MEE5114 Advanced Control for Robotics

Lecture 3: Operator View of Rigid-Body Transformation

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Outline

- Rotation Operation via Differential Equation
- Rotation Operation in Different Frames
- Rigid-Body Operation via Differential Equation
- Homogeneous Transformation Matrix as Rigid-Body Operator
- Rigid-Body Operation of Screw Axis

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Skew Symmetric Matrices $a \in \mathbb{R}^3 \to [a] \in \mathbb{R}^{3\times 3}$

- Recall that cross product is a special linear transformation.
- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n\times n}$ such that $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

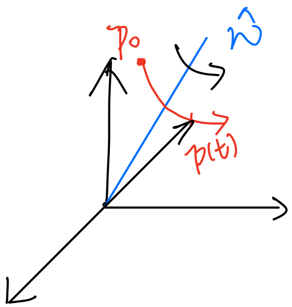
- Note that $[\omega] = -[\omega]^T \leftarrow$ skew symmetric
- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω
- The set of skew-symmetric matrices in: $\underline{so(n)} \triangleq \{S \in \mathbb{R}^{n\times n} : S^T = -S\}$
- We are interested in case $\boxed{n = 2, 3}$

Rotation matrix $SO(3) = \{ R^T R = I,\ det(R) = 1 \}$
Skew symmetric matrix $so(3) =$

Rotation Operation via Differential Equation

- Consider a point initially located at p_0 at time $t = 0$

- Rotate the point with unit angular velocity $\hat{\omega}$. Assuming the rotation axis passing through the origin, the motion is described by



$$\underbrace{\dot{p}(t)}_{\text{Linear velocity at time } t} = \hat{\omega} \times \underbrace{p(t)}_{\overrightarrow{OP}} = \overbrace{[\hat{\omega}]}^{\in \mathbb{R}^{3\times 3}\ so(3)} p(t), \text{ with } p(0) = p_0 \tag{1}$$

Recall $\dot{x} = Ax$, $x(0) = x_0$

$\Rightarrow x(t) = e^{At} \cdot x_0$

$A = [w]$

$\dot{p}(t) = [w] p(t)$

- This is a linear ODE with solution: $\boxed{p(t) = e^{[\hat{\omega}]t} p_0}$

Rotation operator

- After $t = \theta$, the point has been rotated by θ degree. Note $p(\theta) = e^{[\hat{\omega}]\theta} p_0$

- $\text{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}$ can be viewed as a rotation operator that rotates a point about $\hat{\omega}$ through θ degree

The discussion holds for any refrence frame

Rotation Matrix as a Rotation Operator (1/3)

Theorem

- Every rotation matrix R can be written as $\underline{R = \mathrm{Rot}(\hat{\omega}, \theta) \triangleq e^{[\hat{\omega}]\theta}}$, i.e., it represents a rotation operation about $\hat{\omega}$ by θ.

· Any matrix of the form : $e^{[\hat{\omega}]\theta} \in SO(3)$ → a rotation matrix

$= \{ R^T R = I, \det(R) = 1 \}$

$(e^{[\hat{\omega}]\theta})^T (e^{[\hat{\omega}]\theta}) = I$

① $(e^{[\hat{\omega}]\theta})^T = (I + [\hat{\omega}]\theta + \frac{[\hat{\omega}]^2\theta^2}{2!} + \cdots)^T$ → $\sum_{j=0}^{\infty} \frac{([\hat{\omega}]^T\theta)^j}{j!} = e^{-[\hat{\omega}]\theta}$

② $e^A \cdot e^{-A} = I \Rightarrow (e^{[\hat{\omega}]\theta})^T (e^{[\hat{\omega}]\theta}) = I$

- We have seen how to use R to represent frame orientation and change of coordinate between different frames. They are quite different from the operator interpretation of R.

