Advanced Control for Robotics - Homework 2

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$\mathbf{1.}$ Show that for any matrix $A \in R^{n\times n}$, the infinite seriese $e^A = I + A + \frac{A^2}{2!} + \cdots$ converges.

Proof.

We define the sequence of partial sums is $Sn = \sum_{k=0}^{n} \frac{1}{k!}A^k$ and the sum of infinite series is $S = e^A = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$.

We know the sub-multiplicative matrix norm: $\|AB\| \leq \|A\|\|B\|$ and $\|A+B\| \leq \|A\| + \|B\|$ for all matrices A and B in $K^{n\times n}$.

$$\|S - S_n\| = \|\sum_{k=n+1}^{\infty} \frac{1}{k!}A^k\| \leq \sum_{k=n+1}^{\infty} \frac{1}{k!}\|A^k\| \leq \sum_{k=n+1}^{\infty} \frac{1}{k!}\|A\|^k$$

Since the $\|A\|$ is real number $\sum_{0}^{\infty} \frac{1}{k!}\|A\|^k$ converges to $e^{\|A\|}$, the bound on the right approaches zero. So the S_n converges to S.

We can know the infinite seriese $e^A = I + A + \frac{A^2}{2!} + \cdots$ converges.

$\mathbf{2.}$ Given a linear system $x = Ax + Bu$. Its solution is given by

$$x(t) = e^{At}x0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Now assume we have $u(t) \equiv uk$ for $t \in [k\delta t, (k+1)\delta t)$. Please derive the Zero-order-hold discretization rule, namely, derive expressions for A_d and B_d such that

$$x_{k+1} = A_d x_k + B_d u_k$$

where $x_k \triangleq x(k\cdot\delta t)$ and $u_k = u(k\cdot\delta t)$

Solution

According to the solution $x(t) = e^{At}x0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$.

For $t \in [k\delta t, (k+1)\delta t), \quad u(t) \equiv uk$, the solution can be expressed as

$$
\begin{aligned}
x(t) &= e^{A(t-k\delta t)}x(k\delta t) + \int_{k\delta t}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\
&= e^{A(t-k\delta t)}x_k + \int_{k\delta t}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\
&= e^{A(t-k\delta t)}x_k - A^{-1}(e^{A(t-\tau)}|_{k\delta t}^{t})Bu_k
\end{aligned}
$$

At $t = (k+1)\delta t$, the solution is

$$
x((k+1)\delta t) = x_{k+1} = e^{A\delta t}x_k + A^{-1}(e^{A\delta t} - I)Bu_k
$$

Compared to the $x_{k+1} = A_d x_k + B_d u_k$, we can know the

$$
A_d = e^{A\delta t}
$$

$$
B_d = A^{-1}(e^{A\delta t} - I)B
$$

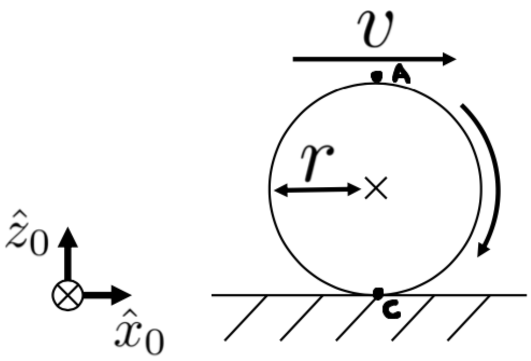
According to the matrix exponential formula, we can get

$$
\begin{aligned}
A_d &= \sum_{n=0}^{\infty} \frac{A^n \delta t^n}{n!} = I + A\delta t + \frac{A^2\delta t^2}{2!} + \dots \\
B_d &= \sum_{n=0}^{\infty} \frac{A^n \delta t^{n+1}}{(n+1)!} B = \left(\delta t + \frac{A\delta t^2}{2!} + \dots\right) B
\end{aligned}
$$

3. Spatial Velocity: A cylinder rolls without slipping in the $\hat{x}_0$ direction on the $\hat{x}_0 - \hat{y}_0$ plane. The cylinder has a radius of r and a constant forward speed of v. Let ${}^0C = [Cx(t), 0, 0]^T$ be the position of the contact point at time t. Let ${}^0A = [Ax(t), 0, 2r]^T$ be the position of the instantaneous top of the cylinder at time t.

a) What is the linear velocity of the point C? (hint: just need to compute $\frac{d}{dt}C_x(t)$) ?
b) What is the linear velocity of the point A?
c) What is velocity of the body-fixed point currently coincides with C?
d) What is velocity of the body-fixed point currently coincides with A?
e) What is the spatial velocity of the cylinder in $\{0\}$-frame?
f) What is the spatial velocity of the cylinder in frame $\{C\}$? ($\{C\}$ has the same orientation as $\{0\}$, while its origin is at the contact point C)

Note: The first 4 questions are all referring to the inertia frame $\{0\}$.



