MEE5114 Spring 2022 **Homework # 5**
Assigned April 27th, 2022 Due: May 5th, 2022

1. Given a linear system $x = Ax$ and a quadratic function $V(x) = x^T Px$, where P is an $n \times n$ symmetric matrix. Derive the conditions for P under which V will be a Lyapunov function for exponential stability that satisfies $\|x(t)\|^2 \leq \beta c^t \|x(0)\|^2$, where $c \in (0, 1)$.

   Solution: If $V(x)$ is a Lyapunov function, and system is asymptotically stable, we know V is PD, which implies $P \succ 0$. $L_f V$ is ND

   $$L_f V = 2(Px)^T Ax = x^T (PA + A^T P)x \tag{1}$$

   This means $PA + A^T P$ is ND.

   In fact we can say V is a Lyapunov function for exponential stability iff for any $Q \succ 0$, there exists unique positive definite matrix P as the solution of

   $$PA + A^T P = -Q \tag{2}$$

   - necessity
     If V is a Lyapunov function for exponential stability, we know the system is exponential stable. So $\text{Re}(\lambda_i) < 0$ for all eigenvalues of A. We can construct a solution for P:

     $$P = \int_0^\infty e^{A^T t} Q e^{At} dt \tag{3}$$

     which is symmetric. Now we show (3) will converge. The system is exponential, so we have $\|x(t)\| \leq c\|x(0)\|e^{-\lambda t}$, and it is the same with system $\dot{x}' = A^T x'$. And we know $x(t) = e^{At}x(0)$ and $x'(t) = e^{At}x'(0)$.

     $$\begin{aligned} P &= \int_0^\infty e^{A^T t} Q e^{At} dt \\ &\leq \int_0^\infty \|e^{A^T t}\|\|Q\|\|e^{At}\| \\ &= \int_0^\infty \alpha_1 e^{-\lambda_1 t}\|Q\|\alpha_2 e^{-\lambda_2 t} < \infty \end{aligned} \tag{4}$$

     And it's not hard to verify (3) is a solution of (2):

     $$\begin{aligned} \int_0^\infty e^{A^T t} Q e^{At} A dt + \int_0^\infty A^T e^{A^T t} Q e^{At} &= \int_0^\infty \frac{d}{dt}(e^{A^T t} Q e^{At}) dt \\ &= e^{A^T t} Q e^{At} \Big|_0^\infty \\ &= -Q \end{aligned} \tag{5}$$

     Furthermore, the solution of the (2) is unique. Let P' be any other solution for (2), we have

$$
\begin{aligned}
P &= \int_0^\infty e^{A^T t} Q e^{At} dt \\
&= -\int_0^\infty e^{A^T t}(P'A + A^T P')e^{At} dt \\
&= -\int_0^\infty \frac{d}{dt}(e^{A^T t} P' e^{At}) dt \\
&= -e^{A^T t} P' e^{At}\Big|_0^\infty \\
&= P'
\end{aligned} \tag{6}
$$

which show the uniqueness of solution of (2). And we can see (3) is well defined for $\forall Q \in S_{++}^n$.

- sufficiency
  If $P \succ 0$, $V(x) \succ 0$. And

$$
L_f V = 2(Px)^T Ax = x^T(PA + A^T P)x = -x^T Q x \prec 0 \tag{7}
$$

Meanwhile, this is a Exponential Lyapunov Function:
$\exists k_1 = \lambda_{\min}(P), k_2 = \lambda_{\max}(P), k3 = \lambda_{\min}(Q)/\lambda_{\max}(P), \alpha = 2$ such that

$$
\begin{aligned}
&k_1\|x\|^\alpha \le V(x) \le k_2\|x\|^\alpha \\
&L_f V(x) \le -k_3 V(x)
\end{aligned} \tag{8}
$$

which satisfies $\|x(t)\|^2 \le \beta c^t \|x(0)\|^2$, where $c = 1/e \in (0,1)$.

$\square$

2. Show that the system $\dot{x} = f(x) = \begin{cases} \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = -x_2 \end{cases}$ is globally asymptotically stable. (hint: try $V(x) = \ln(1 + x_1^2) + x_2^2$ as a Lyapunov function)

Solution: Let $f(x) = 0$, we can get the unique equilibrium $x_1 = 0, x_2 = 0$.

Let $V(x) = \ln(1 + x_1^2) + x_2^2$ be a Lyapunov function candidate.

We know that $V(x) = 0$ iff $x_1 = 0, x_2 = 0$, which means $V \succ 0$.