- To apply the rotation operation, all the vectors/matrices have to be expressed in the the **same reference frame** (this is clear from Eq (1))

Rotation Matrix as a Rotation Operator (2/3)

- For example, assume $R = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}} = \mathrm{Rot}(\hat{\mathrm{x}}; \pi/2)$
- Consider a relation $q = Rp$:
  - **Change reference frame interpretation** : [two frames {A}, {B}, one physical point a]

• R: orientation of {B} relative to {A}
Then p and q are coordinates of the same point in different frames ({B}/{A})

$\Rightarrow$ $p = {}^B a$ $\quad q = {}^A a$ $\quad q = Rp \Rightarrow \underbrace{{}^A a = {}^A R_B \, {}^B a}$

  - **Rotation operator interpretation**:

• Have one frame {A}, and two points. $a \xrightarrow{\mathrm{Rot}(\ )} a'$, $p = {}^A a$, $q = {}^A a'$

${}^A a' = R \, {}^A a$

Rotation Matrix as a Rotation Operator (3/3)

- Consider the frame operation:
  - Change of reference frame: $R_B = RR_A$

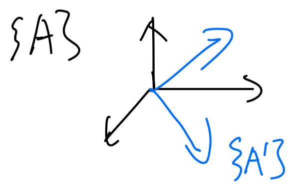
· Have one "frame object", two refience frames

Frame object $\{A\}$, orientation in $\{0\}$, is ${}^0R_A$, ${}^BR_A$

$$\Rightarrow {}^0R_A = {}^0R_B\,{}^BR_A$$

  - Rotating a frame: $R'_A = RR_A$

· two frame object, one refience frame



$$\{A\} \xrightarrow{R} \{A'\}$$

$R'_A = R \cdot R_A$ more precisely $\boxed{{}^0R_{A'} = R\,{}^0R_A}$

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Rotation Matrix Properties $R \in SO(3)$

- $R^T R = I$: definition

- $R_1 R_2 \in SO(3)$, if $R_1, R_2 \in SO(3)$: product of two rotation matrix is also a rotation matrix

- $\|\overset{p'}{Rp} - \overset{q'}{Rq}\| = \|p - q\|$ ← rotation operator preserves distance

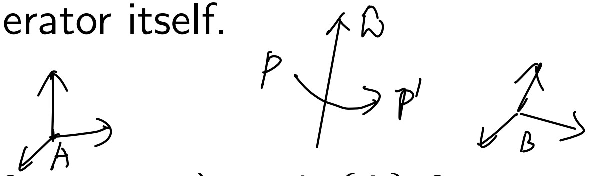
  $p, q \in \mathbb{R}^3$ definition: $\|R(p-q)\|^2 = (p-q)^T R^T R (p-q) = \|p-q\|^2$

- $R(v \times w) = (Rv) \times (Rw)$ ← rotation preserves orientation

- $R[w]R^T = [Rw]$ ← $\dot{X}$

Rotation Operator in Different Frames (1/2)

- Consider two frames {A} and {B}, the actual numerical values of the operator $\mathrm{Rot}(\hat{\omega}, \theta)$ depend on both the reference frame to represent $\hat{\omega}$ and the reference frame to represent the operator itself.



- Consider a rotation axis $\hat{\omega}$ (coordinate free vector), with {A}-frame coordinate ${}^{A}\hat{\omega}$ and {B}-frame coordinate ${}^{B}\hat{\omega}$. We know

$$ {}^{A}\hat{\omega} = {}^{A}R_{B}\, {}^{B}\hat{\omega} $$

- Let ${}^{B}\mathrm{Rot}({}^{B}\hat{\omega}, \theta)$ and ${}^{A}\mathrm{Rot}({}^{A}\hat{\omega}, \theta)$ be the two rotation matrices, representing the same rotation operation $\mathrm{Rot}(\hat{\omega}, \theta)$ in frames {A} and {B}.