Solution

a) Since the point c is the contact point at time t, the linear velocity is 0, $\boldsymbol{v_c} = [0,0,0]^T$. The linear velocity of point C, $\boldsymbol{v_c} = [\frac{dc_x(t)}{dt}, 0, 0]^T = [0,0,0]^T$.

b)The forward velocity of the center of the circle is v, the angular velocity $w = \frac{v}{r}$.
The linear velocity of point C, $\boldsymbol{v_A} = [\frac{dA_x(t)}{dt}, 0, \frac{d2r}{dt}]^T = [2v, 0, 0]^T$.

c)$\boldsymbol{^o v} = \boldsymbol{^o w} \times \boldsymbol{co} = [C_x(t), 0, 0]^T \times [0, \frac{v}{r}, 0]^T = [0, 0, \frac{C_x(t)v}{r}]^T$.
The twist $^0\nu = [0, \frac{v}{r}, 0, 0, 0, \frac{C_x(t)v}{r}]^T$. Linear velocity of c in frame $\{o\}$ is $\boldsymbol{^O v_c} = \boldsymbol{w} \times \boldsymbol{^0 c} + \boldsymbol{^o v} = [0, 0, \frac{(c_x(t) - c_x(t))v}{r}]^T = [0,0,0]^T$.

d)The twist $^0\nu = [0, \frac{v}{r}, 0, 0, 0, \frac{C_x(t)v}{r}]^T$. Linear velocity of A in frame $\{o\}$ is $\boldsymbol{^O v_A} = \boldsymbol{w} \times \boldsymbol{^0 A} + \boldsymbol{^o v} = [2v, 0, \frac{(c_x(t) - A_x(t))v}{r}]^T = [2v, 0, 0]^T$.

e) $\boldsymbol{^o w} = [0, \frac{v}{r}, 0]^T$, so $\boldsymbol{^o v} = \boldsymbol{^o w} \times \boldsymbol{co} = [C_x(t), 0, 0]^T \times [0, \frac{v}{r}, 0]^T = [0, 0, \frac{C_x(t)v}{r}]^T$.
Spatial velocity in $\{o\}$ frame is $^o\nu = [0, \frac{v}{r}, 0, 0, 0, \frac{C_x(t)v}{r}]^T$.

f)Assume the frame$\{C\}$ is the same direction of frame $\{o\}$ with origin point at the point c. Spatial velocity in $\{C\}$ frame is $^o\nu = [0, \frac{v}{r}, 0, 0, 0, 0]^T$.

4. SpatialVelocity: Modern Robotics: Exercise 5.5

a) The position of p, $p = (L + dsin\theta, L - dcos\theta, 0)$.

b) The velocity of p, $\dot{p} = \frac{dp}{dt} = \frac{dp}{d\theta}\frac{d\theta}{dt} = (\dot{\theta}dcos\theta, \dot{\theta}dsin\theta, 0)$.
$\dot{\theta} = 1$, $\dot{p} = \frac{dp}{dt} = \frac{dp}{d\theta}\frac{d\theta}{dt} = (dcos\theta, dsin\theta, 0)$.

c) The configuration of frame $\{b\}$ $T_{sb}(R, p)$.

$$R = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, p = [L + dsin\theta, L - dcos\theta, 0]^T$$

$$T_{sb} = \begin{bmatrix} cos\theta & -sin\theta & 0 & L + dsin\theta \\ sin\theta & cos\theta & 0 & L - dcos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

d) Assume the rotation point is L. The velocity in body coordinates, ${}^{\boldsymbol{b}}\boldsymbol{v} = {}^{\boldsymbol{b}}\boldsymbol{w} \times^b \boldsymbol{Lp} = [0,0,1]^T \times [0,-d,0]^T = [d,0,0]^T$

The twist in body coordinates ${}^b\nu = [0,0,1,d,0,0]^T$.

e) The velocity in spatial coordinates, ${}^{\boldsymbol{s}}\boldsymbol{v} = {}^{\boldsymbol{s}}\boldsymbol{w} \times (-\boldsymbol{OL}) = [0,0,1]^T \times [-L,-L,0]^T = [L,-L,0]^T$

