed particle coincides with o at time t

- So by definition, we have $v_o(t_0) = \dot{q}(t_0)$, however, $\dot{v}_o(t) \neq \ddot{q}(t)$, where $\ddot{q}(t)$ is the conventional acceleration of the body-fixed point q

  - Note: $\dot{v}_o(t_0) \triangleq \lim_{\delta \to 0} \frac{v_o(t_0+\delta) - v_o(t_0)}{\delta}$

At time $t = t_0$, $q(t_0) = 0$, $V(t_0) = \dot{q}(t_0)$

At time $t = t_0 + \delta$, q_1 will be coincided with 0

$q_1(t+\delta) = 0$ $\quad V_0(t+\delta) \cancel{=} \dot{q}(t_0+\delta)$

another body-fixed particle

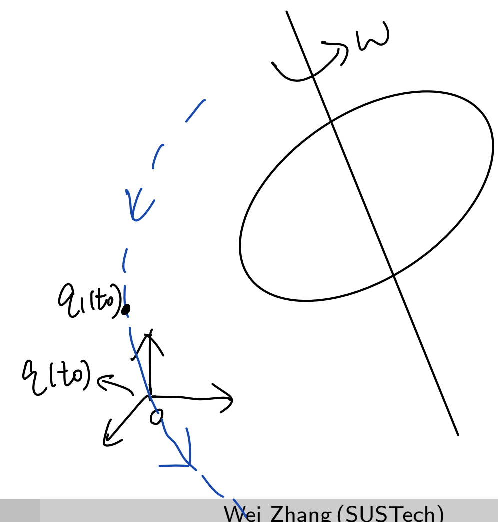
$\dot{q}_1(t+\delta)$

Note: q_1 and q are different points

$\dot{q}_1(t_0+\delta) \neq \dot{q}(t_0+\delta)$

$\ddot{q}_1(t_0+\delta) \neq \ddot{q}(t_0+\delta)$

At time t_0



Spatial vs. Conventional Accel. (2/2)

$$\frac{v_o(t+\delta) - v_o(t_0)}{\delta} \neq \frac{\dot{q}(t_0+\delta) - \dot{q}(t_0)}{\delta} \xrightarrow{\delta \to 0} \ddot{q}(t_0)$$

By definition: $\dot{q}(t) = v_o(t) + \omega(t) \times q(t) \longleftarrow$ holds for all t

$$\ddot{q}(t) = \dot{v}_o(t) + \dot{\omega}(t) \times q(t) + \omega(t) \times \dot{q}(t)$$

At $t = t_0$, $\ddot{q}(t_0) = \dot{v}_o(t_0) + \dot{\omega}(t_0) \times q(t_0) + \omega(t) \times \dot{q}(t_0)$

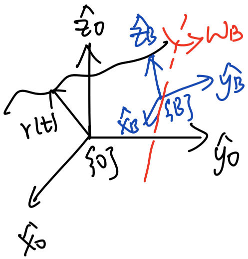
If $q(t_0) = 0$ $\boxed{\ddot{q}(t_0) = \dot{v}_o(t_0) + \omega(t_0) \times \dot{q}(t_0)}$

-
- If $q(t)$ is the body fixed particle coincides with o at time t, then we have

$$\ddot{q}(t) = \dot{v}_o(t) + \omega(t) \times \dot{q}(t)$$

Plücker Coordinate System and Basis Vectors (1/2)

- Recall coordinate-free concept: let $\cancel{w}\, r \in \mathbb{R}^3$ be a free vector with $\{o\}$ and $\{\cancel{A}\, B\}$ frame coordinate ${}^o r$ and ${}^{B} r$



$${}^o r = \begin{bmatrix} {}^o r_x \\ {}^o r_y \\ {}^o r_z \end{bmatrix} \in \mathbb{R}^3 \iff r = [\hat x_0 \ \hat y_0 \ \hat z_0]\, {}^o r$$

$${}^B r = \begin{bmatrix} {}^B r_x \\ {}^B r_y \\ {}^B r_z \end{bmatrix} \in \mathbb{R}^3 \iff r = [\hat x_B \ \hat y_B \ \hat z_B]\, {}^B r$$

$\Downarrow$

express this "physics" using {o}

$$\Rightarrow {}^o r = \underbrace{[{}^o \hat x_B \ {}^o \hat y_B \ {}^o \hat z_B]}_{{}^o R_B}\, {}^B r$$

← change of coordinate

$$-① \Rightarrow \dot r = [\hat x_0 \ \hat y_0 \ \hat z_0]\, \frac{d}{dt}({}^o r)$$

$\Downarrow$ apparent derivative $\triangleq {}^o \dot r$

use {o} frame to express:

$${}^o(\dot r) = \underbrace{[{}^o \hat x_0 \ {}^o \hat y_0 \ {}^o \hat z_0]}_{I_{3\times 3}}\, \frac{d}{dt}({}^o r)$$

$${}^o(\dot r) = \frac{d}{dt}({}^o r)$$

Plücker Coordinate System and Basis Vectors (1/2)

- Recall coordinate-free concept: let $w \in \mathbb{R}^3$ be a free vector with {o} and {A} frame coordinate ${}^{o}w$ and ${}^{A}w$

