MEE5114(Sp22) Advanced Control for Robotics

Lecture 1: Linear Differential Equations and Matrix Exponential

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Homework 20%

Midterm 25%

Mini-project 15%

Final exam 30%

Quiz 10%

Modeling (Kinematics, Dynamics)
Screw-theory

Control
- optimization
- CLF
- motion control
- optimal control

Advanced topic

Outline

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

Motivations

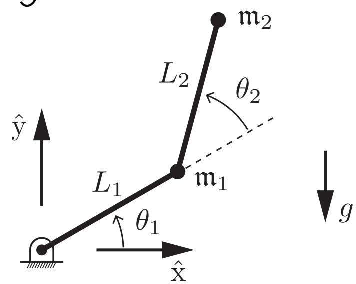
- Most engineering systems (including most robotic systems) are modeled by Ordinary Differential (or Difference) Equations (ODEs)

- Example: Dynamics of 2R robot

differential equation in θ

$$\tau = M(\theta)\ddot{\theta} + \underbrace{c(\theta,\dot{\theta}) + g(\theta)}_{h(\theta,\dot{\theta})},$$

2^{nd}-derivatives of θ



with

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

$$c(\theta,\dot{\theta}) = \begin{bmatrix} -\mathfrak{m}_2 L_1 L_2 \sin\theta_2(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2) \\ \mathfrak{m}_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \end{bmatrix},$$

$$g(\theta) = \begin{bmatrix} (\mathfrak{m}_1 + \mathfrak{m}_2)L_1 g\cos\theta_1 + \mathfrak{m}_2 g L_2\cos(\theta_1+\theta_2) \\ \mathfrak{m}_2 g L_2 \cos(\theta_1+\theta_2) \end{bmatrix},$$

- Screw theory, exponential coordinate, and Product of Exponential (PoE) are based on the (linear) differential equation view of robot kinematics

Linear Differential Equations (Autonomous)

- Linear Differential Equations: ODEs that are linear wrt variables e.g.:

$$① \begin{cases} \dot{x}_1(t) + x_2(t) = 0 \\ \dot{x}_2(t) + x_1(t) + x_2(t) = 0 \end{cases} \qquad ② \begin{cases} \ddot{y}(t) + z(t) = 0 \\ \dot{z}(t) + y(t) = 0 \end{cases}$$

two couple 1 order ODE $x_1(t), x_2(t)$

vector form $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ in $\mathbb{R}^2$

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \Rightarrow \dot{x} = Ax$$

$x_1(t) = y(t)$, $x_2(t) = \dot{y}(t)$, $x_3(t) = z(t)$

$$\underbrace{\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix}}_{\dot{x}(t)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}}_{x(t)}$$

- State-space form (1st-order ODE with vector variables):

Linear: $\dot{x} = Ax$

vector field

General Linear Control Systems

$\Rightarrow$ if $f(x) = Ax$

- General (Autonomous) Dynamical Systems: $\dot{x}(t) = f(x(t))$ $\Rightarrow$ Linear sys: $\dot{x} = Ax$
  - $x(t) \in \mathbb{R}^n$: state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$: *vector field*

"Autonomous" means "f" does not depend on non-x variable

$\begin{cases} \dot{x} = Ax \\ \dot{x} = Ax + b \end{cases}$ only depend on the state-vector

- Non-autonomous: $\dot{x}(t) = f(x(t), t)$ captures all non-x dependence

$\dot{x} = Ax + 2t$

- Control Systems: $\dot{x}(t) = f(x(t), u(t))$
  - vector field $f : \mathbb{R}^n \times \mathbb{R}^m$ depends on external variable $u(t) \in \mathbb{R}^m$

e.g. $\dot{x} = Ax + \sin u$

non-autonomous system

- General Linear Control Systems: $f(x,u) = Ax + Bu$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) & \leftarrow \text{Static relation} \end{cases}$$

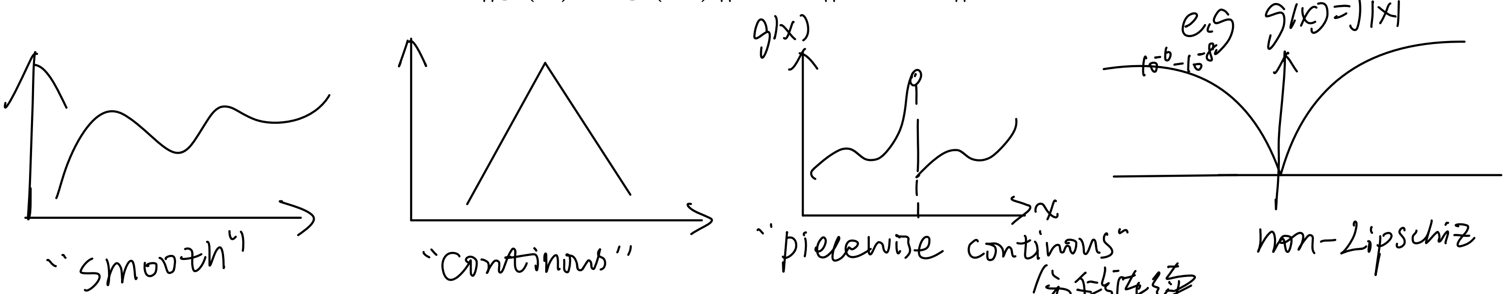
  - $x \in \mathbb{R}^n$: system state, $u \in \mathbb{R}^m$: control input, $y \in \mathbb{R}^p$: system output
  - A, B, C, D are constant matrices with appropriate dimensions