The twist in body coordinates ${}^b\nu_b = [0,0,1,L,-L,0]^T$.

f) We know that ${}^{\boldsymbol{s}}\boldsymbol{w} = R^{\boldsymbol{b}}\boldsymbol{w}$.

$$\begin{aligned} {}^{\boldsymbol{s}}\boldsymbol{v} &= {}^{\boldsymbol{s}}\boldsymbol{v_b} + {}^{\boldsymbol{s}}\boldsymbol{w} \times {}^{\boldsymbol{s}}\boldsymbol{po} \\ &= R_{sb}{}^{\boldsymbol{b}}\boldsymbol{v_b} + (R_{sb}{}^{\boldsymbol{b}}\boldsymbol{w}) \times ({}^{\boldsymbol{s}}\boldsymbol{po}) \\ &= R_{sb}{}^{\boldsymbol{b}}\boldsymbol{v_b} + [{}^{\boldsymbol{s}}\boldsymbol{op}] \cdot (R_{sb}{}^{\boldsymbol{b}}\boldsymbol{w}) \\ &= R_{sb}{}^{\boldsymbol{b}}\boldsymbol{v_b} + [{}^o p] \cdot (R_{sb}{}^{\boldsymbol{b}}\boldsymbol{w}) \end{aligned}$$

$${}^s\nu = \begin{bmatrix} R_{sb} & 0 \\ [\boldsymbol{p}]R_{sb} & R_{sb} \end{bmatrix} {}^b\nu$$

g) The relationship between (b) and (d) is: $\dot{p} = R_{sb}{}^{\boldsymbol{b}}\boldsymbol{v}$.

h) The relationship between (b) and (e) is:

$$\begin{aligned} {}^{\boldsymbol{s}}\boldsymbol{v} &= R_{sb}{}^{\boldsymbol{b}}\boldsymbol{v_b} + [\boldsymbol{p}] \cdot (R_{sb}{}^{\boldsymbol{b}}\boldsymbol{w}) \\ &= R_{sb}R_{sb}^{-1}\dot{p} + [\boldsymbol{p}] \cdot (R_{sb}{}^{\boldsymbol{b}}\boldsymbol{w}) \\ &= \dot{p} + [\boldsymbol{p}] \cdot ({}^{\boldsymbol{s}}\boldsymbol{w}) \end{aligned}$$

$${}^s\nu = \begin{bmatrix} R_{sb} & 0 \\ [\boldsymbol{p}]R_{sb} & I_{3x3} \end{bmatrix} \begin{bmatrix} {}^{\boldsymbol{b}}\boldsymbol{w} \\ \dot{\boldsymbol{p}} \end{bmatrix}$$

$\mathbf{5.}$ Screw axis and its transformation:

a) Draw the screw axis for the twist $V = (0,2,2,4,0,0)$

b) Consider an arbitrary screw axis S. Suppose the axis has gone through a rigid body transformation $T = (R,p)$ and the resulting new screw axis is $S_\prime$. Show that

$$S' = [Ad_T]S$$

(we have given the proof in class, you need to go through it on your own again)

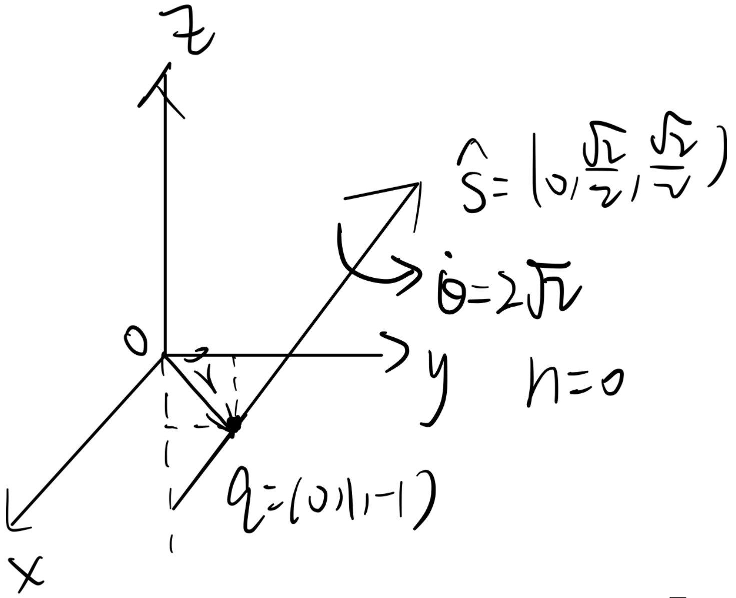
c) Consider a rigid body motion: rotation about z axis counterclockwise by 90° and then translate along negative y-axis by 1m. All the axes are with respect to the fixed inertia frame.

i. Find the numerical values of the corresponding transformation matrix T;

ii. Move the screw axis in part (a) using T. Find the new screw axis S′ after the motion.

