MEE5114 (Sp22) Advanced Control for Robotics

Lecture 2: Rigid Body Configuration and Velocity

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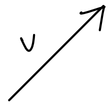
Outline

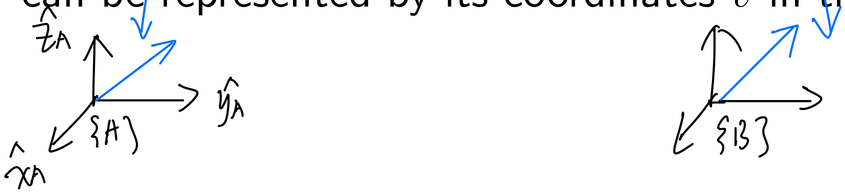
- Rigid Body Configuration
- Rigid Body Velocity (Twist)
- Geometric Aspect of Twist: Screw Motion

Free Vector 自由向量

- **Free Vector**: geometric quantity with length and direction



- Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector v can be represented by its coordinates v in the reference frame.



- v denotes the physical quantity while ${}^A v$ denote its coordinate wrt frame $\{A\}$.

Frame : coordinate system based on basis vector

$\{A\}$ frame : $\{\hat{x}_A, \hat{y}_A, \hat{z}_A\}$

$${}^A\hat{x}_A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$${}^A v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$v = 1\cdot\hat{x}_A + 2\cdot\hat{y}_A + 3\cdot\hat{z}_A$$

Point

- **Point**: p denotes a point in the physical space

$\cdot\, p$

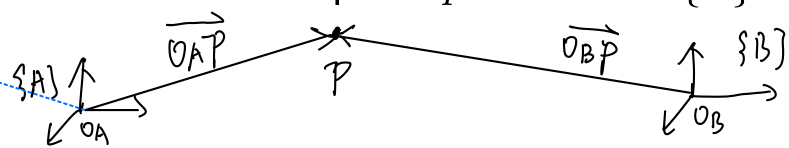
- A point p can be represented by a vector from frame origin to p

${}^A(\overrightarrow{O_B p}) =$

- ${}^A p$ denotes the coordinate of a point p wrt frame {A}

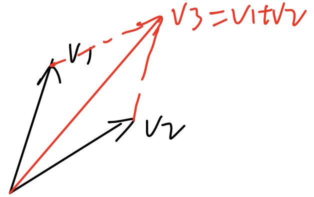
${}^B p = {}^B(\overrightarrow{O_B p})$

${}^A p = {}^A(\overrightarrow{O_A p})$



- When left-superscript is not present, it means the physical vector itself or the coordinate of the vector for which the reference frame is clear from the context. think in "coordinate-free" whenever possible.

e.g.



Coordinate-free:

$v_3 = v_1 + v_2$

But also express "physisc" in different frames.

${}^A v_3 = {}^A v_1 + {}^A v_2$ ${}^B v_3 = {}^B v_1 + {}^B v_2$

${}^A v_3 = {}^A v_1 + {}^B v_2$ ✗

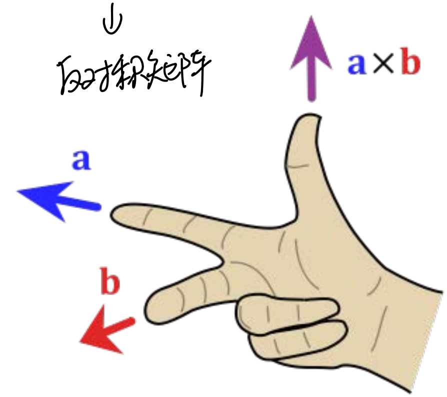
Cross Product linear operator

- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{3\times 1} \tag{1}$$

$a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$3\times 3 \Downarrow$ 反对称矩阵



Properties:

- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$

Skew symmetric representation

$A + A^T = 0$

- It can be directly verified from definition that $a \times b = [a]b$, where

$$[a] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2}$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$