$\dot{\hat{x}}_B = \omega_B \times \hat{x}_B$

② $r = [\hat{x}_B \ \hat{y}_B \ \hat{z}_B] \, {}^{B}r$

$\dot{r} \neq \frac{d}{dt}({}^{B}r)$ (X) {B} frame is rotating

$\Rightarrow \dot{r} = [\dot{\hat{x}}_B \ \dot{\hat{y}}_B \ \dot{\hat{z}}_B] \, {}^{B}r + [\hat{x}_B \ \hat{y}_B \ \hat{z}_B] \frac{d}{dt}({}^{B}r)$

$= \omega_B \times [\hat{x}_B \ \hat{y}_B \ \hat{z}_B] \, {}^{B}r + [\hat{x}_B \ \hat{y}_B \ \hat{z}_B] \frac{d}{dt}({}^{B}r)$

Use {B} frame to express the above equ.

$${}^{B}(\dot{r}) = {}^{B}\omega_B \times {}^{B}r + \frac{d}{dt}({}^{B}r)$$

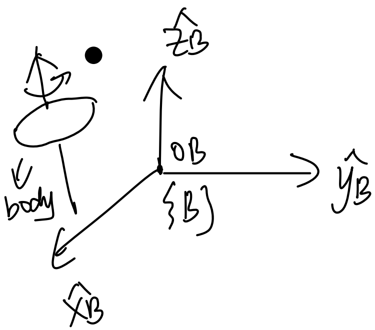
accounts for coordinate frame axis is moving

due to changes in the coordinate

${}^{B}\left(\frac{d}{dt}(r)\right)$

${}^{B}\dot{r}$

Plücker Coordinate System and Basis Vectors (2/2)



- ${}^B\mathcal{V}_{body} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, means rotating about $\hat{x}_B$ at unit speed

  $\Longleftrightarrow \mathcal{V}_{body} = e_{B1}$ motion basis vector

  $= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, means rotating about $\hat{y}_B$ at unit speed

  $\mathcal{V}_{body} = e_{B2}$

  $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, linear motion along $\hat{x}_B$ at unit speed

  $\mathcal{V}_{body} = e_{B3}$

Given $\{B\}$ frame

$\{e_{B1}, e_{B2} \cdots e_{B6}\}$ - 6dim

motion basis vector

coordinate free

$\mathcal{V}_{body} = \alpha_1 e_{B1} + \alpha_2 e_{B2} + \cdots + \alpha_6 e_{B6}$, where $\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_6 \end{bmatrix} = {}^B\mathcal{V}_{body}$

Use $\{O\}$-frame to represent above "physics"

$${}^O\mathcal{V}_{body} = \alpha_1 {}^O e_{B1} + \alpha_2 {}^O e_{B2} + \cdots + \alpha_6 {}^O e_{B6}$$

Plücker Coordinate System and Basis Vectors (2/2)

- $${}^{0}\mathcal{V}_{body} = \left[\, {}^{0}e_{B1} \;\; {}^{0}e_{B2} \;\cdots\; {}^{0}e_{B6} \,\right] {}^{B}\mathcal{V}_{body}$$

${}^{0}e_{Bi}$ can be computed from "physics" and twist definition

— unit speed rotation about $\overrightarrow{O_B x_B}$ expressed in $\{0\}$

$$-\left[\, {}^{0}e_{B1} \;\; {}^{0}e_{B2} \;\cdots\; {}^{0}e_{B6} \,\right] = \underset{6\times 6}{{}^{0}X_B} = Ad_{{}^{0}T_B}$$

$\cdot\; A_{body} = \frac{d}{dt}(\mathcal{V}_{body})$ | $\mathcal{V}_{body} = [\, e_{B1} \cdots e_{B6} \,] \, {}^{B}\mathcal{V}_{body}$

$$\Rightarrow A_{body} = \frac{d}{dt}(\mathcal{V}_{body}) = [\, \dot{e}_{B1} \cdots \dot{e}_{B6} \,] \, {}^{B}\mathcal{V}_{body} + [\, e_{B1} \cdots e_{B6} \,] \underbrace{\frac{d}{dt}({}^{B}\mathcal{V}_{body})}_{{}^{B}\dot{\mathcal{V}}_{body}}$$

If $\{B\}$ does not change:

$$\dot{\mathcal{V}}_{body} = [\, \cancel{\dot{e}_{B1} \cdots e_{B6}} \,] \, {}^{B}\mathcal{V}_{body} + [\, e_1 \cdots e_{B6} \,] \, {}^{B}\dot{\mathcal{V}}_{body} = [\, e_1 \cdots e_{B6} \,] \, {}^{B}\dot{\mathcal{V}}_{body}$$

$\leftarrow$ express in $\{B\}$

special case ${}^{0}A = {}^{0}\dot{\mathcal{V}}_{body}$

$$\underbrace{{}^{B}(\dot{\mathcal{V}}_{body})}_{{}^{B}A_{body} =} = {}^{B}\dot{\mathcal{V}}_{body} = \frac{d}{dt}({}^{B}\mathcal{V}_{body})$$

Work with Moving Reference Frame

- If $\{B\}$ change over time $\underbrace{{}^{B}A \neq {}^{B}\dot{\mathcal{V}}_{body}}$