Consider $L_f V$:

$$
\begin{aligned}
L_f V &= \begin{bmatrix} \frac{2x_1}{1+x_1^2} & 2x_2 \end{bmatrix} \begin{bmatrix} -x_1 + x_1 x_2 \\ -x_2 \end{bmatrix} \\
&= \frac{2x_1(-x_1 + x_1 x_2)}{1+x_1^2} - 2x_2^2 \\
&= \frac{-2x_1^2 - 2x_2^2 - 2x_1^2 x_2^2 + 2x_1^2 x_2}{1+x_1^2} \\
&= -2\frac{x_1^2(1-x_2) + x_2^2 + x_1^2 x_2^2}{1+x_1^2} \\
&= -2\frac{x_1^2\left((x_2 - \frac{1}{2})^2 + \frac{3}{4}\right) + x_2^2}{1+x_1^2}
\end{aligned} \tag{9}
$$

which indicates $L_f V$is ND. So $V(x)$ is a Lyapunov function and this system is asymptotically stable.

Furthermore, $V(x) \to \infty$ as $\|x\| \to \infty$. So the system is globally asymptotically stable. □

3. Consider a discrete time system $x(k+1) = Ax(k) + Bu(k)$, with linear feedback law $u(k) = -Kx(k)$. Write down the closed-loop dynamics, and derive conditions for $V(x) = x^T Px$ to be discrete time Lyapunov function for asymptotic closed-loop stability.

Solution: The closed-loop system is

$$x(k+1) = (A - BK)x(k) \tag{10}$$

And we have

$$
\begin{aligned}
\Delta_f V(x) &= V(f(x)) - V(x) \\
&= x^*(A-BK)^* P(A-BK)x - x^* Px
\end{aligned} \tag{11}
$$

Similar to problem 1. Define $A_c = A - BK$. We set $V(x) = x^T Px$, and we have $V(x)$ is the discrete time Lyapunov function for asymptotic closed-loop stability iff for any $Q \succ 0$, there exists a unique $P \succ 0$ to

$$A_c^T P A_c - P = -Q \tag{12}$$

- necessity
  If the system is asymptotically stable, all eigenvalues of A_c is in the unit circle, i.e. $|\lambda| < 1$. We can construct a solution for (12)

$$P = \sum_{k=0}^{\infty} (A^T)^k Q A^k \tag{13}$$

  which is PD. And it will converge because of follows:
  $A_c = P^{-1}JP$, where is the jordan form of A. A_c has finite dimension, so J^k will be diagonal as $k \to \infty$. So we have

$$\|(A_c^T)^k\| = \|A_c^k\| \le \|P^{-1}\|\|J^k\|\|P\| \le \alpha\|\lambda\|_{\max} \tag{14}$$

where λ is the eigenvalue of A_c. So for P

$$\begin{aligned}\|P\| &\leq \sum_{k=0}^{\infty} \|(A^T)^k\| \|Q\| \|A^k\| \\ &\leq \sum_{k=0}^{\infty} \|\lambda\|_{\max}^2 \|Q\| \\ &= \frac{\alpha^2 \|Q\|}{1 - \|\lambda\|_{\max}^2} < \infty\end{aligned} \tag{15}$$

It is not hard to verify (13) is a solution of (12):

$$A_c^T P A - P = \sum_{k=1}^{\infty} (A^T)^k Q A^k - \sum_{k=0}^{\infty} (A^T)^k Q A^k = -Q \tag{16}$$

To show (13) is unique, let P' be another solution of (12), we have

$$\begin{aligned}P &= \sum_{k=0}^{\infty} (A^T)^k Q A^k \\ &= \sum_{k=0}^{\infty} (A^T)^k (P' - A_c^T P' A_c) A^k \\ &= \sum_{k=0}^{\infty} (A^T)^k P' A^k - \sum_{k=1}^{\infty} (A^T)^k P' A^k \\ &= P'\end{aligned} \tag{17}$$

which show the uniqueness.

- sufficiency
  If $P \succ 0$, $V(x) \succ 0$. And

$$L_f V = 2(Px)^T A x = x^T (A_c^T P A_c - P) x = -x^T Q x \prec 0 \tag{18}$$

which shows $L_f V$ is ND.