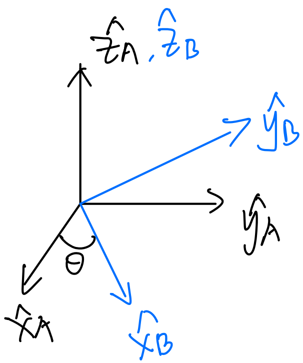
- $[a] = -[a]^T$ (called skew symmetric) $A = -A^T$

- $[a][b] - [b][a] = [a \times b]$ (Jacobi's identity)

Rotation Matrix

- **Frame**: 3 coordinate vectors (unit length) $\hat{x}, \hat{y}, \hat{z}$, and an origin
  - $\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}$ mutually orthogonal
  - $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$ (right hand rule)

- **Rotation Matrix**: specifies orientation of one frame relative to another



$$ {}^A R_B = \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B \end{bmatrix} $$

$$ {}^A R_B = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

- A valid rotation matrix R satisfies: (i) $R^T R = I$; (ii) $\det(R) = 1$

$$ {}^A\hat{x}_B^T \underbrace{({}^A\hat{y}_B \times {}^A\hat{z}_B)}_{{}^A\hat{x}_B} = 1 $$

Special Orthogonal Group

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

SO(3)
$$SO(n) = \{R \in \mathbb{R}^{n \times n} : R^T R = I, \det(R) = 1\}$$

- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set G, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$
  - **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c), \forall a, b, c \in G$
  - **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.
  - **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Use of Rotation Matrix (1/2)

- Representing an orientation ${}^{A}R_{B}$ ← from definition

- Changing the reference frame ${}^{A}R_{B}$: Given vector v, its coordinate in $\{A\}$, $\{B\}$ are ${}^{A}v$, ${}^{B}v$

$$ {}^{A}v = {}^{A}R_{B}\, {}^{B}v $$

"coordinate-free" proof:

– Same physics vector v, suppose ${}^{A}v = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$

"physics" $\Rightarrow$ $v = \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A$

$$ v = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B $$

$$ \Rightarrow \quad \alpha_1 \hat{x}_A + \alpha_2 \hat{y}_A + \alpha_3 \hat{z}_A = \beta_1 \hat{x}_B + \beta_2 \hat{y}_B + \beta_3 \hat{z}_B $$

State this "physics" in $\{A\}$ frame: $\alpha_1 {}^{A}\hat{x}_A + \alpha_2 {}^{A}\hat{y}_A + \alpha_3 {}^{A}\hat{z}_A = \beta_1 {}^{A}\hat{x}_B + \beta_2 {}^{A}\hat{y}_B + \beta_3 {}^{A}\hat{z}_B$

Use of Rotation Matrix (2/2)

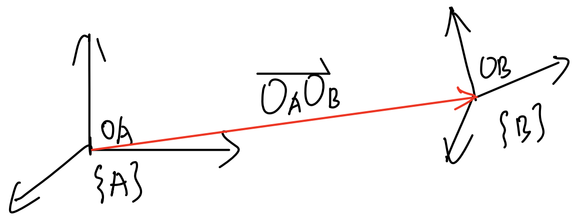
$$\begin{bmatrix} {}^A\hat{x}_A & {}^A\hat{y}_A & {}^A\hat{z}_A \end{bmatrix} \begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix} = \begin{bmatrix} {}^A\hat{x}_B & {}^A\hat{y}_B & {}^A\hat{z}_B\end{bmatrix}\begin{bmatrix}\beta_1\\ \beta_2\\ \beta_3\end{bmatrix}$$

$$I \cdot {}^A v = {}^A R_B \cdot {}^B v$$

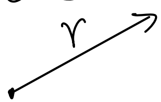
- Rotating a vector or a frame $\mathrm{Rot}(\hat{\omega}, \theta)$: will be discussed in next lecture.

