MEE5114 Advanced Control for Robotics

Lecture 7: Velocity Kinematics: Geometric and Analytic Jacobian of Open Chain

Prof. Wei Zhang

CLEAR Lab

Department of Mechanical and Energy Engineering

Southern University of Science and Technology, Shenzhen, China

https://www.wzhanglab.site/

Outline

- Background
- Geometric Jacobian Derivations
- Analytic Jacobian

Velocity Kinematics

FK: Find the func of $T_b(\theta_1, \theta_2 \cdots \theta_n)$

$\theta_1, \cdots \theta_n \to T_b(\theta_1, \theta_2 \cdots \theta_n)$

Result: $T_b(\theta_1, \cdots \theta_n) = e^{[{}^0S_1]\theta} e^{[{}^0S_2]\theta} \cdots e^{[{}^0S_n]\theta} M$

${}^0S_i$: screw axis i when $\theta_i = 0$, express in $\{0\}$

- **Velocity Kinematics**: How does the velocity of $\{b\}$ relate to the joint velocities $\dot{\theta}_1, \ldots, \dot{\theta}_n$ Note: $\{b\}$'s velocity is due to joint velocity
- This depends on how to represent $\{b\}$'s velocity → Can we use $\dot{T}_b$ to represent velocity of $\{b\}$ → 4×4 NO
  - Twist representation → **Geometric Jacobian**

$\mathcal{V}_b = \begin{bmatrix} \omega \\ v \end{bmatrix}$ $\mathcal{V}_b(\theta, \dot{\theta})$: it turns out $\mathcal{V}_b$ is a linear func of $\dot{\theta}$

$\Rightarrow \mathcal{V}_b(\theta, \dot{\theta}) = J(\theta)\dot{\theta}$ → Geometric Jacobian

$J(\theta) \in \mathbb{R}^{6 \times n}$

  - Local coordinate of SE(3) → **Analytic Jacobian**

$\theta_1, \theta_2 \cdots \theta_n \xrightarrow{FK} T_b(\theta_1, \cdots \theta_n) = (R, p) \xrightarrow{(x,y,z)} \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \in \mathbb{R}^6$

RPY: α, β, γ

$g(\cdot)$

$x \in \mathbb{R}^6$

$x = g(\theta_1 \cdots \theta_n)$

$\dot{x} = \left[\frac{\partial g}{\partial \theta}\right] \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$ $\quad \frac{\partial g}{\partial \theta} = \frac{\partial g_i}{\partial \theta_i}$

analytic jacobian $\in \mathbb{R}^{6 \times n}$

Outline

For example

$$p(\theta) = \begin{bmatrix} p_x(\theta) \\ p_y(\theta) \\ p_z(\theta) \end{bmatrix}$$

$$\dot{p}(\theta) = \left[\frac{\partial p}{\partial \theta}\right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

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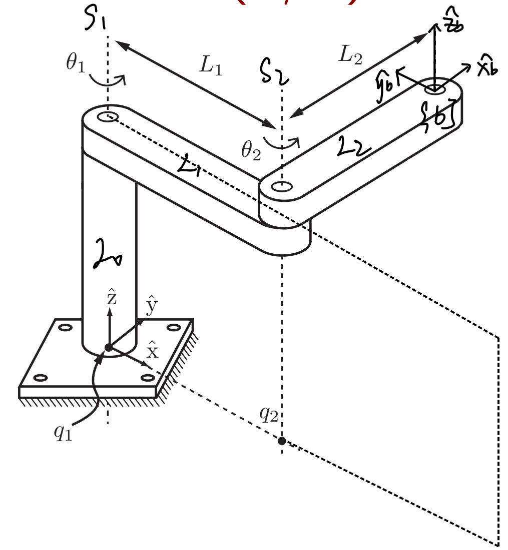
Simple Illustration Example: Geometric Jacobian (1/2)



– coordinate-free   Joint 1   Joint 2

screw axis: S_1 S_2

independent θ_1, θ_2

Spatial velocity of each link:

