MEE5114 Advanced Control for Robotics

Lecture 5: Instantaneous Velocity of Moving Frames

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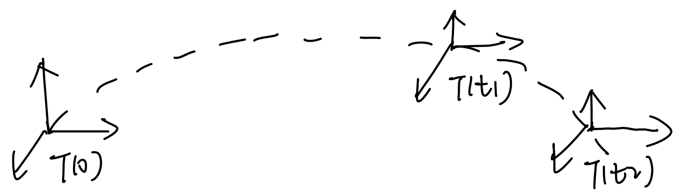
Outline

- Instantaneous Velocity of Rotating Frames

- Instantaneous Velocity of Moving Frames

Given

- Frame trajectory $T(t) = (R(t), p(t))$ (wrt $\{0\}$)



Question: What's the velocity of frame at time t?

twist / spatial velocity $\in \mathbb{R}^6$

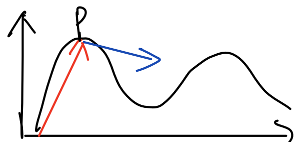
Outline

- $T(t) \in \mathbb{R}^{4\times 4}$: 4×4 matrix $\in SE(3)$
- $\log(T(t)) \longrightarrow S\theta$: Is S the velocity of T ?

twist

No

e.g.



- Instantaneous Velocity of Rotating Frames

$\begin{cases} p(t): \text{"position" vector} \\ \dot{p}(t): \text{"velocity" vector} \end{cases}$

- Instantaneous Velocity of Moving Frames

$S\theta \longleftrightarrow T(t)$: "position" coordinate

$\dot{T}(t)$ is velocity of $T(t)$?

4×4 matrix

No

How about $\log(\dot{T}(t))$?

$\dot{T}(t) \notin SE(3)$

Instantaneous Velocity of Rotating Frame (1/2)

- {A} frame is rotating with orientation $R_A(t)$ and velocity $\omega_A(t)$ at time t (Note: everything is wrt {O}-frame)

  orientation of A relative {o} at time t

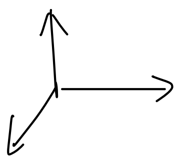
- Let $\hat{\omega}\theta = \log(R_A(t))$ be its exp. coordinate.
  - Note: $\hat{\omega}\theta$ means $R_A(t)$ can be obtained from the reference frame (say {O}-frame) by rotating about $\hat{\omega}$ by θ degree.
  - $\hat{\omega}\theta$ only describes the current orientation of {A} relative to {O}, it does not contain info about how the frame is rotating at time t.

Instantaneous Velocity of Rotating Frame (2/2)

- What is the relation between $\omega_A(t)$ and $R_A(t)$?

$$\frac{d}{dt}R_A(t) = [\omega_A(t)]R_A(t) \Rightarrow [\omega_A(t)] = \dot{R}_A(t)R_A^{-1}(t)$$

$R_A(t) = [\,{}^0\hat{x}_A(t) \quad {}^0\hat{y}_A(t) \quad {}^0\hat{z}_A(t)\,]$ $\qquad \dot{R}_A = [\,\dot{\hat{x}}_A \quad \dot{\hat{y}}_A \quad \dot{\hat{z}}_A\,]$



$\dot{\hat{x}}_A = w_A \times \hat{x}_A = [w_A]\hat{x}_A$, $\dot{\hat{y}}_A = [w_A]y_A$, $\dot{\hat{z}}_A = [w_A]z_A$

$\dot{R}_A = [w_A]R_A \Rightarrow [w_A] = \dot{R}_A R_A^{-1}$

More precise : $[{}^0w_A] = {}^0\dot{R}_A\,{}^0R_A^{-1}$

What's ${}^Aw_A$? $\quad {}^Aw_A = {}^AR_0 \cdot {}^0w_A \Rightarrow [{}^Aw_A] = [{}^AR_0\,{}^0w_A] = {}^AR_0 \cdot [{}^0w_A]\,{}^AR_0^T$

$= {}^AR_0\,{}^0\dot{R}_A\,{}^0R_A^{-1}\,{}^0R_A = {}^AR_0\,{}^0\dot{R}_A$

${}^Aw_A$: velocity of A frame relative {0}, expressed in {A}

$= {}^0R_A^{-1} \cdot \dot{R}_A$

$[{}^Aw_A] = {}^0R_A^{-1} \cdot {}^0\dot{R}_A$

Outline

- Instantaneous Velocity of Rotating Frames

- Instantaneous Velocity of Moving Frames

Instantaneous Velocity of Moving Frame (1/2)

- {A} moving frame with configuration $T_A(t)$ at time t undergoes a rigid body motion with velocity $\mathcal{V}_A(t) = (\omega, v)$ (Note: everything is wrt {O}-frame)

- The exponential coordinate $\hat{\mathcal{S}}\theta = \log(T_A(t))$ only indicates the current configuration of {A}, and does not tell us about how the frame is moving at time t.