Rotation Operator in Different Frames (2/2)

- We have the relation:

$$^{A}\mathrm{Rot}(^{A}\hat{\omega},\theta) = {}^{A}R_B\,{}^{B}\mathrm{Rot}(^{B}\hat{\omega},\theta)\,{}^{B}R_A$$

· Approach 1 : two points $p \xrightarrow{Rot()} p' \underset{\{A\}}{\Rightarrow} {}^{A}p' = {}^{A}Rot({}^{A}\hat{\omega};\theta)\,{}^{A}p$

$\underset{\{B\}}{\Rightarrow} {}^{B}p' = {}^{B}Rot({}^{B}\hat{\omega};\theta)\,{}^{B}p$

${}^{A}R_B\,{}^{B}p' = {}^{A}R_B\,{}^{B}Rot({}^{B}\hat{\omega};\theta)\,{}^{B}p$

$= {}^{A}R_B\,{}^{B}Rot({}^{B}\hat{\omega};\theta)\,{}^{B}R_A\,{}^{A}p$

$\Downarrow$

${}^{A}p' = \left({}^{A}R_B\,{}^{B}Rot({}^{B}\hat{\omega};\theta)\,{}^{B}R_A\right){}^{A}p$

Therefore, $\underline{{}^{A}Rot({}^{A}\hat{\omega};\theta) = {}^{A}R_B\,{}^{B}Rot({}^{B}\hat{\omega};\theta)\,{}^{B}R_A}$

Approach 2: Recall Fact:

for $a \in \mathbb{R}^3$ $[a] \in so(3)$, for any $R \in SO(3)$,

$\boxed{[Ra] \in \mathbb{R}^3 \Rightarrow [Ra] = R[a]R^T}$

$Rot({}^{A}\hat{\omega};\theta) = e^{[{}^{A}\hat{\omega}]\cdot\theta} = e^{[\underbrace{{}^{A}R_B\,{}^{B}\hat{\omega}}_{{}^{A}\hat{\omega}}]\theta}$

$= e^{{}^{A}R_B[{}^{B}\hat{\omega}]\,{}^{A}R_B^T\cdot\theta} = {}^{A}R_B\,e^{[{}^{B}\hat{\omega}]\theta}\,{}^{B}R_A$

Recall fact: $e^{PAP^{-1}} = Pe^{A}P^{-1}$, also ${}^{A}R_B^T = {}^{A}R_B^{-1}$

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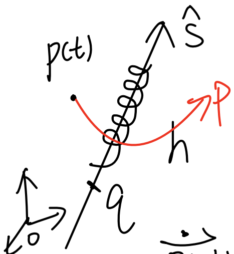
Rigid-Body Operation via Differential Equation (1/3)

- Recall: Every $R \in SO(3)$ can be viewed as the state transition matrix associated with the rotation ODE(1). It maps the initial position to the current position (after the rotation motion)
  - $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta)p_0$ viewed as a solution to $\dot{p}(t) = [\hat{\omega}]p(t)$ with $p(0) = p_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can obtain similar ODE characterization for $T \in SE(3)$, which will lead to exponential coordinate of $SE(3)$

Rigid-Body Operation via Differential Equation (2/3)

- Recall: Theorem (Chasles): Every rigid body motion can be realized by a screw motion

- Consider a point p undergoes a screw motion with screw axis $\mathcal{S}$ and unit speed $\underbrace{(\dot{\theta} = 1)}$. Let the corresponding twist be $\mathcal{V} = \mathcal{S} = (\omega, v)$. The motion can be described by the following ODE.



$$\dot{p}(t) = \overbrace{\omega \times p(t) + v}^{[\omega]p(t)+v} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}}_{\dot{\tilde{p}}(t)} = \overbrace{\begin{array}{cc} 3\times3 & 3\times1 \\ \left[\begin{matrix} [\omega] & v \\ 0 & 0 \end{matrix}\right] & \\ 1\times3 & 1\times1 \end{array}}^{\tilde{A}} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \in \mathbb{R}^4 \quad (2)$$

$\dot{\overrightarrow{Op}}(t) = V_0 + \omega \times \overrightarrow{Op}(t)$ definition of twist

$\tilde{p}(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix}$

$\dot{\tilde{p}}(t) = \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix}$

- Solution to (2) in homogeneous coordinate is:

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$$

$\dot{\tilde{p}}(t) = \tilde{A} \cdot \tilde{p}(t)$

$\tilde{p}(t) = e^{\tilde{A}t} \cdot \tilde{p}(0)$

$\tilde{p}(t) = e^{\hat{A}t} \hat{p}(0)$

↳ rigid body operator

Rigid-Body Operation via Differential Equation (3/3)

- For any twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation of twist $\mathcal{V}$

$\mathbb{R}^6$

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$e^{[S]t} = I + \begin{bmatrix} [\omega] & v \\ 0 & 1 \end{bmatrix} t + \frac{\begin{bmatrix} [\omega] & v \\ 0 & 1 \end{bmatrix}^2}{2!} t^2 \cdots$$

$(\hat{s}, q, h)$

- The above definition also applies to a screw axis $\mathcal{S} = (\omega, v)$ $\quad [S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$

- With this notation, the solution to (2) is $\tilde{p}(t) = e^{[\mathcal{S}]t}\tilde{p}(0)$ $\quad e^{\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$

$\tilde{p}(0) = e^{[S]0}\tilde{p}(0)$

- Fact: $e^{[\mathcal{S}]t} \in SE(3)$ is always a valid homogeneous transformation matrix.

$e^{[S]t} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$ $R \in SO(3)$, $p \in \mathbb{R}^3$ 齐次变换矩阵

← can be proved using definition of matrix exponential

- Fact: Any $T \in SE(3)$ can be written as $T = e^{[\mathcal{S}]t}$, i.e., it can be viewed as an operator that moves a point/frame along the screw axis at unit speed for time t

$\hat{S}$

as action

$se(3)$

$\forall w \in \mathbb{R}^3 \to [w] \in so(3) \to e^{[w]\theta} \in SO(3)$

$\forall S \in \mathbb{R}^6 \to [S]_{4\times4} \in se(3) \to e^{[S]t} \in SE(3)$

齐次变换矩阵

- Similar to $so(3)$, we can define $se(3)$:

$$se(3) = \{\underbrace{([\omega], v)}_{\begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix}} : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

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Homogeneous Transformation as Rigid-Body Operator

- ODE for rigid motion under $\mathcal{V} = (\omega, v)$

$$\dot{p} = v + \omega \times p \quad \Rightarrow \dot{\tilde{p}}(t) = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \tilde{p}(t) \Rightarrow \tilde{p}(t) = e^{[\mathcal{V}]t}\tilde{p}(0)$$

- Consider "unit velocity" $\mathcal{V} = \mathcal{S}$, then time t means degree

if not unit speed, $\mathcal{V} = S\dot{\theta}$

- $\tilde{p}' = T\tilde{p}$: "rotate" p about screw axis $\mathcal{S}$ by θ degree

$T = e^{[S]\theta}$

two points: $\tilde{p} \to \tilde{p}'$

more precisely: ${}^{0}\tilde{p}' = {}^{0}T\, {}^{0}\tilde{p}$

- TT_A: "rotate" {A}-frame about $\mathcal{S}$ by θ degree

$T = e^{[S]\theta}$

For $T \in SE(3)$

- configuration representation

${}^{A}T_{B}$: config of {B} relative to {A}

${}^{A}\tilde{p} = {}^{A}T_{B}\, {}^{B}\tilde{p}$

same physical point

but two different frames

Rigid-Body Operator in Different Frames

- Expression of T in another frame (other than {O}):

$$\underset{\text{operation in }\{O\}}{T} \quad \leftrightarrow \quad \underset{\text{operation in }\{B\}}{T_B^{-1} T T_B}$$

T_B means ${}^{O}T_B$

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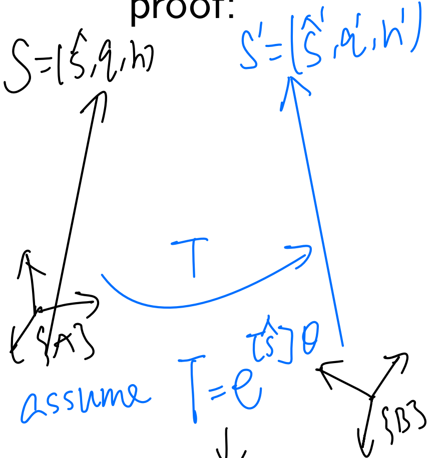
Rigid Operation on Screw Axis