Then the key to compute $[\dot{e}_{B_1} \ \dot{e}_{B_2} \cdots \dot{e}_{B_6}]$ ← can be compute purely by physics (see featherstone)

Now let's work with $\{0\}$ frame to find the derivate

⇒ we need to compute $[{}^{0}\dot{e}_{B_1} \cdots {}^{0}\dot{e}_{B_6}] = {}^{0}\dot{X}_B = \frac{d}{dt}\left[Ad_{{}^{0}T_B}\right]$

Let's denote: ${}^{0}T_B = (R, p) \Rightarrow \frac{d}{dt}\left(\begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}\right) = \begin{bmatrix} \dot{R} & 0 \\ ([p]R)' & \dot{R} \end{bmatrix}$

$\{B\}$-frame has instantaneous velocity $\mathcal{V}_B = \begin{bmatrix} w \\ v \end{bmatrix}$

Note: $\dot{R} = w \times R \qquad \dot{p} = v + w \times p \qquad [Rw] = R[w]R^T$

$[w_1 \times w_2] = [w_1][w_2] - [w_2][w_1]$ ·· Jacobian's

⇒ After some computation, $\frac{d}{dt}(Ad_{{}^{0}T_B}) = \begin{bmatrix} [w] & 0 \\ [v] & [w] \end{bmatrix} (Ad_{{}^{0}T_B})$

$$\underset{6\times 6}{{}^{0}\dot{X}_B} = \underset{6\times 6}{\begin{bmatrix} [w] & 0 \\ [v] & [w] \end{bmatrix}} \underset{6\times 6}{{}^{0}X_B}$$

Work with Moving Reference Frame

- $$\begin{bmatrix} [\dot{\omega}]R & 0 \\ [\dot{p}]R + [p]\dot{R} & [\omega]R \end{bmatrix}$$

$[\dot{\omega}]R + [\omega][p][R]$

$$= \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

$[v + \omega \times p] R + [p] \omega \times R$

$[\dot{v}]R + [\omega \times p]R + [p][\omega]R$

$= [\dot{v}]R + [\omega][p]R - [p][\omega][R] + [p][\omega][R]$

Work with Moving Reference Frame

- Define $\begin{bmatrix} [\omega] & [v] \\ [\omega] & 0 \end{bmatrix} \triangleq [\mathcal{V}_B \times]$, $\boxed{\begin{array}{l} {}^0\dot{X}_B = \mathcal{V}_B \times {}^0X_B \\ {}^0\dot{R}_B = \omega_B \times {}^0R_B \end{array}}$

In coordinate free: $\dot{e}_{B1} = \mathcal{V}_B \times e_{B1}$, $\dot{e}_{B2} = \mathcal{V}_B \times e_{B2}$ ......

Derivative of Adjoint

- Suppose a frame $\{A\}$'s pose is $T_A = (R_A, p_A)$, and is moving at an instantaneous velocity $\mathcal{V}_A = (\omega, v)$. Then

$$\boxed{\frac{d}{dt}\left([\mathrm{Ad}_{T_A}]\right) = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} [\mathrm{Ad}_{T_A}]}$$

$${}^{0}\dot{X}_A = \mathcal{V}_A \times {}^{0}X_A$$

Spatial Cross Product

- Given two spatial velocities (twists) $\mathcal{V}_1$ and $\mathcal{V}_2$, their spatial cross product is:

$$\mathcal{V}_1 \times \mathcal{V}_2 = \begin{bmatrix} \omega_1 \\ v_1 \end{bmatrix} \times \begin{bmatrix} \omega_2 \\ v_2 \end{bmatrix} \triangleq \begin{bmatrix} \omega_1 \times \omega_2 \\ \omega_1 \times v_2 + v_1 \times \omega_2 \end{bmatrix}$$

Lie Bracket

- Matrix representation: $\mathcal{V}_1 \times \mathcal{V}_2 = [\mathcal{V}_1 \times]\mathcal{V}_2$, where

$$[\mathcal{V}_1 \times] \triangleq \begin{bmatrix} [\omega_1] & 0 \\ [v_1] & [\omega_1] \end{bmatrix}$$

- Roughly speaking, when a motion vector $\mathcal{V}$ is moving with a spatial velocity $\mathcal{Z}$ (e.g. it is attached to a moving frame) but is otherwise not changing, then

$$\boxed{\dot{\mathcal{V}} = \mathcal{Z} \times \mathcal{V}}$$

Spatial Cross Product: Properties (1/1)

- Assume A is moving wrt to O with velocity $\mathcal{V}_A$

$$ {}^O\dot{X}_A = [{}^O\mathcal{V}_A \times]\, {}^O X_A $$

- $\underbrace{[X\mathcal{V}\times] = X[\mathcal{V}\times]X^T}$, for any transformation X and twist $\mathcal{V}$

$[R\omega] = R[\omega]R^T$

Spatial Acceleration with Moving Reference Frame

Consider a body with velocity $\mathcal{V}_{body}$ (wrt inertia frame), and ${}^O\mathcal{V}_{body}$ and ${}^B\mathcal{V}_{body}$ be its Plücker coordinates wrt {O} and {B}:

- $${}^B\mathcal{A}_{body} = \underbrace{\frac{d}{dt}\left({}^B\mathcal{V}_{body}\right)}_{\text{apparent derivative}} + \underbrace{{}^B\mathcal{V} \times {}^B\mathcal{V}_{body}}$$ ← due to frame {B} is moving

  $${}^B\left(\frac{d}{dt}(\mathcal{V}_{body})\right) = {}^B\dot{\mathcal{V}}_{body}$$

- ${}^O\mathcal{A} = {}^OX_B\,{}^B\mathcal{A}$

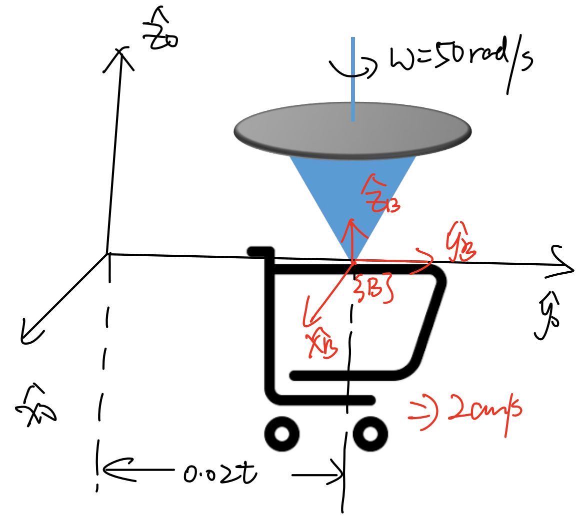
$${}^O\mathcal{A}_{body} = \frac{d}{dt}({}^O\mathcal{V}_{body}) = \frac{d}{dt}({}^OX_B\,{}^B\mathcal{V}_{body}) = {}^O\dot{X}_B\,{}^B\mathcal{V}_{body} + {}^OX_B\,{}^B\mathring{\mathcal{V}}_{body}$$

$$= {}^O\mathcal{V}_B \times {}^OX_B\,{}^B\mathcal{V}_{body} + {}^OX_B\,{}^B\mathring{\mathcal{V}}_{body}$$

$$= {}^OX_B\left\{\underbrace{{}^BX_O\left[{}^O\mathcal{V}_B\times\right]{}^OX_B}_{\left[{}^BX_O\mathcal{V}_B\times\right] = \left[{}^B\mathcal{V}_B\times\right]}\,{}^B\mathcal{V}_{body} + {}^B\mathring{\mathcal{V}}_{body}\right\} = {}^OX_B\underbrace{\left\{{}^B\mathring{\mathcal{V}}_{body} + {}^B\mathcal{V}_B \times {}^B\mathcal{V}_{body}\right\}}_{{}^B\mathcal{A}_{body}}$$

Spatial Acceleration Example

Find ${}^B A_{top}$



Method ${}^B A_{top} = {}^B X_0 \, {}^0 A_{top} = {}^0 X_B \frac{d}{dt}({}^0 V_{top})$

$${}^0 V_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ t \\ 0.02 \\ 0 \end{bmatrix} \qquad {}^0 V_{top} = \begin{bmatrix} 0 \\ 0.02 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix} \times \begin{bmatrix} -0.02t \\ 0 \\ 0 \end{bmatrix}$$

$${}^0 A_{top} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$${}^B A_{top} = {}^B X_0 \, {}^0 A_{top} = \begin{bmatrix} I & 0 \\ [{}^B p_0] & I \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Method 2

$${}^B V_{top} = \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} \qquad {}^B A_{top} = {}^B \dot{V}_{top} + {}^B V_{B\text{-}frame} \times {}^B V_{top}$$

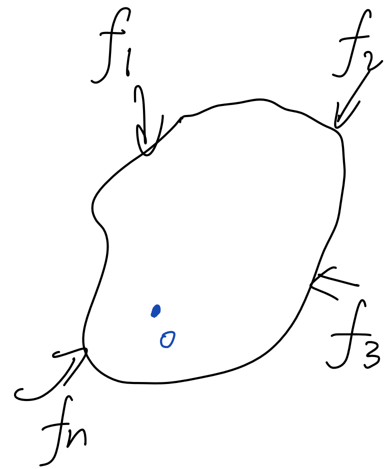
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \\ 0.02 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Outline

- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum
- Newton-Euler Equation using Spatial Vectors

Spatial Force (Wrench)

- Consider a rigid body with many forces on it and fix an arbitrary point O in space



- The net effect of these forces can be expressed as
  - A force f, acting along a line passing through O

    $f = \sum f_i$

  - A moment n_O about point O

    $n_O \in \mathbb{R}^3$ $\qquad$ $n_O = \sum_i (\overrightarrow{Op_i}) \times f_i$

- **Spatial Force (Wrench):** is given by the 6D vector

$$\mathcal{F} = \begin{bmatrix} n_O \\ f \end{bmatrix}$$

What if we choose reference point to q

$$n_q = \sum_i (\overrightarrow{qp_i}) \times f_i = n_O + \sum_i (\overrightarrow{qp_i} - \overrightarrow{Op_i}) \times f_i = n_O + \overrightarrow{qO} \times \sum_i f_i = n_O + \overrightarrow{qO} \times f$$