Rigid Body Configuration ${}^A R_B = [{}^A x_B \ {}^A y_B \ {}^A z_B]$ ${}^A o_B$

- Given two coordinate frames {A} and {B}, the configuration of B relative to A is determined by
  - ${}^A R_B$ and ${}^A o_B$



- For a (free) vector r, its coordinates ${}^A r$ and ${}^B r$ are related by:



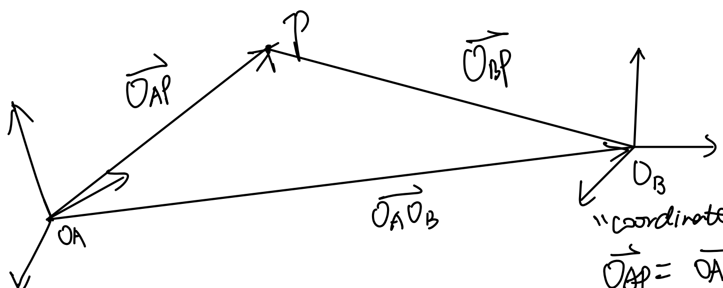
${}^A r = {}^A R_B {}^A r_B$ (可移动, 只是 orientation 不同)

$\Rightarrow \tilde{r}_A = \begin{bmatrix} r_A \\ 1 \end{bmatrix}$ ✗ $\tilde{r}_B = \begin{bmatrix} r_B \\ 1 \end{bmatrix}$ ✗

${}^A \tilde{r} = {}^A T_B {}^B \tilde{r}$

$\Rightarrow {}^A \tilde{r} = {}^A o_B + {}^A R_B {}^B \cancel{p}$

- For a point p, its coordinates ${}^A p$ and ${}^B p$ are related by:



"coordinate free"

$\overrightarrow{O_AP} = \overrightarrow{O_AO_B} + \overrightarrow{O_BP}$

${}^A \overrightarrow{O_AP} = {}^A \overrightarrow{O_BP} + {}^A \overrightarrow{O_AO_B}$

$\Rightarrow {}^A p = {}^A R_B \cdot {}^B p + {}^A o_B$

Homogeneous Transformation Matrix

- **Homogeneous Transformation Matrix**: ${}^{A}T_{B}$

$$\underset{\uparrow}{{}^{A}p} = \underbrace{{}^{A}o_{B}}_{\in \mathbb{R}^3} + \underbrace{{}^{A}R_{B}}_{3\times 3}\, \underbrace{{}^{B}p}_{\in \mathbb{R}^3} \Rightarrow \begin{bmatrix} {}^{A}p \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}o_{B} \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}p \\ 1 \end{bmatrix}$$

affine relation

$${}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}o_{B} \\ 0 & 1 \end{bmatrix}$$

$T(R,p)$: configuration of $\{B\}$ relative to $\{A\}$

- Homogeneous coordinates: 齐次坐标

Given a point $p \in \mathbb{R}^3$, its homogeneous coordinate is given by $\tilde{p} = \begin{bmatrix} p \\ 1 \end{bmatrix} \in \mathbb{R}^4$

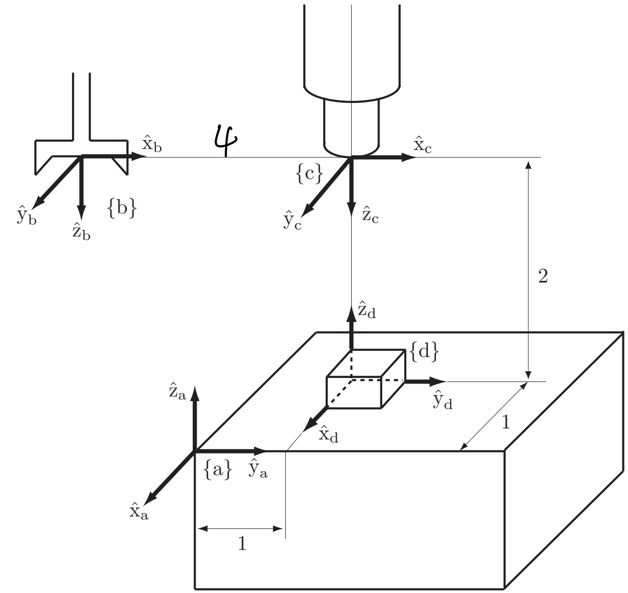
$${}^{A}\tilde{p} = {}^{A}T_{B}\, {}^{B}\tilde{p}$$

