MEE5114 Spring 2022 **Homework # 2**

Assigned Feb 28, 2022 Due: March 9th, 2022

1. **(5 points)** Show that for any matrix $A \in \mathbb{R}^{n\times n}$, the infinite series $e^A = I + A + \frac{A^2}{2!} + \cdots$ converges.

Solution:

The norm of matrix A is defined to be

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \tag{1}$$

Equivalently, $\|A\|$ is the smallest number λ such that $\|Ax\| \le \lambda\|x\|$ for all $x \in \mathbb{R}^n$.

So we know that $\|A\|$ is finite. Furthermore, it is easy to get

$$\|AB\| \le \|A\|\|B\| \tag{2}$$

which leads

$$\|A^m\| \le \|A\|^m \tag{3}$$

We know for the space of matrix as $\mathbb{R}^{n\times n}$, every Cauchy sequence converges. And any series which converges absolutely also converges.

So we can get

$$\sum_{m=0}^{\infty} \left\|\frac{X^m}{m!}\right\| \le \sum_{m=0}^{\infty} \frac{\|X\|^m}{m!} = e^{\|X\|} < \infty \tag{4}$$

which means $e^A = I + A + \frac{A^2}{2!} + \dots$ also converges. □

2. **(5 points)** Given a linear system $\dot{x} = Ax + Bu$. Its solution is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Now assume we have $u(t) \equiv u_k$ for $t \in [k\delta t, (k+1)\delta t)$. Please derive the Zero-order-hold discretization rule, namely, derive expressions for A_d and B_d such that

$$x_{k+1} = A_d x_k + B_d u_k$$

where $x_k \triangleq x(k\cdot\delta t)$ and $u_k = u(k\cdot\delta t)$

Solution: We know

$$x_k = e^{Ak\delta t}x(0) + \int_0^{k\delta t} e^{A(k\delta t-\tau)}Bu(\tau)d\tau \tag{5}$$

and

$$\begin{aligned} x_{k+1} &= e^{A(k+1)\delta t}x(0) + \int_0^{(k+1)\delta t} e^{A((k+1)\delta t-\tau)}Bu(\tau)d\tau \\ &= e^{A\delta t}\left(e^{Ak\delta t}x(0) + \int_0^{k\delta t} e^{A(k\delta t-\tau)}Bu(\tau)d\tau\right) + \int_{k\delta t}^{(k+1)\delta t} e^{A((k+1)\delta t-\tau}Bu(\tau)d\tau \end{aligned} \tag{6}$$

Substitute (5) into (6) and let $(k+1)\delta t - \tau = \alpha$ we can get

$$\begin{aligned} x_{k+1} &= e^{A\delta t}x_k + \int_0^{\delta t} e^{A\alpha}Bu((k+1)\delta t - \alpha)d\alpha \\ &= e^{A\delta t}x_k + \int_0^{\delta t} e^{A\tau}d\tau Bu_k \end{aligned} \tag{7}$$

So

$$A_d = e^{A\delta t} \qquad B_d = \int_0^{\delta t} e^{A\tau}d\tau B$$

and

$$\begin{aligned} \int_0^{\delta t} e^{A\tau}d\tau &= \int_0^{\delta t} \left(I + A\tau + \frac{A^2}{2!}\tau^2\right) d\tau \\ &= \delta t I + \frac{A}{2!}\delta t^2 + \frac{A^2}{3!}\delta t^3 + \dots \end{aligned} \tag{8}$$

□

3. **Spatial Velocity:** (2 × 6 **points**) A cylinder rolls without slipping in the $\hat{x}_0$ direction on the $\hat{x}_0 - \hat{y}_0$ plane. The cylinder has a radius of r and a constant forward speed of v. Let ${}^0C{=}[C_x(t),0,0]^T$ be the position of the contact point at time t. Let ${}^0A{=}[A_x(t),0,0]^T$ be the position of the instantaneous top of the cylinder at time t.

   (a) What is the linear velocity of the point C? (hint: just need to compute $\frac{d}{dt}C_x(t)$)?

   (b) What is the linear velocity of the point A?

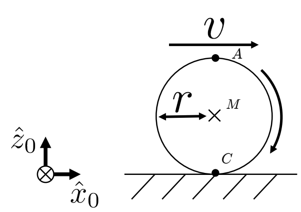
   (c) What is velocity of the body-fixed point currently coincides with C?

   (d) What is velocity of the body-fixed point currently coincides with A?

   (e) What is the spatial velocity of the cylinder in {0}-frame?

   (f) What is the spatial velocity of the cylinder in frame {C}? ({C} has the same orientation as {0}, while its origin is at the contact point C)

   Note: The first 4 questions are all referring to the inertia frame {0}



Solution: (a): The constant forward speed is v. So

$$\frac{dC_x(t)}{dt} = v \tag{9}$$

The linear velocity of the point C is $[v, 0, 0]^T$.

