Advanced Control for Robotics - Homework 5

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$\mathbf{1.}$ Given a linear system $x' = Ax$ and a quadratic function $V(x) = x^T Px$, where P is an $n \times n$ symmetric matrix. Derive the conditions for P under which V will be a Lyapunov function for exponential stability that satisfies $\|x(t)\|^2 \leq \beta c^t \|x(0)\|^2$, where $c \in (0, 1)$.

Solution:

As for the definition of globally exponentially stability, there exit posotive constants δ, λ, c

$$|x(t)| \leq C|x(0)|e^{-\lambda t}, \forall |x(0)| \leq \delta$$

$\|x(t)\|^2 \leq \beta c^t \|x(0)\|^2$, When

For linear system $\dot{x} = Ax$, its solution is $x(t) = e^{At}x(0)$.

For $\|x(t)\|^2 \leq \beta c^t \|x(0)\|^2$, when $c = e^{-2\lambda}, \beta = C^2$, the equation is satisfied, so it is equivilent to proove the exponential stability.

For a linear system $x' = Ax$, $e^{At} = Te^{\lambda t}T^{-1}$, if the eigen value λ satisfy the condition $Re(\lambda) < 0$, $\lim\limits_{t \to 0} e^{At}x(0) = 0$ then the system is exponentially stable.

Therefore, for linear system which is expoentially stable, that is equivilent to symptotically stable.

The condition of P is (1) : P is positive definite. (2) : $\dot{V}(x) = \dot{x}^T Px + x^T P\dot{x} = x^T(A^T P + PA)x < 0$, that is $A^T P + PA$ is negative definite.

$\mathbf{2.}$ Show that the system $\dot{x} = f(x) = \begin{cases} \dot{x_1} = -x_1 + x_1 x_2 \\ \dot{x_2} = -x_2 \end{cases}$ is globally asymptotically stable (hint: try $V(x) = ln(1 + x_1^2) + x_2^2$ as a Lyapunov function

Solution

$f(x) = 0$, for equilibrium, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

For $V(x) = ln(1 + x_1^2) + x_2^2$, check lyapunov conditions.

(1) $\forall x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0, V(x) > 0$. For $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0, V(x) = 0$, the scalar function $V(x)$ is positive definite (PD).

(2)

$$
\begin{aligned}
LfV(x) = (\frac{\partial V}{\partial x})^T f(x) &= \begin{bmatrix} \frac{2x_1}{x_1^2+1} & 2x_2 \end{bmatrix} \begin{bmatrix} -x_1 + x_1x_2 \\ -x_2 \end{bmatrix} \\
&= \frac{-2x_1^2 + 2x_1^2x_2}{x_1^2+1} - 2x_2^2 \\
&= \frac{-2x_1^2 - 2x_2^2}{x_1^2+1}
\end{aligned}
$$

$LfV(x)$ is ND.

Therefore, we can prove the system is globally asymptotically stable.

3. Consider a discrete time system $x(k+1) = Ax(k) + Bu(k)$, with linear feedback law $u(k) = -Kx(k)$. Write down the closed-loop dynamics, and derive conditions for $V(x) = x^T Px$ to be discrete time Lyapunov function for asymptotic closed-loop stability.

Solution

The closed-loop discrete time system is $x(k+1) = (A - BK)x(k)$.

The initial condition of the system is $x(0)$, when $k = 1$, $x(1) = (A - BK)x(0)$; when $k = 2$, $x(2) = (A - BK)^2 x(0)$.

We can know the dynamics of cloosed-loop system, $x(K) = (A - BK)^k x(0)$.

For system to be asymptotically stable, the conditions is $(1): V(x)$ is PD. (2) Rate of change of a function $V(x)$ along the system trajectory $\triangle_f V(x) < 0$.

That means for (1): P is PD matrix.
For (2):

$$
\begin{aligned}
\triangle_f V(x) &= V(x(K+1)) - V(x(k)) = V((A-BK)x) - V(x) \\
&= x^T \left[(A-BK)P(A-BK) - P \right] x < 0
\end{aligned}
$$

that is $\left[(A-BK)P(A-BK) - P \right]$ is ND matrix.

4. Show that the PSD cone is acute, i.e., $\forall A, B \in Sn^+$, we have $tr(AB) \geq 0$. (Hint: decompose A using unitary matrix Q, i.e. $A = Q\Lambda Q^T$, and then use the same Q to define another matrix $C =$

QBQ^T . The trace $tr(AB)$ can be computed directly in terms of the entries in C and Λ)

Proof

According to the "Spectral decomposition", for $\forall A \in Sn$, there exit a unitary matrix Q and a diagonal matrix Λ satisfy $A = Q\Lambda Q^T$.

Then $A = Q\Lambda Q^T$ and use the same Q matrix to define a matrix C, $C = QBQ^T$.

According to the trace property, $tr(A) = tr(\Lambda)$, then $tr(AC) = tr(Q\Lambda Q^T QBQ^T) = tr(QABQ^T)$, for $QQ^T = I$.

Similarity transformation do not change the trace, then $tr(AC) = tr(QABQ^T) = tr(AB)$.

Since the A, B are the positive semidefinite matrix, then C is also PSD, that means the eigenvalues of A, C are non-negative.

$$tr(AB) = tr(AC) \geq 0$$

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5. Given a symmetric matrix $A \in S_n$, let λmin(A) and λmax(A) be the smallest and largest eigenvalues of A. Show that

$$\begin{cases} \lambda min(A) \geq \mu \\ \lambda max(A) \leq \beta \end{cases} \Leftrightarrow \mu I \preceq A \preceq \beta I$$

Proof

For the left inequality, $\mu I \preceq A$ is equivilent to show $A - \mu I \geq 0$.
The eigenvalue of $A - \mu I$ is $eig(A - \mu I) = \{\lambda_1 - \mu, \lambda_2 - \mu, ..., \lambda_n - \mu\}$.
The minimum of the eigenvalue is

$$\underset{i}{min}(\lambda - \mu) = \underset{i}{min}\, \lambda - \mu = 0$$

Therefore, $eig(A - \mu I) \geq 0$, $A - \mu I$ is PSD, that is $\mu I \preceq A$.

For the right inequality, $A \preceq \beta I$ is equivilent to show $\beta I - A \geq 0$.
The eigenvalue of $\beta I - A$ is $eig(\beta I - A) = \{\beta - \lambda_1, \beta - \lambda_2, ..., \beta - \lambda_n\}$.
The minimum of the eigenvalue is

$$\underset{i}{min}(\beta - \lambda) = \beta - \underset{i}{min}\, \lambda = 0$$

Therefore, $eig(\beta I - A) \geq 0$, $\beta I - A$ is PSD, that is $A \preceq \beta I$.

$\mathbf{6.}$ Suppose $fi : R_n \to R, i = 1, 2$ are convex. Show that the pointwise maximum function $f(x) = max f_1(x), f_2(x)$ is also convex.

Proof

Pointwise maximum function $f(x) = max\ f_1(x), f_2(x)$

For convex function f_i, for $\alpha \in (0, 1)$, pick any $x_1, x_2 \in D$,

(for some $i \in 1, 2$)

$$\begin{aligned} f(\alpha x_1 + (1-\alpha)x_2) &= f_i(\alpha x_1 + (1-\alpha)x_2) \\ &< \alpha f_i(x_1) + (1-\alpha)f_i(x_2) \\ &< \alpha f(x_1) + (1-\alpha)f(x_2) \end{aligned}$$

Therefore, pointwise maximum function $f(x) = max\ f_1(x), f_2(x)$ is convex function.