Given vector $\bar{v}$, its homogeneous coordinate is $\tilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$

free vector $\bar{v} = p_1 - p_2$ $\qquad \tilde{v} = \tilde{p}_1 - \tilde{p}_2$

Example of Homogeneous Transformation Matrix

Fixed frame {a}; end effector frame {b}, the camera frame {c}, and the workpiece frame {d}. Suppose $\|p_c - p_b\| = 4$



1. Camera "Location?

$$^{A}T_{C} = \begin{bmatrix} ^{A}R_{C} & ^{A}p_{C} \\ 0 & 1 \end{bmatrix} \qquad ^{A}R_{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$^{A}p_{C} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$^{A}T_{C} = \begin{bmatrix} ^{A}R_{C} & ^{A}p_{C} \\ 0_{1\times 3} & 1 \end{bmatrix}$$

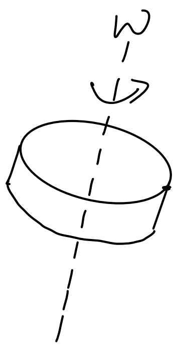
2. end-effector frame $^{A}T_{B}$

$$^{A}R_{B} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad ^{C}p_{B} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix}$$

$$\boxed{^{A}T_{B} = \underbrace{^{a}T_{c}} \cdot \underbrace{^{c}T_{b}}_{\text{绕当前轴}}}$$

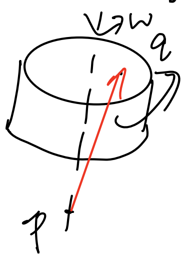
Outline

- Rigid Body Configuration
- Rigid Body Velocity (Twist) $\Leftarrow$
- Geometric Aspect of Twist: Screw Motion



Rigid Body Velocity (1/2)

- Consider a rigid body with angular velocity: ω (this is a free vector).
- Suppose the actual rotation axis passes through a point p; Let v_p be the velocity of the point p.



$V_q = V_p + w \times (\overrightarrow{pq})$

Question: A rigid body contains infinitely many points with different velocities. How to parameterize all of their velocities?

- Consider an arbitrary body-fixed point q (means that the point is rigidly attached to the body, and moves with the body), we have:

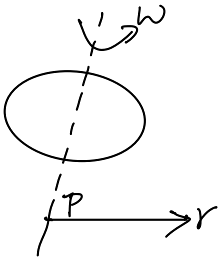
$$v_q = v_p + \omega \times (\overrightarrow{pq}) \tag{3}$$

- The velocity of an arbitrary body-fixed point depends only on (ω, v_p, p) and the location of the point q.

Rigid Body Velocity (2/2)

与p的取法

- Fact: The representation form (3) is <u>independent of the reference point p.</u>



任意点

- Consider an arbitrary point r in space
  - r may not be on the rotation axis
  - r may be a stationary point in space (does not move)
  - Let v_r be the **velocity of the body-fixed point currently coincides with** r

    恰好与r重合

    rigidly attached to the body (与刚体固连)
  - We still have: $v_q = v_r + \omega \times (\overrightarrow{rq})$

$V_q = V_p + \omega \times (\overrightarrow{pq})$, $V_r = V_p + \omega \times (\overrightarrow{pr})$

$= V_r - \omega \times (\overrightarrow{pr}) + \omega \times (\overrightarrow{pq})$

$= V_r + \omega (\overrightarrow{pq} - \overrightarrow{pr})$

$= V_r + \omega \overrightarrow{rq}$

- The body can be regarded as translating with a linear velocity v_r, while rotating with angular velocity ω about an axis passing through r