Solution

a) From twist to screw motion, $\hat{\boldsymbol{s}} = \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = [0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]^T$, $\dot{\theta} = \|\boldsymbol{w}\| = 2\sqrt{2}$, $\boldsymbol{r} = q = \frac{\boldsymbol{w}\times\boldsymbol{v}}{\|\boldsymbol{w}\|^2} = [0, 1, -1]^T$, $h = \frac{\boldsymbol{w}^T\cdot\boldsymbol{v}}{\|\boldsymbol{w}\|^2} = [0, 1, -1]^T = \frac{0}{8} = 0$.



b)Assume the twist of the screw motion S and the S' is $\nu = [\boldsymbol{w}, \boldsymbol{v}]^T$ and $\nu' = [\boldsymbol{w'}, \boldsymbol{v'}]^T$ expressed in the same frame. The parameter of S is $\{\hat{\boldsymbol{S}}, h, q, \dot{\theta}\}$, and the parameter of S' is $\{\hat{\boldsymbol{S'}}, h, q', \dot{\theta}\}$.

We can easily know that $\hat{\boldsymbol{S'}} = R\hat{\boldsymbol{S}}$, $\boldsymbol{oq'} = \boldsymbol{oq} + \boldsymbol{p}$.

For the screw axis S, ${}^{\boldsymbol{0}}\boldsymbol{v} = h\dot{\theta}\hat{\boldsymbol{S}} + \boldsymbol{oq} \times (\hat{\boldsymbol{S}}\dot{\theta})$.

For the new screw axis S',

$$\begin{aligned}
{}^{\boldsymbol{0}}\boldsymbol{v'} &= h\dot{\theta}\hat{\boldsymbol{S'}} + \boldsymbol{oq'} \times (\hat{\boldsymbol{S'}}\dot{\theta}) \\
&= h\dot{\theta}R\hat{\boldsymbol{S}} + (\boldsymbol{oq} + \boldsymbol{p}) \times (R\hat{\boldsymbol{S}}\dot{\theta}) \\
&= h\dot{\theta}R\hat{\boldsymbol{S}} + \boldsymbol{oq} \times (R\hat{\boldsymbol{S}}\dot{\theta}) + \boldsymbol{p} \times (R\hat{\boldsymbol{S}}\dot{\theta}) \\
&= R^{\boldsymbol{o}}\boldsymbol{v} + [\boldsymbol{p}]R\hat{\boldsymbol{S}}\dot{\theta}
\end{aligned}$$

In matrix form,

$$\begin{bmatrix} \hat{\boldsymbol{S'}}\dot{\theta} \\ {}^{\boldsymbol{0}}\boldsymbol{v'} \end{bmatrix} = \begin{bmatrix} R & 0 \\ [\boldsymbol{p}]R & R \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{S}}\dot{\theta} \\ {}^{\boldsymbol{0}}\boldsymbol{v} \end{bmatrix}$$

$$S'\dot{\theta} = [Ad_T]S\dot{\theta}$$

That is to prove :

$$S' = [Ad_T]S$$

c)

i)The rotation matrix

$$\begin{aligned} Rot(\hat{\boldsymbol{w}}, \theta) = e^{[\boldsymbol{w}]\theta} &= I + [\boldsymbol{w}]\theta + \frac{1}{2!}([\boldsymbol{w}]\theta)^2 + ... \\ &= I + [\boldsymbol{w}]sin\theta + [\boldsymbol{w}]^2(1 - cos\theta) \end{aligned}$$

Therefore,

$$Rot(\hat{[\mathbf{0}, \mathbf{0}, \mathbf{1}]^{\boldsymbol{T}}}, 90°) = I + [\boldsymbol{w}]sin(-90°) + [\boldsymbol{w}]^2(1 - cos(-90°))$$

$$Rot(\hat{[\mathbf{0}, \mathbf{0}, \mathbf{1}]^{\boldsymbol{T}}}, 90°) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix is

$$T(R, p) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii) We can know $[Ad_T] = \begin{bmatrix} R & 0 \\ [\boldsymbol{p}]R & R \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$S' = [Ad_T] \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ \sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ \sqrt{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$$