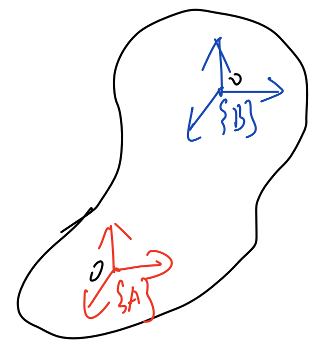
$$= n_O + f \times \overrightarrow{Oq}$$

Spatial Force in Plücker Coordinate Systems

- Given a frame $\{A\}$, the Plücker coordinate of a spatial force $\mathcal{F}$ is given by

$$^B\mathcal{F} = \begin{bmatrix} ^B n_{O_B} \\ ^B f \end{bmatrix} \qquad ^A\mathcal{F} = \begin{bmatrix} ^A n_{O_A} \\ ^A f \end{bmatrix} \qquad ^AT_B = (^AR_B, {^Ap_B})$$

- Coordinate transform: $^A\mathcal{F} = {^AX_B^*}\,{^B\mathcal{F}}$ where $^AX_B^* = {^BX_A}^T$



$$^Af = {^AR_B}\,{^Bf}$$

moment: coordinate free $\quad n_{O_A} = n_{O_B} + \overrightarrow{O_AO_B} \times f$

Choose {A} to express $\quad {^An_{O_A}} = {^An_{O_B}} + {^A\overrightarrow{O_AO_B}} \times {^Af}$

$$= {^AR_B}\,{^Bn_{O_B}} + {^AR_B}\left(-{^BO_A} \times {^Bf}\right)$$

$$= {^AR_B}\,{^Bn_{O_B}} - {^AR_B}\left[{^BO_A}\right]{^Bf}$$

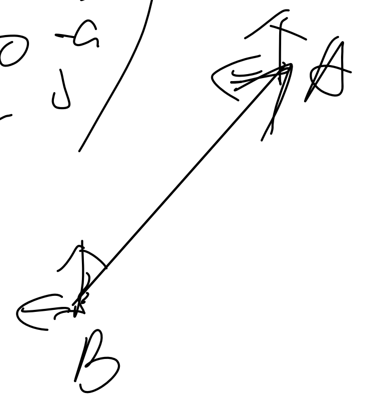
$$\begin{bmatrix} ^An_{O_A} \\ ^Af \end{bmatrix} = \begin{bmatrix} ^AR_B & {^AR_B}[{^Bp_A}] \\ 0 & ^AR_B \end{bmatrix} \begin{bmatrix} ^Bn_O \\ ^Bf \end{bmatrix}$$

More Discussions

- $\triangleq {}^A X_B^*$ $\qquad {}^A X_B^* = \underline{\left({}^B X_A\right)^T}$

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & c & J \end{bmatrix}$$

$$\begin{bmatrix} I & {}^B p_A \\ 0 & I \end{bmatrix}$$



Wrench-Twist Pair and Power

- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$

- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)

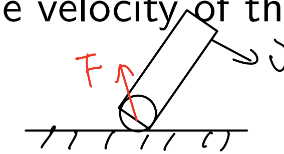
- Suppose a rigid body has a twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$ and a wrench ${}^A\mathcal{F} = ({}^A n_{o_A}, {}^A f)$ acts on the body. Then the power is simply

$$P = \underset{(6\times 1)^T}{({}^A\mathcal{V})^T} \underset{6\times 1}{{}^A\mathcal{F}} = {}^A\mathcal{F}^T\, {}^A\mathcal{V}$$

$$= \underbrace{({}^A\omega^T)\, {}^A n_{oA}}_{\text{rotational power}} + \underbrace{{}^A v_{oA}^T\, {}^A f}$$

Joint Torque

- Consider a link attached to a 1-dof joint (e.g. revolute or prismatic). Let $\hat{\mathcal{S}}$ be the screw axis of the joint. The velocity of the link induced by joint motion is given by: $\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$



- $\mathcal{F}$ be the wrench provided by the joint. Then the power produced by the joint is

$$P = \mathcal{V}^T\mathcal{F} = (\hat{\mathcal{S}}^T\mathcal{F})\dot{\theta} \triangleq \boxed{\tau\dot{\theta}}$$

$$(\hat{S}\dot{\theta})^T \mathcal{F} = \hat{S}^T \mathcal{F}\dot{\theta}$$

$$\hat{S}^T \mathcal{F} \triangleq \tau$$

scalar

- $\tau = \hat{\mathcal{S}}^T\mathcal{F} = \mathcal{F}^T\hat{\mathcal{S}}$ is the projection of the wrench onto the screw axis, i.e. the effective part of the wrench.