Rigid Body Velocity: Spatial Velocity (Twist)

Featherstone mapping

- **Spatial Velocity (Twist)**: $\mathcal{V}_r = (\omega, v_r) \in \mathbb{R}^6$
  - ω: angular velocity
  - v_r: velocity of the body-fixed point currently coincides with r
  - For any other body-fixed point q, its velocity is

$$v_q = v_r + \omega \times (\overrightarrow{rq})$$

- Twist is a "physical" quantity (just like linear or angular velocity)
  - It can be represented in any frame for any chosen reference point r

- A rigid body with $\mathcal{V}_r = (\omega, v_r)$ can be "thought of" as translating at v_r while rotating with angular velocity ω about an axis passing through r
  - This is just one way to interpret the motion.

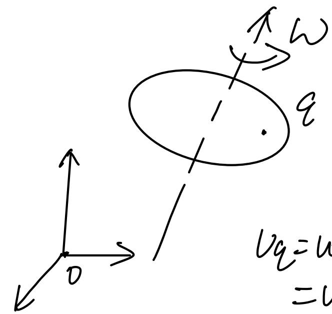
Spatial Velocity Representation in a Reference Frame

- Given frame $\{O\}$ and a spatial velocity $\mathcal{V}$

- Choose o (the origin of $\{O\}$) as the reference point to represent the rigid body velocity

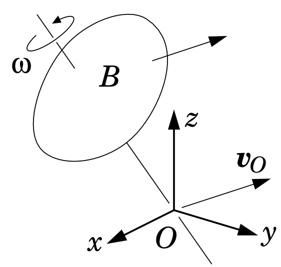
- Coordinates for the $\mathcal{V}$ in $\{O\}$:

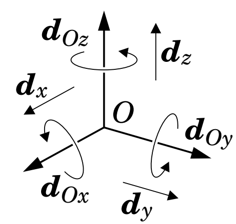
$$^{O}\mathcal{V}_o = (^{O}\omega, {}^{O}v_o)$$



$v_q = v_o + \omega \times (\vec{oq})$
$= v_o + \omega \times {}^{o}q$

- By default, we assume the origin of the frame is used as the reference point: $^{O}\mathcal{V} = {}^{O}\mathcal{V}_o$



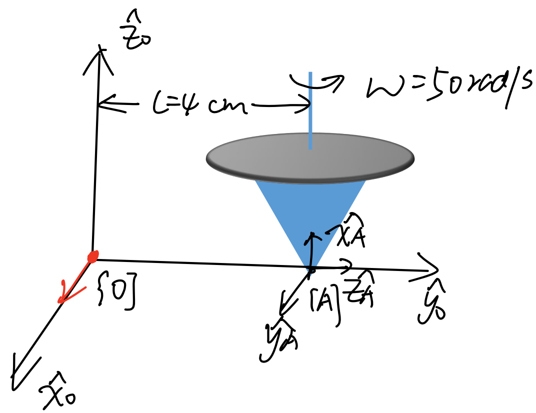


Example of Twist I

$^{0}V_0$: body fixed point coincides with O

- Example I: What's the twist of the spining top?

Choose $\{0\}$-frame : $^{0}\mathcal{V}_{top} = \begin{bmatrix} ^{0}\omega \\ \hline ^{0}V_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 50\,rad/s \\ \hline 0.04\times 50 = 2 \\ 0 \\ 0 \end{bmatrix}$



choose $\{A\}$ frame: $^{A}\mathcal{V} = \begin{bmatrix} ^{A}\omega \\ \\ ^{A}V_{0A} \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

In matlab X-A-to-0 = Adjoint(T)

V.E.Y $\underset{\mathbb{R}^6}{^{0}\mathcal{V}_{top}} = \underset{6\times 6}{^{0}X_A} \cdot \underset{\mathbb{R}^6}{^{A}\mathcal{V}_{top}}$