- Consider an arbitrary screw axis $\mathcal{S}$, suppose the axis has gone through a rigid transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$, then

$$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

proof:



assume $T = e^{[S]\theta}$

↓

is S or S' ?

not at all

Let's work on an arbitrary frame {A} rigidly attached to the screw axis)

Let {B} be the frame obtained by applying T operation

The coordinate of S in {A} is the same as coordinate of S' in {B} (body 坐标不变)

i.e. ${}^{A}S = {}^{B}S'$ ---- ①

- We also know ${}^{A}T_{B} = T \cdot {}^{A}T_{A} \Rightarrow T = {}^{A}T_{B}$

More Space

Multiply ${}^{A}X_B$ to equation ①

$$T = [{}^{A}R_B \ {}^{A}P_B]$$

$${}^{A}X_B = \begin{bmatrix} {}^{A}R_B & 0 \\ [{}^{A}P_B]\,{}^{A}R_B & {}^{A}R_B \end{bmatrix}$$

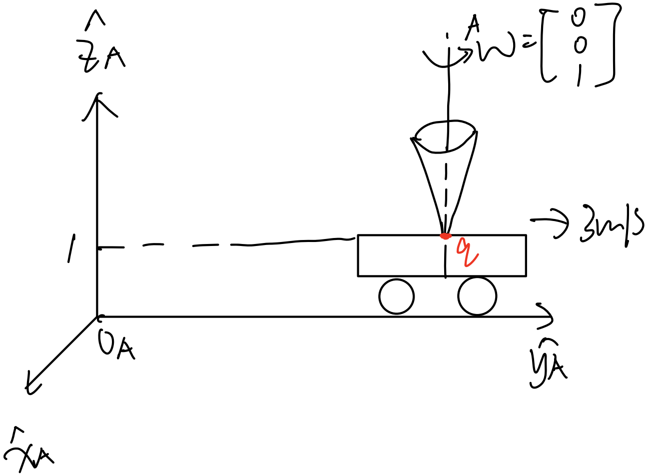
$$= [Ad_T] \triangleq \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

$${}^{A}X_B {}^{A}S = {}^{A}X_B {}^{B}S' = {}^{A}S'$$

$${}^{A}S' = {}^{A}X_B {}^{A}S$$

$$\boxed{S' = Ad_T S}$$

More Space



1. What ${}^A\mathcal{V}_{top}$?

$${}^A\mathcal{V} = \begin{bmatrix} {}^A w \\ {}^A v_{OA} \end{bmatrix} \qquad {}^A w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q(t) = \begin{bmatrix} 0 \\ 3t \\ 1 \end{bmatrix}$$

$$v_{OA} = v_q + w \times \vec{qO_A}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -3t \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3t \\ 3 \\ 0 \end{bmatrix}$$

$${}^A\mathcal{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3t \\ 3 \\ 0 \end{bmatrix}$$

2. What's screw axis? $(\hat{S}, q, h)\dot{\theta}$

$$\hat{S} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad q = \begin{bmatrix} 0 \\ 3t \\ 1 \end{bmatrix}$$

Suppose we use equation(5) $\cancel{\Rightarrow}$ ${}^A\mathcal{V}_{top} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3t \\ 0 \\ 0 \end{bmatrix}$

↳ only consider the velocity due to only "screw motion"

- Velocity is always w.r.t some inertial/reference frame

- ${}^{C}v_{A/B}$ velocity of object A relative to B, expressed in C

equ5 is computing velocity of the body relative to screw axis

$$ {}^{\{A\}}v_{top/\{A\}} = {}^{A}v_{top/screw\ axis} + {}^{\{A\}}v_{top/\{A\}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3t \\ 3 \\ 0 \end{bmatrix} $$