Instantaneous Velocity of Moving Frame (2/2)

- What is the relation between $\mathcal{V}_A(t)$ and $T_A(t)$?

$$\frac{d}{dt}T_A(t) = [\mathcal{V}_A(t)]T_A(t) \Rightarrow [\mathcal{V}_A(t)] = \dot{T}_A(t)T_A^{-1}(t)$$

- A frame can be determined by $\underbrace{\text{direction vector of axis}}_{\hat{x}_A \ \hat{y}_A \ \hat{z}_A}$, and the origin O_A $\Downarrow$

$$T_A = [\tilde{\hat{x}}_A \ \tilde{y}_A \ \tilde{z}_A \ \tilde{p}_A]$$

$$\tilde{\hat{x}}_A = \begin{bmatrix} \hat{x}_A \\ 0 \end{bmatrix}, \quad \hat{O}_A(t) = \begin{bmatrix} O_A \\ 1 \end{bmatrix}$$

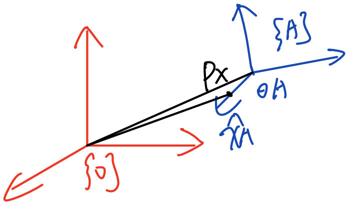
we want to show:

$$\dot{T}_A = [\dot{\tilde{\hat{x}}}_A \ \dot{\tilde{y}}_A \ \dot{\tilde{z}}_A \ \dot{\tilde{p}}_A] = [\mathcal{V}_A] \cdot T_A$$

$$[\mathcal{V}_A] = \begin{bmatrix} \omega \\ v_0 \end{bmatrix}$$

① $\dot{\tilde{\hat{x}}}_A = \begin{bmatrix} \dot{\hat{x}}_A \\ 0 \end{bmatrix}$, $\dot{\hat{x}}_A = \omega \times \hat{x}_A$

More Space



Prove it

$$\hat{x}_A = \overrightarrow{O_A P_x} = P_x - O_A$$

$$\dot{\hat{x}}_A = \dot{P}_x - \dot{O}_A = v_o + \omega \times (\overrightarrow{oP_x}) - v_o - \omega \times (\overrightarrow{oO_A})$$

$$= \omega \times (\overrightarrow{oP_x} - \overrightarrow{oO_A}) = \omega \times (\overrightarrow{O_A P_x}) = \omega \times \hat{x}_A$$

$$\dot{\hat{y}}_A = \omega \times \hat{y}_A \qquad \dot{\hat{z}}_A = \omega \times \hat{z}_A$$

$$\dot{\tilde{O}}_A = \begin{bmatrix} \dot{O}_A \\ 0 \end{bmatrix} = \begin{bmatrix} v_o + \omega \times O_A \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v_o \\ 0 & 0 \end{bmatrix} \begin{bmatrix} O_A \\ 1 \end{bmatrix}$$

$${}^0\mathcal{V}_A = [Ad_T]\,{}^A\mathcal{V}_A$$

$$\dot{T}_A = [\mathcal{V}_A]\,T_A \quad \Rightarrow \quad [{}^0\mathcal{V}_A] = \dot{T}_A \cdot T_A^{-1}$$

$$T_A = {}^0T_A \qquad [{}^A\mathcal{V}_A] = T_A^{-1} \cdot \dot{T}_A$$

Review / Summary of Rigid body velocity & operation

- spatial velocity / twist $\mathcal{V} = \begin{bmatrix} \omega \\ v_o \end{bmatrix}$

ref point "o" may / may not move with the body

ω: angular velocity

v_o: velocity of the body-fixed particle currently coincides with O

- Given the twist $\mathcal{V} = \begin{bmatrix} \omega \\ v_o \end{bmatrix}$, any body fixed point p, its velocity is

$$\boxed{v_p = v_o + \omega \times \vec{op}}$$

- Twist in frames: Given frame $\{B\}$ and $\{O\}$ with relation ${}^{O}T_B = (R, p)$