$^{0}X_A = \begin{bmatrix} ^{A}R_B & 0 \\ [^{A}O_B]\,^{A}R_B & ^{A}R_B \end{bmatrix}$

Example of Twist II

- Example II: What's ${}^b V_{car}$ and ${}^s V_{car}$

Car rotates about W

$${}^b V_{car} = \begin{bmatrix} {}^b W \\ {}^b V_{O_b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ \hline 2.8 \\ 4 \\ 0 \end{bmatrix}$$

指向不同

$${}^b V_{O_b} = {}^b W \times {}^b \vec{r}_{O_b} = {}^b W \times (-{}^b r)$$

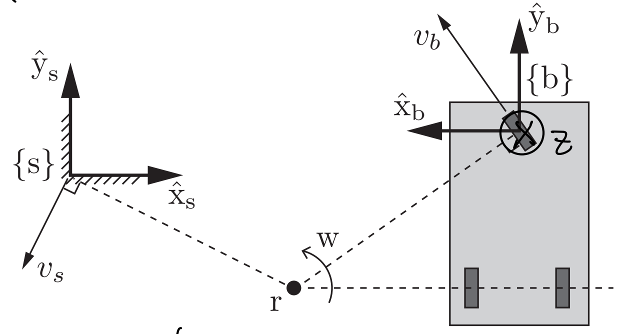
$$= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 4 \\ 0 \end{bmatrix}$$

$${}^s V_{car} = \begin{bmatrix} {}^s W \\ {}^s V_{O_s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ \hline -2 \\ -4 \\ 0 \end{bmatrix}$$

body fixed point

$${}^s V_{O_s} = {}^s W \times \vec{r}_{O_s} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}$$

$$= {}^s W \times (-{}^s r)$$



$({}^s r)$ $({}^b r)$

$r_s = (2, -1, 0)$, $r_b = (2, -1.4, 0)$, w=2 rad/s

Change Reference Frame for Twist (1/2)

- Given a twist $\mathcal{V}$, let ${}^A\mathcal{V}$ and ${}^B\mathcal{V}$ be their coordinates in frames $\{A\}$ and $\{B\}$

$$ {}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^Av_A \end{bmatrix}, \qquad {}^B\mathcal{V} = \begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix} $$

- They are related by ${}^A\mathcal{V} = {}^AX_B\, {}^B\mathcal{V}$

① ${}^A\omega = {}^AR_B\,{}^B\omega = \begin{bmatrix} {}^AR_B & 0 \end{bmatrix} \begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix}$

${}^AT_B = \begin{bmatrix} {}^AR_B & {}^Ap_B \\ 0 & 1 \end{bmatrix}$

② "coordinate free"

v_A: body fixed point coincides with O_A

v_B: ------ with O_B

choose {A} to express physics

$v_A = \omega \times {}^Ar_A$

$v_B = \omega \times {}^Br_B$

$v_A = v_B + \omega \times (\overrightarrow{BA})$

Choose {A} to express v_A

$${}^Av_A = {}^Av_B + {}^A\omega \times {}^A(\overrightarrow{BA})$$
$$= {}^AR_B\,{}^Bv_B + {}^AR_B\,{}^B\omega \times {}^A(-\overrightarrow{AB})$$
$$= {}^AR_B\,{}^Bv_B + {}^AR_B\,{}^B\omega \times (-{}^AO_B)$$
$$= {}^AR_B\,{}^Bv_B + {}^AO_B \times ({}^AR_B\,{}^B\omega)$$
$$= {}^AR_B\,{}^Bv_B + [{}^AO_B]({}^AR_B\,{}^B\omega)$$

↳ skew matrix

$$= \begin{bmatrix} [{}^AO_B]\,{}^AR_B & {}^AR_B \end{bmatrix} \begin{bmatrix} {}^B\omega \\ {}^Bv_B \end{bmatrix} \to {}^B\mathcal{V}$$