(b): Similarly to (a)

$$\frac{dA_x(t)}{dt} = v \tag{10}$$

The linear velocity of the point A is $[v, 0, 0]^T$.

(c): We can get the angular velocity of the cylinder as

$$^0\omega = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \end{bmatrix} \tag{11}$$

So the velocity of the body-fixed point currently coincides with C is

$$^0v_C = \begin{bmatrix} v - |^0\omega| r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

Or $^0v_C = {}^0v_M + {}^0\omega \times {}^0(\overrightarrow{MC}) = [0, 0, 0]^T$

(d):Similarly to (c), the velocity of the body-fixed point currently coincides with A is

$$^0v_A = \begin{bmatrix} 2v \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Or $^0v_A = {}^0v_M + {}^0\omega \times {}^0(\overrightarrow{MA}) = [2v, 0, 0]^T$

(e): The velocity of the body-fixed point currently coincides with the origin of frame $\{0\}$ is

$$0v_0 = \begin{bmatrix} v - |^0\omega| r \\ 0 \\ |^0\omega| C_x(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{vC_x(t)}{r} \end{bmatrix} \tag{14}$$

So the spatial velocity of the cylinder in $\{0\}$-frame is

$$^0\mathcal{V} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ 0 \\ 0 \\ \frac{vC_x(t)}{r} \end{bmatrix} \tag{15}$$

(f): Similarly to (e), the spatial velocity of the cylinder in frame $\{C\}$ is

$$ {}^{C}\mathcal{V} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{16} $$

$\square$

4. **Spatial Velocity: (2 × 8 points)** Modern Robotics: Exercise 5.5

Solution: We can construct the twist (in spatial coordinates) for the revolute joint as

$$ \mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L \\ L \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ L \\ -L \\ 0 \end{bmatrix} \tag{17} $$

(a): The position of P is

$$ \begin{aligned} {}^{s}P(t) &= e^{[\mathcal{V}]\theta s}P(0) \\ &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L + L\sin\theta - L\cos\theta \\ \sin\theta & \cos\theta & 0 & L - L\sin\theta - L\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L \\ L \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} L + d\sin\theta \\ L - d\cos\theta \\ 0 \\ 1 \end{bmatrix} \end{aligned} \tag{18} $$

(b): The velocity of point P in terms of the fixed frame is

$$\begin{aligned}\dot{P} &= [V_s]P(t) \\ &= \dot{\theta}\begin{bmatrix}0 & -1 & 0 & L \\ 1 & 0 & 0 & -L \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}L + d\sin\theta \\ L - d\sin\theta \\ 0 \\ 1\end{bmatrix} \\ &= \begin{bmatrix}d\cos\theta \\ d\sin\theta \\ 0 \\ 0\end{bmatrix}\end{aligned} \tag{19}$$

or

$$\dot{P} = \frac{d^s P(t)}{dt} = \begin{bmatrix}d\cos\theta \\ d\sin\theta \\ 0 \\ 0\end{bmatrix} \tag{20}$$

(c):

$$\begin{aligned}T_{sb}(t) &= e^{[\mathcal{V}]\theta}T_{sb}(0) \\ &= \begin{bmatrix}\cos\theta & -\sin\theta & 0 & L + d\sin\theta \\ \sin\theta & \cos\theta & 0 & L - d\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{bmatrix}\end{aligned} \tag{21}$$

(d): The twist of T_{sb} in body coordinates is

$$\mathcal{B} = \begin{bmatrix}0 \\ 0 \\ 1 \\ d \\ 0 \\ 0\end{bmatrix} \tag{22}$$

(e): The twist of T_{sb} in spatial coordinates is

$$\mathcal{V} = \begin{bmatrix}0 \\ 0 \\ 1 \\ L \\ -L \\ 0\end{bmatrix} \tag{23}$$

(f):

$$
\begin{aligned}
\mathcal{V} &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & L - d\cos\theta & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & -L - d\sin\theta & \sin\theta & \cos\theta & 0 \\ L(\sin\theta - \cos\theta) + d & L(\cos\theta + \sin\theta) & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{B} \\
&= \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \mathcal{B} \\
&= [\mathrm{Ad}_T]\mathcal{B}
\end{aligned} \tag{24}
$$

(g): Let

$$\mathcal{B} = \begin{bmatrix} \omega^b \\ v^b \end{bmatrix} \tag{25}$$

We have

$$^aR_b^T \dot{P} = v^b \tag{26}$$

And we also have

$$T_{sb}^{-1} \dot{P} = [\mathcal{B}]^b P(t) \tag{27}$$

(h): Let

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \tag{28}$$

We have

$$-[\omega]^s P(t) + \dot{P} = v \tag{29}$$

And we also have

$$\dot{P} = [\mathcal{V}]^s P(t) \tag{30}$$

□

5. **Screw axis and its transformation:** **(3×3 points)**
   (a) Draw the screw axis for the twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$
   (b) Consider an arbitrary screw axis $\mathcal{S}$. Suppose the axis has gone through a rigid body transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$. Show that

   $$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

   (we have given the proof in class, you need to go through it on your own again)

(c) Consider a rigid body motion: rotation about z axis counterclockwise by 90^o and then translate along negative y-axis by 1m. All the axes are with respect to the fixed inertia frame.

i. Find the numerical values of the corresponding transformation matrix T;

ii. Move the screw axis in part (a) using T. Find the new screw axis $\mathcal{S}'$ after the motion.