- $\mathcal{V}$ is a twist of some rigid body

$${}^{O}\mathcal{V} = \begin{bmatrix} {}^{O}\omega \\ {}^{O}v_o \end{bmatrix} = \qquad {}^{B}\mathcal{V} = \begin{bmatrix} {}^{B}\omega \\ {}^{B}v_B \end{bmatrix}$$

origin of $\{O\}$ — origin of $\{B\}$

$${}^{O}\mathcal{V} = {}^{O}X_B \, {}^{B}\mathcal{V}$$

change of coordinate of twist

$${}^{O}X_B = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

For any given $T = (R, p) \Rightarrow [Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$

- Screw axis $S = (\hat{s}, q, h) \longleftrightarrow \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ h\hat{s} - \hat{s} \times q \end{bmatrix} \dot{\theta}$

$\dot{\theta} = 1$

$\Rightarrow$ All rigid body motion can be "thought of" as screw motion

$\Rightarrow$ We typically write $\mathcal{V} = S\dot{\theta}$

"unit" normalized twist

- Rotation Operator / Exponential coordinate

- ODE for Rotation $\dot{p} = \omega \times p = [\omega]p \Rightarrow p(t) = e^{[\omega]t} p(0)$

if $\omega = \hat{\omega}$, unit vector, $t = \theta$

$so(3) = \{[\omega], \omega \in \mathbb{R}^3\}$

$\hat{\omega}\theta \Longleftrightarrow R = e^{[\hat{\omega}]\theta} \in SO(3)$, $\hat{\omega}\theta$ is called exponential coordinate denoted by $\log(R)$

- $p' = \underbrace{e^{[\hat{\omega}]\theta}}_{R} p$

Given a frame $\{A\}$, $R_A = [\hat{x}_A \ \hat{y}_A \ \hat{z}_A]$ then

$$\underbrace{R}_{\text{action}} \cdot R_A = e^{[\hat{\omega}]\theta} R_A .$$

↳ Rotate $\{A\}$ by $\hat{\omega}$ axis by θ degree

Expression of Rotation Operator R in $\{0\}$ and $\{B\}$

R in $\{0\}$ $\qquad$ ${}^0R_B^{-1} R \, {}^0R_B$ Same rotation operation in $\{B\}$

- Rigid body transformation and exp coordinate (wrt $\{0\}$)

- ODE $\dot{p} = v + \omega \times p$

$$\dot{\hat{p}} = \underbrace{\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}}_{\text{twist 矩阵表示}} \hat{p} \Rightarrow \hat{p}(t) = e^{[\omega]t} \hat{p}(0), \qquad \hat{p} = \begin{bmatrix} p \\ 1 \end{bmatrix}$$

matrix representation of $\mathcal{V}$, $se(3) = \{\mathcal{V}, \mathcal{V} \in \mathbb{R}^6\}$

$$\underset{\hat{S}\theta}{\quad} \ \underset{[\hat{S}]}{se(3)} \xrightarrow{\exp} \exp^{[\hat{S}]\theta} \in SE(3) \rightarrow T$$

$\hat{S}\theta$ is the exponential coordinate of T $\quad \log(T)$

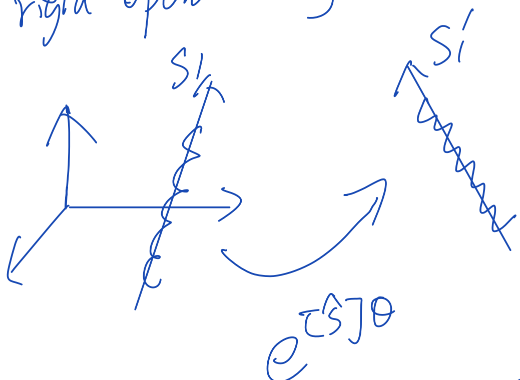
$e^{[\hat{S}]\theta}$

- $\tilde{p}' = T\tilde{p}$

$T \cdot T_A$: rotate from {A} about the screw axis $\hat{S}$ by θ degree

T_A: configuration of {A}

rigid operation of screw axis :



$S_1' = [Ad_T] S_1$ means rotate about axis S_1 along $\hat{S}$ by θ degree

- expression of T in {B} $T_B^{-1} T T_B$

— velocity of moving frames $T(t)$

$\mathcal{V}_{frame}$ satisfies $[\mathcal{V}_T] = \dot{T} T^{-1}$