Change Reference Frame for Twist (2/2)

$${}^A\mathcal{V} = \begin{bmatrix} {}^A\omega \\ {}^Av_A \end{bmatrix} = \underbrace{\begin{bmatrix} {}^AR_B & 0 \\ [{}^Ao_B]\,{}^AR_B & {}^AR_B \end{bmatrix}}_{6\times 6 \text{ matrix}} \begin{bmatrix} \omega_B \\ {}^B\mathcal{V}_B \end{bmatrix}$$

6×6 matrix

$$\triangleq {}^AX_B$$

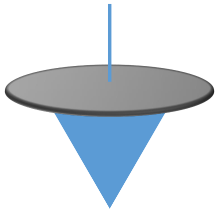
从齐次变换矩阵得到

$${}^AT_B \triangleq (R, p) \Rightarrow {}^AX_B = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

- If configuration {B} in {A} is $T = (R, p)$, then

$${}^AX_B = [\mathrm{Ad}_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

Example I Revisited



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speed (scalar)

- Recall: linear velocity $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \underbrace{\|v\|}_{\text{speed (scalar)}} \cdot \underbrace{\hat{v}}_{\text{direction}} = \sqrt{5} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

- angular velocity: $\omega = \hat{\omega} \cdot \dot{\theta}$

scalar

rotation axis (unit vector)

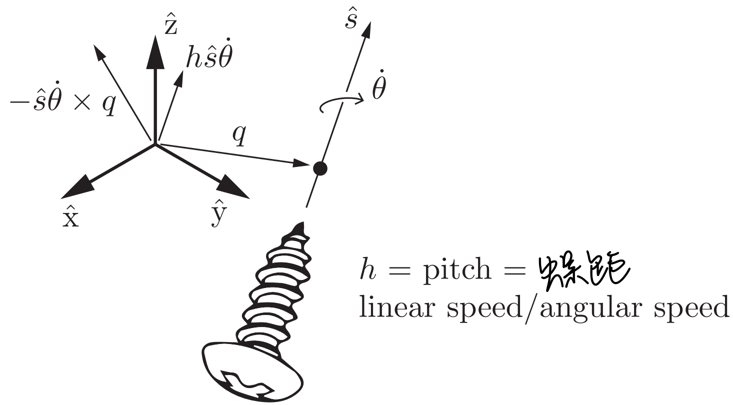
- rigid-body velocity: twist $\mathcal{V} = (v, \omega)$

Screw Motion: Definition

standard / canonical motion for rigid body motion

螺旋运动

- Rotating about an axis while also translating along the axis



- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$ → $h\dot{\theta}$ linear velocity
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

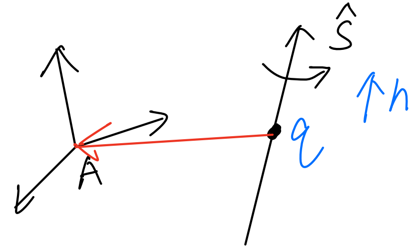
- Theorem (Chasles): Every rigid body motion can be realized by a screw motion.

From Screw Motion to Twist

axis direction $\quad$ pitch

- Consider a rigid body under a screw motion with screw axis $\{\hat{s}, h, q\}$ and (rotation) speed $\dot{\theta}$

point



- Fix a reference frame $\{A\}$ with origin o_A.

- Find the twist ${}^A\mathcal{V} = ({}^A\omega, {}^A v_{o_A})$

${}^A\omega = {}^A\hat{S} \cdot \dot{\theta}$

coordinate free: pick q as reference point

$v_{o_A} = v_q + \omega \times (\overrightarrow{q o_A})$

$= (h\dot{\theta})\hat{s} + \omega \times (\overrightarrow{q o_A})$

${}^A v_{o_A} = (h\dot{\theta})\,{}^A\hat{S} + {}^A\omega \times (\overrightarrow{q o_A})$ $\quad$ $-{}^Aq$