Link 0: $\mathcal{V}_{L0} = 0 \in \mathbb{R}^6$

Link 1: $\mathcal{V}_{L1} = S_1 \dot{\theta}$

Link 2: $\mathcal{V}_{L2} = \mathcal{V}_{L2/L1} + \mathcal{V}_{L1/L0} = S_2\dot{\theta}_2 + S_1\dot{\theta}_1$

$\mathcal{V}_{L2} \Downarrow \mathcal{V}_{L2/L0}$

$= [S_1 \;\; S_2(\theta_1)] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

$\{b\}: \mathcal{V}_b = \mathcal{V}_{L2} = [S_1 \; ; \; S_2(\theta_1)] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = [J_1(\theta) \;\; J_2(\theta)] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$

1st column of Geometric Jacobian

$J_i(\theta)$: the twist of $\{b\}$ when $\theta_i = 1$, $\theta_j = 0$, $i \neq j$

Simple Illustration Example: Geometric Jacobian (2/2)

- Computation: Let's work with $\{0\}$ frame, ${}^0S_1 = {}^0S_1(\theta=0) = {}^0\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

${}^0S_2(\theta_1) =$

Let $\theta_1 = 0$, ${}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -l_1 \\ 0 \end{bmatrix}$

$\theta_1 \neq 0$, ${}^0\bar{S}_2 = {}^0S_2(0) \xrightarrow{\hat{T}(\theta_1) = e^{[{}^0\bar{S}_1]\theta_1}}$

$$\boxed{\underbrace{{}^0S_2(\theta_1)}_{6\times 1} = \underbrace{\left[Ad_{\hat{T}(\theta_1)}\right]}_{6\times 6} \underbrace{{}^0\bar{S}_2}_{6\times 1}}$$

$\Rightarrow J(\theta) = \left[{}^0\bar{S}_1 \;\middle|\; Ad_{\hat{T}(\theta_1)} {}^0\bar{S}_2 \right]$

Geometric Jacobian: General Case (1/3)

- Let $\mathcal{V} = (\omega, v)$ be the end-effector twist (coordinate-free notation), we aim to find $J(\theta)$ such that

$\mathcal{V} = J \dot{\theta}$

We have n joints

$$\mathcal{V} = J(\theta)\dot{\theta} = J_1(\theta)\dot{\theta}_1 + \cdots + J_n(\theta)\dot{\theta}_n$$

$$= [J_1(\theta) \mid J_2(\theta) \mid \cdots \mid J_n(\theta)] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \vdots \\ \dot{\theta}_n \end{bmatrix}$$

Let $\dot{\theta}_i = 1$, $\dot{\theta}_j = 0$

- The ith column $J_i(\theta)$ is the end-effector *velocity* when the robot is rotating about $\mathcal{S}_i$ at unit speed $\dot{\theta}_i = 1$ while all other joints do not move (i.e. $\dot{\theta}_j = 0$ for $j \neq i$).

- Therefore, in **coordinate free** notation, J_i is just the screw axis of joint i:

$$J_i(\theta) = \mathcal{S}_i(\theta)$$

$[Ad_{T_1} S_1 \quad Ad_{T_2} S_2 \quad Ad_{T \cdots} S]$

${}^0\mathcal{V} = {}^0J \dot{\theta}$

Geometric Jacobian: General Case (2/3)

$\tau = {}^0J^T\,{}^0F$

linear

- The actual coordinate of $\mathcal{S}_i$ depends on θ as well as the reference frame.
- The simplest way to write Jacobian is to use local coordinate:

$$ {}^iJ_i = {}^iS_i, \quad i = 1, \dots, n $$

independent of θ

$${}^0J = [{}^0J_1\ {}^0J_2 \cdots {}^0J_n]$$

- In fixed frame $\{0\}$, we have

$$ {}^0J_i(\theta) = \boxed{{}^0X_i(\theta)}\,{}^iS_i, \quad i = 1, \dots, n \tag{1} $$

- Recall: ${}^0X_i$ is the change of coordinate matrix for spatial velocities.
- Assume $\theta = (\theta_1, \dots, \theta_n)$, then

$$ {}^0T_i(\theta) = e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_i]\theta_i} M \quad \Rightarrow \quad {}^0X_i(\theta) = \left[\mathrm{Ad}_{{}^0T_i(\theta)}\right] \tag{2} $$

pose of fram{i} to inertial frame {0}

Geometric Jacobian: General Case (3/3)