- Often times, τ is referred to as joint "torque" or generalized force

q

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- Spatial Momentum $\longleftarrow$
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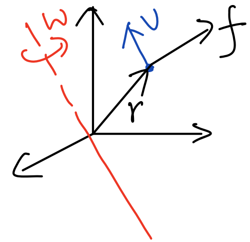
Rotational Inertia (1/2)

point mass: m



- Recall momentum for point mass:

velocity: $v=\dot{r}$, $a=\dot{v}=\ddot{r}\in\mathbb{R}^3$ $\longleftrightarrow$ $\omega=\hat{\omega}\cdot\dot{\theta}$, $v=\omega\times r$

force: $f=ma=m\dot{v}=m\ddot{r}$ $\longleftrightarrow$ moment: $n=r\times f$

Linear momentum: $p=L=m\cdot v$ $\longleftrightarrow$ angular momentum: $\phi=r\times p=r\times L$

$$=r\times(mv)=r\times(m\omega\times r)$$
$$=m(r\times\omega\times r)$$
$$=m\ r\times(-r)\times\omega$$
$$=\boxed{m[r][-r]}\,\omega$$

3x3 matrix

Inertia matrix for this point mass

Rotational Inertia (2/2)

density function

$\phi = \sum m_i [r_i][r_i] w$

- Rotational Inertia: $\bar{I} \triangleq \int_V \rho(r)[r][r]^T dr$

this matrix depends on coordinate sys to represents r_i

$[-r_i] = [r_i]^T$

  - $\rho(\cdot)$ is the density function of the body
  - $\bar{I}$ depends on coordinate system
  - It is a constant matrix if the origin coincides with CoM

What's the definition for CoM?

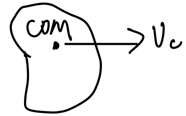
$C \triangleq CoM = \frac{1}{m}\int \rho(r) r \, dv$

$C = \sum_i m_i r_i$

If C is CoM, then $\frac{1}{m}\sum_i m_i(\vec{cr_i}) = 0$

$\boxed{\sum_i m_i \vec{cr_i} =} \Rightarrow \boxed{\sum_i m_i [cr_i] = 0}$

Spatial Momentum



- Consider a rigid body with spatial velocity $\mathcal{V}_C = (\omega, v_C)$ expressed at the center of mass C
  - Linear momentum: $L = m V_{com}$ $= m v_c$

    Why? $L = \sum_i m v_i = \sum_i m_i (v_c + \omega \times c r_i)$
    $= (\sum_i m_i) v_c + \sum_i m_i (-c r_i) \times \omega$ $\to 0$
    $= m v_c$

  - Angular momentum about CoM:

    $\phi_c = \bar{I}_c \omega$ $\quad \phi_c = \sum_i \vec{cr_i} \times m v_i = \sum_i \vec{cr_i} \times (m_i v_c + m_i \omega \times \vec{cr_i})$ $\to 0$
    $= (\sum_i \vec{cr_i} \times m_i v_c + \sum_i m_i c r_i \times \omega \times \vec{cr_i})$
    $= \bar{I}_c \omega$

  - Angular momentum about a point O:

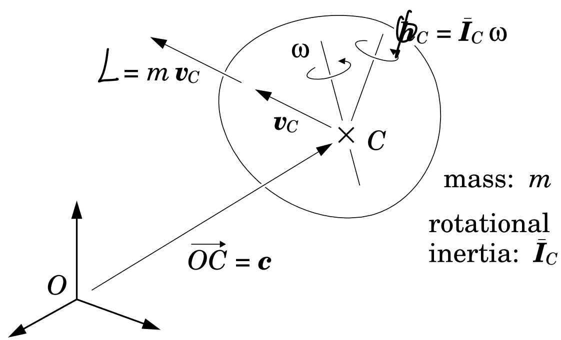
$\phi_o = \sum_i \vec{or_i} \times m_i v_i = \phi_c + \vec{oc} \times L$ HW

$\boxed{\phi_o = \phi_c + \vec{oc} \times L}$

coordinate free

- Spatial Momentum:

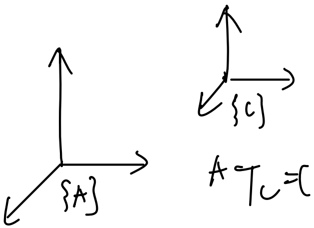
$h = \begin{bmatrix} \phi \\ L \end{bmatrix} \in \mathbb{R}^6$



Change Reference Frame for Momentum

- Spatial momentum transforms in the same way as spatial forces:

$$^{A}h = {}^{A}X_{C}^{*}\,{}^{C}h$$



$^{A}T_{C} = ({}^{A}R_{C}, {}^{A}p_{C})$

$^{A}v_{com} = {}^{A}R_{C}\,{}^{C}v_{com}$

$^{C}h = \begin{bmatrix} {}^{C}\phi_{oC} \\ {}^{C}L \end{bmatrix}$

$^{A}h = \begin{bmatrix} {}^{A}\phi_{oA} \\ {}^{A}L \end{bmatrix}$

$\left\{ \begin{array}{l} \underset{3\times1}{{}^{A}L} = \underset{3\times3}{{}^{A}R_{C}}\,\underset{3\times1}{{}^{C}L} \\ \\ \end{array} \right.$

$\Longleftrightarrow$ coordinate free

$\underbrace{\phi_{oA} = \phi_{oC} + \overrightarrow{O_A O_C} \times L}$

$\{A\} \Rightarrow \; {}^{A}\phi_{oA} = {}^{A}R_{C}\,{}^{C}\phi_{oC} + {}^{A}R_{C}(-{}^{C}p_{A} \times {}^{C}L)$

