MEE5114 Spring 2022 | **Homework # 4**
Assigned April 13th, 2022 | Due: April 20th, 2022

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1. **(5 points)** Let ${}^OT_A = (R, p)$ be the pose of frame A. Suppose A is moving with velocity ${}^O\mathcal{V}_A = (\omega, v)$. Show that

$$\frac{d}{dt}[{}^OX_A] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} {}^OX_A$$

Solution:

$$\frac{d}{dt}[{}^OX_A] = \begin{bmatrix} \dot{R} & 0 \\ \frac{d}{dt}[p]R & \dot{R} \end{bmatrix} \tag{1}$$

We know that $\dot{R} = [\omega]R$ and

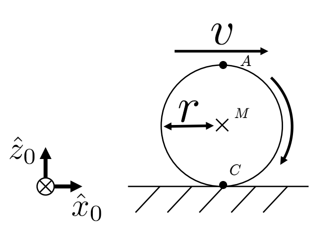
$$\begin{aligned} \frac{d}{dt}[p]R &= [\dot{p}]R + p\dot{R} \\ &= [\dot{p}R] + [p][\omega]R \\ &= [v + \omega \times p]R + [p][\omega]R \\ &= [v]R + [\omega][p]R - [p][\omega]R + [p][\omega]R \\ &= [v]R + [\omega][p]R \end{aligned} \tag{2}$$

So

$$\begin{aligned} \frac{d}{dt}[{}^OX_A] &= \begin{bmatrix} [\omega]R & 0 \\ [v]R + [\omega][p]R & [\omega]R \end{bmatrix} \\ &= \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} {}^OX_A \end{aligned} \tag{3}$$

□

2. **(3+3 points)** A cylinder rolls without slipping in the $\hat{x}_0$ direction. The cylinder has a radius of r and a constant forward speed of v. What is the spatial acceleration of this cylinder expressed in $\{o\}$, ${}^o\mathcal{A}$ and expressed in $\{C\}$,${}^C\mathcal{A}$, where frame $\{C\}$ has the same orientation as frame $\{o\}$ and its origin is at the contact point C.



Solution: In frame $\{o\}$

We know that

$${}^o\mathcal{V}_{body} = \begin{bmatrix} 0 & v/r & 0 & 0 & 0 & vC_x(t)/r \end{bmatrix}^T \tag{4}$$

so

$$
\begin{aligned}
{}^{o}\mathcal{A}_{body} &= {}^{o}\mathring{\mathcal{V}}_{body} \\
&= \begin{bmatrix} 0 & \dot{v}/r & 0 & 0 & 0 & (\dot{v}C_x(t) + v^2)/r \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & v^2/r \end{bmatrix}
\end{aligned} \tag{5}
$$

In frame $\{C\}$

We know that

$$
{}^{C}\mathcal{V}_{body} = \begin{bmatrix} 0 & v/r & 0 & 0 & 0 & 0 \end{bmatrix}^T \tag{6}
$$

so

$$
\begin{aligned}
{}^{C}\mathcal{A}_{body} &= {}^{C}\mathring{\mathcal{V}}_{body} + {}^{C}\mathcal{V}_C \times {}^{C}\mathcal{V}_{body} \\
&= \begin{bmatrix} 0 \\ \dot{v}/r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ [{}^{C}v_C] & \mathbf{0} \end{bmatrix} \begin{bmatrix} 0 \\ v/r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{7}
$$

That is

$$
\begin{aligned}
{}^{C}\mathcal{A}_{body} &= \begin{bmatrix} 0 & \dot{v}/r & 0 & 0 & 0 & v^2/r \end{bmatrix}^T \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & v^2/r \end{bmatrix}
\end{aligned} \tag{8}
$$

Or you can use adjoint transformation

$$
{}^{C}\mathcal{A}_{body} = {}^{C}X_o {}^{o}\mathcal{A}_{body} \tag{9}
$$

□

3. **(5 points)** Given our derivation in class, we have $M(\theta) = \sum_i J_i^T \mathcal{I}_i J_i$ and $c(\theta, \dot{\theta}) = \sum_i J_i^T (\mathcal{I}_i \dot{J}_i + \mathcal{I}_i V_i \times J_i + V_i \times^* \mathcal{I}_i J_i)$. Prove that $\dot{M} - 2c$ is skew symmetric.

Solution: We have

$$
\dot{M} = \dot{J}_i \mathcal{I}_i J_i + J_i^T \mathcal{I}_i \dot{J}_i \tag{10}
$$

which leads

$$
\dot{M} - 2c = \dot{J}_i \mathcal{I}_i J_i - J_i^T \mathcal{I}_i \dot{J}_i - 2J_i^T \left(\mathcal{I}_i [\mathcal{V}_i \times] + [\mathcal{V}_i \times^*] \mathcal{I}_i \right) J_i \tag{11}
$$

Using $\mathcal{I}_i^T = \mathcal{I}_i$ and $[\mathcal{V}_i \times^*] = -[\mathcal{V}_i \times]^T$

$$
\left(\dot{J}_i \mathcal{I}_i J_i - J_i^T \mathcal{I}_i \dot{J}_i \right)^T = -\left(\dot{J}_i \mathcal{I}_i J_i - J_i^T \mathcal{I}_i \dot{J}_i \right) \tag{12}
$$

and

$$
\begin{aligned}
\left(\mathcal{I}_i [\mathcal{V}_i \times] + [\mathcal{V}_i \times^*] \mathcal{I}_i \right)^T &= [\mathcal{V}_i \times]^T \mathcal{I}_i - \mathcal{I}_i [\mathcal{V} \times] \\
&= -\left(\mathcal{I}_i [\mathcal{V} \times] + [\mathcal{V}_i \times^*] \mathcal{I}_i \right)
\end{aligned} \tag{13}
$$

So $\dot{M} - 2c$ is skew-symmetric. □

4. **(3+3+3+3 points)** Download the "DynamicsExample1" from github folder, which contains a Drake model of a double pendulum.

   (a) Select an initial configuration of your interest, and simulate the free fall for 4 seconds. Show a few snapshots of the Meshcat visualization, and also plot the joint trajectories for the simulation.

   (b) Write your own inverse dynamics function ID(...) by implementing the Recursive Newton Euler Algorithm from scratch (not allowed to use existing packages). Attach your codes with sufficient explanations of each line of codes.

   (c) Based on your RNEA, and follow the procedure introduced in class, write your own forward dynamics function FD(...).

   (d) Using your own FD function to simulate the free fall of the double pendulum from the same initial condition. Compare your result with Drake simulation. Do the comparisons for at least another two initial conditions.