$= {}^A\hat{S}(h\dot{\theta}) - {}^A\omega \times {}^Aq$

- **Result**: given screw axis $\{\hat{s}, h, q\}$ with rotation speed $\dot{\theta}$, the corresponding twist $\mathcal{V} = (\omega, v)$ is given by

$$\omega = \hat{s}\dot{\theta} \qquad v = -\underbrace{\hat{s}\dot{\theta}}_{\omega} \times q + h\hat{s}\dot{\theta}$$

  - The result holds as long as all the vectors and the twist are represented in the same reference frame

From Twist to Screw Motion

对应的螺旋运动

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ we can always find the corresponding screw motion $\{q, \hat{s}, h\}$ and $\dot{\theta}$
  - If $\omega = 0$, then it is a pure translation $(h = \infty)$

$$\hat{s} = \frac{v}{\|v\|}, \quad \dot{\theta} = \|v\|, h = \infty, q \text{ can be arbitrary}$$

  - If $\omega \neq 0$:

$$\hat{s} = \frac{\omega}{\|\omega\|}, \quad \dot{\theta} = \|\omega\|, \quad q = \frac{\omega \times v}{\|\omega\|^2}, \quad h = \frac{\omega^T v}{\|\omega\|}$$

you can plug into the equation on previous slide to verify the result

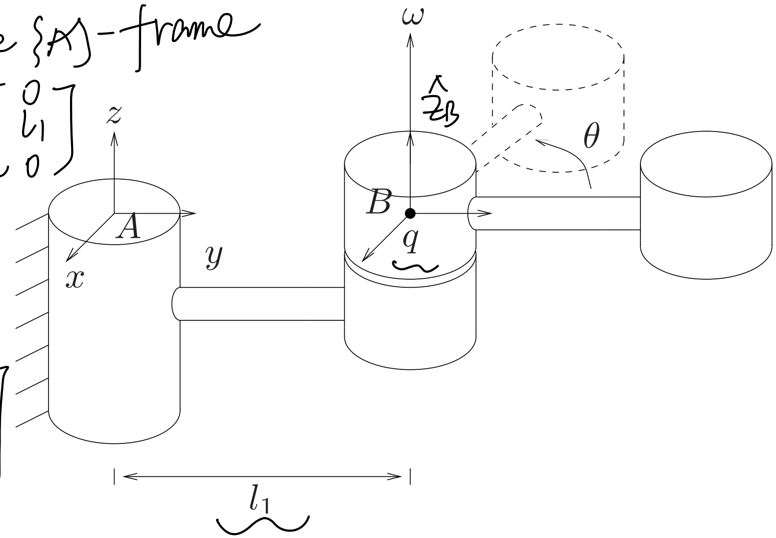
Examples: Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_B$ with $\dot{\theta} = 2$?

Choose $\{A\}$-frame

· screw axis $\quad {}^A\hat{S} = {}^A\hat{z}_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad {}^Aq = \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix}$

$h = 0$ (pure rotation) $\quad , \dot{\theta} = 2$



$$\begin{cases} {}^A\omega = {}^A\hat{S} \cdot \dot{\theta} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \\ {}^A v_{oA} = 0 - {}^A\omega \cdot {}^A q = - \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \times \begin{bmatrix} 0 \\ l_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2l_1 \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

- What is the screw axis for twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$?

$\omega = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow q = \dfrac{\omega \times v}{\|\omega\|^2}$

Screw Representation of a Twist

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\hat{\mathcal{S}}$ and a velocity $\dot{\theta}$ about the screw axis

- Consider a rigid body motion along a screw axis $\hat{\mathcal{S}} = \{\hat{s}, h, q\}$ with speed $\dot{\theta}$. With slight abuse of notation, we will often write its twist as

$$\mathcal{V} = \hat{\mathcal{S}}\dot{\theta}$$

spatial velocity (ω, v)

  - In this notation, we think of $\boxed{\hat{\mathcal{S}} \text{ as the twist}}$ associated with a unit speed motion along the screw axis $\{\hat{s}, h, q\}$

$\dot{\theta}$ rate of motion

More Discussions