$= \begin{bmatrix} {}^{A}R_{C} & {}^{A}R_{C}[{}^{A}p_{C}] \end{bmatrix} \begin{bmatrix} \phi_{oC} \\ {}^{C}L \end{bmatrix}$

$\boxed{{}^{A}h = {}^{A}X_{C}^{*}\,{}^{C}h}$

$\downarrow$

$\in \mathbb{R}^{6}$

Spatial Inertia

think about inertial matrix as mapping

$M \longrightarrow F$

- Inertia of a rigid body defines linear relationship between velocity and momentum.

twist $\mathcal{V}$ wrench momentum

- Spacial inertia $\mathcal{I}$ is the one such that

$$\boxed{\underbrace{h}_{6\times 1} = \underset{6\times 6}{\mathcal{I}}\,\underset{6\times 1}{\mathcal{V}}}$$

- Let $\{C\}$ be a frame whose origin coincide with CoM. Then

$${}^{C}\mathcal{V} = \begin{bmatrix} {}^{C}\omega \\ {}^{C}v_c \end{bmatrix}, \quad {}^{C}v_{com} = {}^{C}v_c \quad \Rightarrow {}^{C}l = m\,{}^{C}v_c$$

$${}^{C}\mathcal{I} = \begin{bmatrix} {}^{C}\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix}$$

$${}^{C}\phi_c = \bar{I}_c\,{}^{C}\omega$$

$${}^{C}h = \begin{bmatrix} {}^{C}\bar{I}_c\,{}^{C}\omega \\ m\,{}^{C}v_c \end{bmatrix} = \begin{bmatrix} {}^{C}\bar{I}_c & 0 \\ 0 & mI_3 \end{bmatrix} \begin{bmatrix} {}^{C}\omega \\ {}^{C}v \end{bmatrix}$$

Spatial Inertia

- Spatial inertia wrt another frame $\{A\}$: ${}^C X_A^T \, {}^C\mathcal{I} \, {}^C X_A$

$$ {}^A\mathcal{I} = {}^A X_C^* \, {}^C\mathcal{I} \, {}^C X_A $$

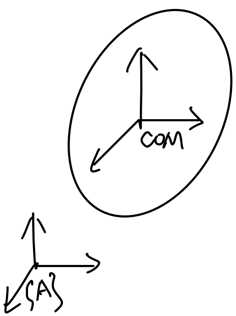
$$ {}^A h = {}^A\mathcal{I}\, {}^A v = {}^A X_C^* \, {}^C h = {}^A X_C^* \, {}^C\mathcal{I}\, {}^C v = {}^A X_C^* \, {}^C\mathcal{I}\, {}^C X_A \, {}^A v $$

$$ {}^A\mathcal{I} = {}^A X_C^* \, {}^C\mathcal{I} \, {}^C X_A $$

When $\{A\}$ has same orientation of com frame ${}^C\bar{\mathcal{I}}_C$

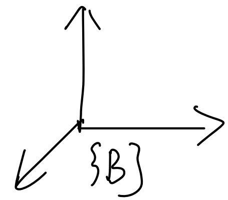
- Special case: ${}^A R_C = I_3$

we know ${}^A X_C = \begin{bmatrix} I_3 & 0 \\ [{}^A p_C] & I_3 \end{bmatrix} \Rightarrow$



$$ {}^A\mathcal{I} = \begin{bmatrix} {}^C\bar{\mathcal{I}} + m[{}^A p_C][{}^A p_C]^T & m[{}^A p_C] \\ m[{}^A p_C] & m I_{3\times 3} \end{bmatrix} $$

Outline



- Spatial Acceleration
- Spatial Force (Wrench)
- Spatial Momentum

X_B^*

$$\tilde{F} = \begin{bmatrix} e_{B1}^* & e_{B2}^* & \cdots & e_{B6}^* \end{bmatrix} {}^B\tilde{F} \quad \hookrightarrow \begin{bmatrix} \vdots \end{bmatrix}$$

force space basic vector 力基向量

elementary "force/torque"

- **Newton-Euler Equation using Spatial Vectors**

$$\dot{\tilde{F}} = \dot{X}_B^* \, {}^B\tilde{F} + X_B^* \underbrace{({}^B\tilde{F})'}_{{}^B\mathring{\tilde{F}}}$$

It turns out (if {B} has velocity v_B)

Cross Product for Spatial Force and Momentum

Assume frame A is moving with velocity ${}^{A}\mathcal{V}_{A}$

- ${}^{A}\left[\frac{d}{dt}\mathcal{F}\right] = \underbrace{\frac{d}{dt}\left({}^{A}\mathcal{F}\right)}_{{}^{A}\mathring{\mathcal{F}}\text{: apparent}} + {}^{A}\mathcal{V}_{A} \times^{*} {}^{A}\mathcal{F}_{A}$

  coordinate free

- ${}^{A}\left[\frac{d}{dt}h\right] = \frac{d}{dt}\left({}^{A}h\right) + {}^{A}\mathcal{V} \times^{*} {}^{A}h$