$\|g(x_1)-g(x_2)\| \leq L\| x_1-x_2\|$

Existence and Uniqueness of ODE Solutions

- Function $g : \mathbb{R}^n \to \mathbb{R}^p$ is called *Lipschitz* over domain $\mathcal{D} \subseteq \mathbb{R}^n$ if $\exists\, L < \infty$

$$\|g(x) - g(x')\| \leq L\|x - x'\|, \forall x, x' \in \mathcal{D}$$



分段连续

- **Theorem [Existence & Uniqueness]** Nonlinear ODE

$$\dot{x}(t) = f(x(t), t), \quad \text{I.C. } x(t_0) = x_0$$

has a *unique* solution if $f(x, t)$ is Lipschitz in x and piecewise continuous in t

$\|f(x,t)-f(x',t)\| \leq L\|x-x'\|, \ \forall t \in [t_0, t_f]$

Solution to ① means
$$\begin{cases} \langle 1\rangle \ \text{I.C. } x(t_0)=x_0 \ \text{初始条件} \\ \langle 2\rangle \ \dot{x}(t)=f(x(t),t), \ \forall t \end{cases}$$

Existence and Uniqueness of Linear Systems

- **Corollary**: Linear system

推论:

$$\Downarrow f(x,t) = Ax + Bu(t)$$

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \text{I.C. } x(t_0) = x_0, \; t(0) = t_0$$

has a unique solution for any piecewise continuous input $u(t)$

prof: Check condition

① $\|f(x,t) - \hat{f}(x',t)\| = \|A\| \, \|(x-x')\| \leq \|A\| \, \|x - x'\|$

② $\hat{f}(x,t) = Ax + Bu(t)$ is also piece-wise continous in t

x: fixed

because $u(t)$ is piecewise continuous in t.

- *Homework*: Suppose A becomes time-varying $A(t)$, can you derive conditions to ensure existence and uniqueness of $\dot{x}(t) = A(t)x(t) + Bu(t)$?

Outline

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

How to Solve Linear Differential Equations?

- General linear ODE: $\dot{x}(t) = Ax(t) + d(t)$

- The key is to derive solutions to the autonomous linear case: $\dot{x}(t) = Ax(t)$, with $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, and initial condition (IC) $x(0) = x_0$.

- By existence and uniqueness theorem, the ODE $\dot{x} = Ax$ admits a unique solution.

- It turns out that the solution can be found analytically via the *Matrix Exponential*

What is the "Euler's Number" e?

- Consider a scalar linear system: $z(t) \in \mathbb{R}$ and $a \in \mathbb{R}$ is a constant

$$\dot{z}(t) = az(t), \quad \text{with initial condition } z(0) = z_0 \tag{1}$$

- The above ODE has a unique solution: $z(t) = e^{at} \cdot z_0$

prof. check <1> I.C. $z(0) = z_0$

<2> vector field $\dot{z}(t) = a e^{at} \cdot z_0 = a z(t)$

- What is the number "e"?

Euler's number

Defined as the number such that $\underbrace{(e^x)' = e^x}$

$$\Rightarrow \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \Rightarrow \frac{e^h - 1}{h} \xrightarrow{h \to 0} 1$$

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- This can be extended to complex variables:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

  This power series is well defined for all $z \in \mathbb{C}$

- In particular, we have $e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2} - j\frac{\theta^3}{3!} + \cdots$

- Comparing with Taylor expansions for $\cos(\theta)$ and $\sin(\theta)$ leads to the Euler's Formula

$$\left.\begin{aligned} \sin\theta &= x - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} \\ \cos\theta &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} \cdots \end{aligned}\right\} \Rightarrow e^{i\theta} = \cos\theta + j\sin\theta$$

Euler's formular

Matrix Exponential Definition

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

E.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$e^A = I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- This power series is well defined for any finite square matrix $\underline{A \in \mathbb{R}^{n \times n}}$.