$\square$

4. Show that the PSD cone is acute, i.e., $\forall A, B \in S_+^n$, we have $tr(AB) \geq 0$. (Hint: decompose A using unitary matrix Q, i.e. $A = Q \Lambda Q^T$, and then use the same Q to define another matrix $C = QBQ^T$. The trace tr(AB) can be computed directly in terms of the entries in C and Λ)

Solution:

We know A can be decomposed as

$$A = Q^T \Lambda Q \tag{19}$$

where $QQ^T = I$. Define $C = QBQ^T$, we have $B = Q^T CQ$, so

$$AB = Q^T \Lambda QQ^T CQ = Q^T \Lambda CQ \tag{20}$$

Consider $x^T ABx$, set $y = Qx$

$$x^T ABx = x^T Q^T \Lambda CQx = y^T \Lambda Cy = y^T \begin{bmatrix} \lambda_1 & & \\ & \vdots & \\ & & \lambda_n \end{bmatrix} Cy \geq \lambda_{\min} y^T Cy \tag{21}$$

where $\lambda_k \geq 0$ is the k-th pivot in Λ. We know C is also PD, which means $x^T ABx \geq 0$.

We set $x = e_i$, where e_i is basic unit vector i.e. $e_i = [0, \dots, \overset{i-th}{1}, \dots, 0]$. So we have

$$tr(AB) = \sum_{i=1}^{n} e_i^T ABe_i \geq 0 \tag{22}$$

So the PSD cone is acute.

□

5. Given a symmetric matrix $A \in S^n$, let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ be the smallest and largest eigenvalues of A. Show that

$$\begin{cases} \lambda_{\min}(A) \geq \mu \\ \lambda_{\max}(A) \leq \beta \end{cases} \Leftrightarrow \mu I \preceq A \preceq \beta I$$

Solution:

First we need know $\lambda_{\min}(A)x^T x \leq x^T Ax \leq \lambda_{\max}(A)x^T x$, which can be related with spectral norm of matrix. Here we give the proof by lagrange multiplier (or you can directly use $A = Q\Lambda Q^T$ to proof). we can see $\alpha = x^T Ax/(x^T x)$ is only changed by the direction of x and A is finite, so α is bounded. Without loss of generality, set $x^T x = 1$

$$L = x^T Ax + \lambda(x^T x - 1) \tag{23}$$

Let the partial derivative equals to zero we can get the condition for extreme value:

$$Ax = -\lambda x \tag{24}$$

which means the extreme value is the eigenvalues of A when x is the eigenvector. So we have

$$\lambda_{\min}(A)x^T x \leq x^T Ax \leq \lambda_{\max}(A)x^T x$$

- If $\begin{cases} \lambda_{\min}(A) \geq \mu \\ \lambda_{\max}(A) \leq \beta \end{cases}$, we have

$$x^T(A - \mu I)x = x^T Ax - \mu x^T x \geq \lambda_{\min}(A)x^T x - \mu x^T x \geq 0 \tag{25}$$

  and

$$x^T(\beta I - A)x = \beta x^T x - x^T Ax \geq \beta x^T x - \lambda_{\max}(A)x^T x \geq 0 \tag{26}$$

  So $\mu I \preceq A \preceq \beta I$.

- If $\mu I \preceq A \preceq \beta I$, we have

$$x^T(A - \mu I)x \geq 0 \tag{27}$$

Set x as the eigenvector of A

$$\lambda x^T x - \mu x^T x \geq 0 \tag{28}$$

which leads $\lambda \geq \mu$, so we have $\lambda_{\min}(A) \geq \mu$. Have the same procedure on $A \preceq \beta I$ we can get $\lambda_{\max}(A) \leq \beta$.

$\square$

6. Suppose $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, 2$ are convex. Show that the pointwise maximum function $f(x) = \max\{f1(x), f2(x)\}$ is also convex.

Solution: Pick any $x_1, x_2 \in \mathbb{R}^n$, $\alpha \in (0, 1)$

$$\begin{aligned} \alpha f(x_1) + (1-\alpha) f(x_2) &= \alpha \max\{f_1(x_1), f_2(x_1)\} + (1-\alpha)\max\{f_1(x_2), f_2(x_2)\} \\ &\geq \alpha f_1(x_1) + (1-\alpha) f_1(x_2) \\ &\geq f_1(\alpha x_1 + (1-\alpha)x_2) \end{aligned} \tag{29}$$

similarly

$$\begin{aligned} \alpha f(x_1) + (1-\alpha) f(x_2) &= \alpha \max\{f_1(x_1), f_2(x_1)\} + (1-\alpha)\max\{f_1(x_2), f_2(x_2)\} \\ &\geq \alpha f_2(x_1) + (1-\alpha) f_2(x_2) \\ &\geq f_2(\alpha x_1 + (1-\alpha)x_2) \end{aligned} \tag{30}$$

That means

$$\alpha f(x_1) + (1-\alpha) f(x_2) \geq \max\{f_1(\alpha x_1 + (1-\alpha)x_2), f_2(\alpha x_1 + (1-\alpha)x_2)\} = f(\alpha x_1 + (1-\alpha)x_2) \tag{31}$$

which verifies this is a convex function. $\square$