Fact $[\mathcal{V}\times^{*}] = -[\mathcal{V}\times]^{T}$

$\dot{\mathcal{F}} = X_B^{*}\, {}^{B}\mathring{\mathcal{F}} + [\mathcal{V}_B \times^{*}] X_B^{*}\, {}^{B}\mathcal{F}$

then $\dot{e}_{Bi}^{*} = \boxed{\mathcal{V}_B \times^{*}} e_{Bi}^{*}$

Where "$\times^{*}$" defines as

$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix}, \quad \mathcal{F} = \begin{bmatrix} n \\ f \end{bmatrix}$

$\mathcal{V} \times^{*} \mathcal{F} = \begin{bmatrix} \omega \times n + v \times f \\ \omega \times f \end{bmatrix}$

or equivalently

$[\mathcal{V}\times^{*}] = \begin{bmatrix} [\omega] & [v] \\ 0 & [\omega] \end{bmatrix}$

6×6

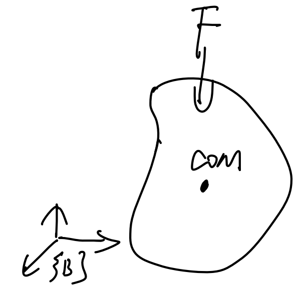
$\Rightarrow \quad \dot{X}_B^{*} = [\mathcal{V}_B \times^{*}] X_B^{*}$

Newton-Euler Equation

- Newton-Euler equation:

net external wrench

$$\boxed{\mathcal{F} = \frac{d}{dt}h = \underline{\mathcal{I}\mathcal{A} + \mathcal{V} \times^* \mathcal{I}\mathcal{V}}}$$



- Adopting spatial vectors, the Newton-Euler equation has the same form in any frame

• $\underline{{}^B\mathcal{F} = {}^B\mathcal{I}\,{}^B\mathcal{A} + {}^B\mathcal{V} \times^* {}^B\mathcal{I}\,{}^B\mathcal{V}}$ (even when {B} is away from COM)

• ${}^B(\mathcal{F}) = {}^B\left(\frac{d}{dt}h\right) = {}^B\left(\frac{d}{dt}(\mathcal{I}\mathcal{V})\right) = {}^B(\mathcal{I}\mathcal{A} + \dot{\mathcal{I}}\mathcal{V})$

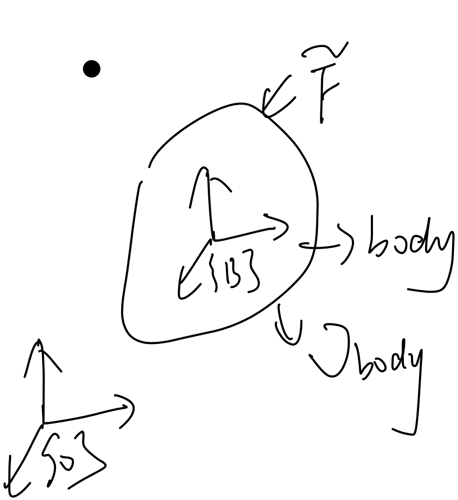
$= \underline{\mathcal{I}\mathcal{A}} + \mathcal{V} \times^* \mathcal{I}\mathcal{V}$

due to $\mathcal{V}$ (velocity) is changing

Accounts for the fact that inertial is moving

let's work with inertial frame {0} to derive the NE-Equation

Derivations of Newton-Euler Equation

- 

Assume $\{B\}$ attached to the body

$\Rightarrow v_B = v_{body}$, ${}^B I$ is constant

(只与形状有关)

$$\frac{d}{dt}({}^0 h) = \frac{d}{dt}({}^0 I\, {}^0 v) = {}^0 \dot{I}\, {}^0 v + {}^0 I A$$

$$= \frac{d}{dt}({}^0 X_B^{*}\, {}^B I\, {}^B X_0)\, {}^0 v + {}^0 I\, {}^0 A$$

$$= {}^0 \dot{X}_B^{*}\, {}^B I\, {}^B X_0\, {}^0 v + {}^0 X_B^{*}\, {}^B I\, {}^B \dot{X}_0\, {}^0 v + {}^0 I\, {}^0 A$$

${}^0 v = v_B$

$$= [{}^0 v_B \times^{*}]\, {}^0 X_B^{*}\, {}^B I\, {}^B X_0\, {}^0 v + \underbrace{{}^0 X_B^{*}\, {}^B I \left(-{}^B X_0 [{}^0 v_B \times]\right) {}^0 v}_{=0} + {}^0 I\, {}^0 A$$

$$= [{}^0 v_B \times^{*}]\, {}^0 I\, {}^0 v + {}^0 I\, {}^0 A = {}^0 I\, {}^0 A + [{}^0 v_B \times^{*}]\, {}^0 I\, {}^0 v$$

Note: ${}^0 \dot{X}_B = [{}^0 v_B \times]\, {}^0 X_B$ ${}^0 X_B\, {}^B X_0 = I \Rightarrow {}^0 \dot{X}_B\, {}^B X_0 + {}^0 X_B \cdot {}^B \dot{X}_0 = 0$

${}^B \dot{X}_0 = -{}^B X_0\, {}^0 \dot{X}_B\, {}^B X_0$

$= -{}^B X_0 [{}^0 v_B \times]$

More Discussions

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More Discussions

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