If $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ $\quad e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + \begin{bmatrix} \frac{\lambda_1^2}{2} & 0 \\ 0 & \frac{\lambda_2^2}{2} \end{bmatrix} + \cdots$

Some Important Properties of Matrix Exponential

① • $Ae^A = e^A A$ proof: By definition $Ae^A = A\sum_{i=1}^{\infty}\frac{A^i}{i!} =$

But remember $\underbrace{Ae^B \neq e^B \cdot A}$, if $AB \neq AB$

② • $e^A e^B = e^{A+B}$ if $\underbrace{AB = BA}$ ← See next page

③ • If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$ (P is nonsingular)

A is similar to D

$e^A = I + PDP^{-1} + \frac{PDP^{-1}PDP^{-1}}{2!} + \cdots = Pe^D P^{-1}$ → PD^2P^{-1}

④ • For every $t, \tau \in \mathbb{R}$, $e^{At}e^{A\tau} = e^{A(t+\tau)}$

from ②: $e^{At+A\tau} =$

⑤ • $\left(e^A\right)^{-1} = e^{-A}$

From ② ⇒ $e^A \cdot e^{-A} = e^{A+(-A)} = e^0 = I$

Outline

proof of ②: $e^{A+B} = e^A e^B$ $(AB=BA)$

property $\binom{N}{k} = \frac{N!}{k!(N-k)!}$

$$e^{A+B} = \sum_{N=0}^{\infty} \frac{(A+B)^N}{N!} \overset{?}{=} \left(\sum_{0}^{\infty} \frac{A^N}{N!}\right)\left(\sum_{0}^{\infty} \frac{B^N}{N!}\right)$$

二项式定理

$$= \sum_{N=0}^{\infty} \frac{\sum_{k=0}^{N} \binom{N}{k} A^k B^{N-k}}{N!} = \sum_{k=0}^{\infty} \sum_{N=k}^{\infty} \frac{A^k B^{N-k}}{k!(N-k)!}$$

holds $AB=BA$

- Linear System Model
- Matrix Exponential
- Solution to Linear Differential Equations

Autonomous Linear Systems

$x \in \mathbb{R}^n$

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{2}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.

- With the definition of matrix exponential, we can show that the solution to (2) is given by

$$x(t) = e^{At}x_0 \quad \leftarrow \text{function of } t$$

proof: ① check I.C.

$t=0 \quad x(0) = e^{A \cdot 0}x_0 = I \cdot x_0 = x_0$ satifies the Initial condition

② check the vector field $\frac{d}{dt}(e^{At}x_0) \overset{?}{=} A(e^{At} \cdot x_0)$, we need to show this

By definition $e^{At}x_0 = \left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} \cdots\right)x_0$

$\frac{d}{dt}(e^{At}x_0) = \left(A + A^2 t + A^3 \frac{t^2}{2!} + \cdots\right)x_0$

$= A\left(I + At + \frac{A^2t^2}{2!} + \cdots\right)x_0 = Ae^{At}x_0$, satisfies the ....

$= Ax(t)$

Computation of Matrix Exponential (1/2)

- Directly from definition $e^{At} = \sum_{i=0}^{\infty} \frac{(At)^i}{i!}$
  - It's bad to compute
  - For special case, this series have analytical form.

  e.g. $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $e^{At} = I + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ not general

- For diagonalizable matrix:

Example $A = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix}$

$\Rightarrow A = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{P^{-1}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{P^{-1}}$

$= P^{-1} D P^{-1}$

$\Rightarrow$ By property ③ $e^{At} = P e^{Dt} P^{-1}$

$= P \begin{bmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{bmatrix} P^{-1} =$

Computation of Matrix Exponential (2/2)

- Using Laplace transform

$\hat{X}(s) = \int x(t) e^{-st} dt$

$\dot{x} = Ax$, I.C. $x_0 \in \mathbb{R}^n$

Laplace transform $x(t) \Longleftrightarrow \hat{X}(s) \in \mathbb{R}^n$

$\dot{x}(t) \Leftrightarrow s\hat{X}(s) - x(0)$

Apply $s\hat{X}(s) - x(0) = A\hat{X}(s) \Rightarrow (sI - A)\hat{X}(s) = x_0$

$\Rightarrow \hat{X}(s) = (sI - A)^{-1} x_0$

$\Rightarrow x(t) = \mathcal{L}^{-1}[(sI - A)^{-1} x_0]$

$x(t) = e^{At} x_0 \Rightarrow e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$

Solution to General Linear Systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad \text{with } x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{3}$$

- $x \in \mathbb{R}^n$ is system state, $u \in \mathbb{R}^m$ is control input, $y \in \mathbb{R}^p$ is the system output
- A, B, C, D are constant matrices with appropriate dimensions
- **Homework:** The solution to the linear system (3) is given by

$$\begin{cases} x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \end{cases}$$

More Discussions