- The Jacobian formula (1) with (2) is conceptually simple, but can be cumbersome for calculation. We now derive a recursive Jacobian formula

- Note: ${}^0J_i(\theta) = {}^0S_i(\theta)$
  - For $i = 1$, ${}^0S_1(\theta) = {}^0S_1(0) = {}^0\bar{\mathcal{S}}_1$ (independent of θ)
  - For $i = 2$, ${}^0S_2(\theta) = {}^0S_2(\theta_1) = \left[\mathrm{Ad}_{\hat{T}(\theta_1)}\right] {}^0\bar{\mathcal{S}}_2$, where $\hat{T}(\theta_1) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1}$

    $\theta_1 = 0$

  - For $i = 3$, ${}^0S_3(\theta) = {}^0S_3(\theta_1, \theta_2) = \left[\mathrm{Ad}_{\hat{T}(\theta_1,\theta_2)}\right] {}^0\bar{S}_3$

    ${}^0\bar{S}_3 = {}^0S_3(0,0) \xrightarrow{e^{[{}^0\bar{S}_1]\theta_1} e^{[{}^0\bar{S}_2]\theta_2}} \left[\mathrm{Ad}_{\hat{T}(\theta_1,\theta_2)}\right] {}^0\bar{S}_3$

  - For general i, we have

$$
{}^0J_i(\theta) = {}^0S_i(\theta) = \left[\mathrm{Ad}_{\hat{T}(\theta_1,\ldots,\theta_{i-1})}\right] {}^0\bar{\mathcal{S}}_i
$$

$$
\text{where} \quad \hat{T}(\theta_1,\ldots,\theta_{i-1}) \triangleq e^{[{}^0\bar{\mathcal{S}}_1]\theta_1} \cdots e^{[{}^0\bar{\mathcal{S}}_{i-1}]\theta_{i-1}} \tag{3}
$$

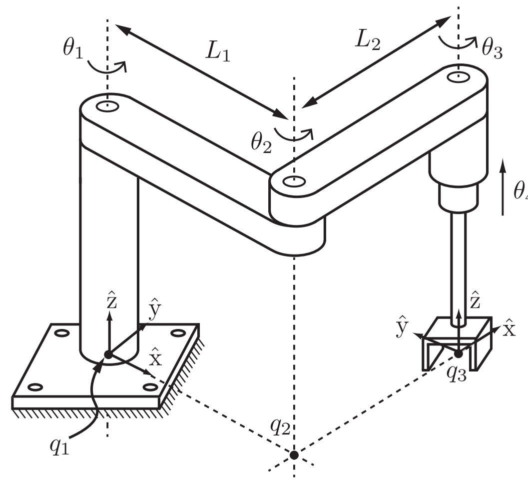
Geometric Jacobian Example

$J(\theta) = [\, S_1(\theta) \;\; S_2(\theta) \;\; S_3(\theta) \;\; S_4(\theta) \,]$

1° Find screw axis at home position $(\theta_1, \theta_2 \cdots \theta_n = 0)$

$${}^0\bar{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad {}^0\bar{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix} \quad {}^0\bar{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1+L_2) \\ 0 \end{bmatrix} \quad {}^0\bar{S}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$${}^0 w_3 = \omega \times (-{}^0 q_3)$$



$$2°: \; {}^0J(\theta) = \left[\, {}^0\bar{S}_1 \;\middle|\; [Ad_{\hat{T}_1}]\,{}^0\bar{S}_2 \;\middle|\; [Ad_{\hat{T}_2}]\,{}^0\bar{S}_3 \;\middle|\; [Ad_{\hat{T}_3}]\,{}^0\bar{S}_4 \,\right]$$

$$\hat{T}_1 = e^{[{}^0\bar{S}_1]\theta_1} \qquad \hat{T}_3 = e^{[{}^0\bar{S}_1]\theta_1}\, e^{[{}^0\bar{S}_2]\theta_2}\, e^{[{}^0\bar{S}_3]\theta_3}$$