Solution: (a): The axis is

$$l = \left\{ \frac{\omega \times v}{\|w\|^2} + \lambda\omega : \lambda \in \mathbb{R} \right\}$$

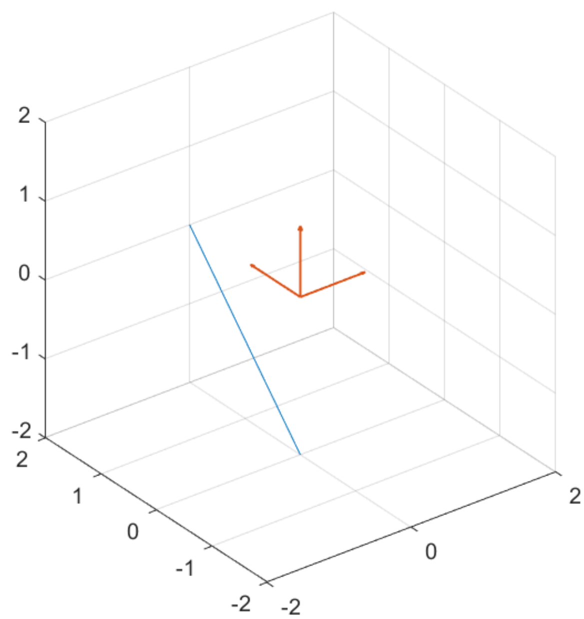


Figure 1: Axis for $\mathcal{V}$

(b): We define the original frame is {A} and there is a screw axis $\mathcal{S}$ in {A}. After the transformation (a rigid motion) {A} becomes {B}, the corresponding screw axis is $\mathcal{S}'$. But the coordinate represented in their own frame is the same which is

$$^{A}\mathcal{S} = {}^{B}\mathcal{S}' \tag{31}$$

We already know there exist a twist corresponding to a screw motion. In some sense we can regard it as a "same" thing which means the adjoint transformation for change of coordinates can be applied on a screw axis. With slight abuse of notation, left multiply $[\mathrm{Ad}_{{}^AT_B}]$ in (31)

$$[\mathrm{Ad}_{{}^AT_B}]^A\mathcal{S} = [\mathrm{Ad}_{{}^AT_B}]^B\mathcal{S}' \tag{32}$$

So we have

$$[\mathrm{Ad}_{{}^AT_B}]^A\mathcal{S} = {}^A\mathcal{S}' \tag{33}$$

Omit superscript and subscript

$${}^A\mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S} \tag{34}$$

(c):

(i):

$$T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

(ii):

$$\mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \mathcal{S} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ -2 \\ 4 \\ -2 \end{bmatrix} \tag{36}$$

The screw coordinates of $\mathcal{S}'$ is

$$\begin{aligned} &\text{Pitch:} \qquad h = \frac{\omega^T v}{\|\omega\|^2} = 0 \\ &\text{Axis:} \qquad l = \{[-1, -1, -1]^T + \lambda[-2, 0, 2] : \lambda \in \mathbb{R}\} \\ &\text{Magnitude:} \qquad M = 2\sqrt{2} \end{aligned} \tag{37}$$

See Fig. 2.

□

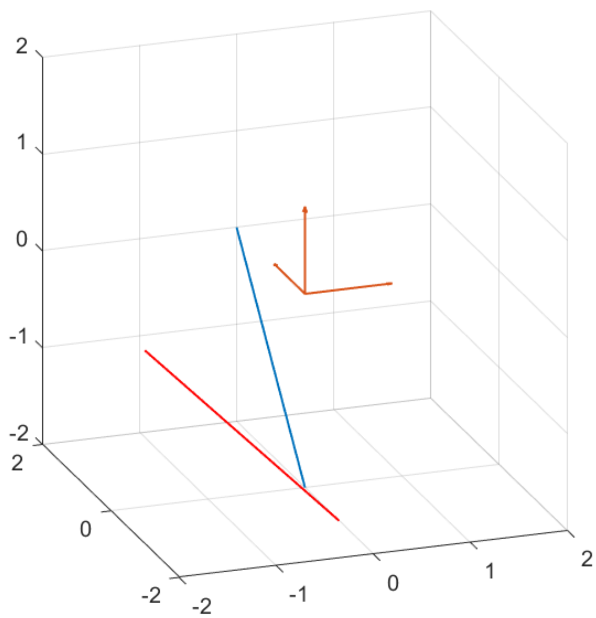


Figure 2: Screw axis after transformation