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Analytic Jacobian

- Let $x \in \mathbb{R}^p$ be the task space variable of interest with desired reference x_d
  - E.g.: x can be Cartesian + Euler angle of end-effector frame

    ZYZ, ZYX, XYZ,
  - $p < 6$ is allowed, which means a partial parameterization of SE(3), e.g. we only care about the position or the orientation of the end-effector frame

$x = g(\theta)$ $\quad J_a = \frac{\partial g}{\partial \theta}$

- Analytic Jacobian: $\dot{x} = J_a(\theta)\dot{\theta}$
- Recall Geometric Jacobian: $\boxed{\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = J(\theta)\dot{\theta}}$
- They are related by:

$$J_a(\theta) = E(x)J(\theta) = \boxed{E(\theta)}J(\theta)$$

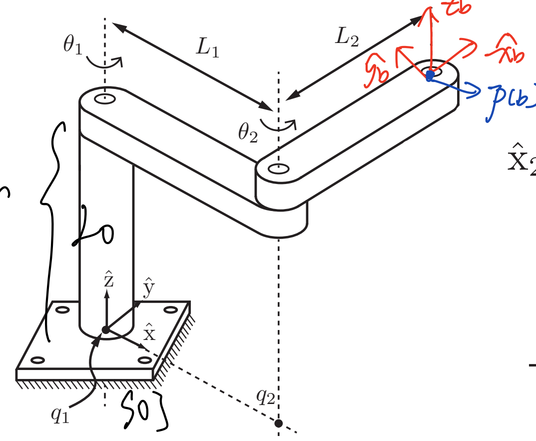
  - $E(x)$ can be easily found with given parameterization x

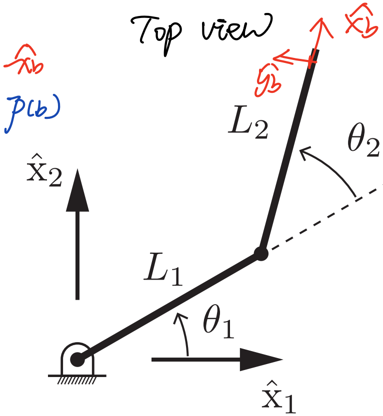
Simple Illustration Example: Analytic Jacobian (1/3)

Task variable

$$ {}^0\dot{p}_b = \left[\frac{\partial g}{\partial \theta}\right]\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} $$

J_a: analytical Jacobian





$$ J_a(\theta) = \left[\frac{\partial g}{\partial \theta}\right] = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} \\ \frac{\partial g_3}{\partial \theta_1} & \frac{\partial g_3}{\partial \theta_2} \end{bmatrix} $$

$$ \begin{cases} {}^0p_{b,x} = L_1\cos(\theta_1) + L_2\cos(\theta_1+\theta_2) \\ {}^0p_{b,y} = L_1\sin(\theta_1) + L_2\sin(\theta_1+\theta_2) \\ {}^0p_{b,z} = L_0 \end{cases} $$

$$ = \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1+\theta_2) & -L_2\sin(\theta_1+\theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) & L_2\cos(\theta_1+\theta_2) \\ 0 & 0 \end{bmatrix}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} $$

Simple Illustration Example: Analytic Jacobian (2/3)

- Let ${}^0J(\theta)$ denote the Geometric Jacobian

$$ {}^0\mathcal{V}_b = {}^0J(\theta)\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} $$

$$ {}^0\dot{p}_b = {}^0v + {}^0\omega \times (\overrightarrow{Op_b}) = -{}^0p_b \times {}^0\omega + {}^0v = \begin{bmatrix} -[{}^0p_b] & I_{3\times3}\end{bmatrix}\begin{bmatrix}{}^0\omega\\ {}^0v\end{bmatrix} $$

$$ = \begin{bmatrix} -[{}^0p_b] & I_{3\times3}\end{bmatrix} {}^0J(\theta)\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix} $$

Simple Illustration Example: Analytic Jacobian (3/3)

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More Discussions

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