

The twisted conjugacy problem for endomorphisms of polycyclic groups

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Abstract. An algorithm is constructed that, when given an explicit presentation of a polycyclic group G , decides for any endomorphism $\psi \in \text{End}(G)$ and any pair of elements $u, v \in G$, whether or not the equation $(x\psi)u = vx$ has a solution $x \in G$. Thus it is shown that the problem of the title is decidable. Also we present an algorithm that produces a finite set of generators of the subgroup $\text{Fix}_\psi(G) \leq G$ of all ψ -invariant elements of G .

1 Introduction

Let G be a group, and $u, v \in G$. Given an endomorphism $\psi \in \text{End}(G)$, one says that u and v are ψ -twisted conjugate, and one writes $u \sim_\psi v$, if and only if there exists $x \in G$ such that $u = (x\psi)^{-1}vx$, or equivalently $(x\psi)u = vx$. This notion has important applications in modern Nielsen fixed-point theory. The recognition of twisted conjugacy classes with respect to a given endomorphism $\psi \in \text{End}(G)$ in the case of any polycyclic group G is the main concern of this paper.

The class \mathcal{P} of all polycyclic groups has a satisfactory algorithmic theory. A basic fact about polycyclic groups due to Hall is that every such group is finitely presented (see [18] or [23]). Polycyclic groups are residually finite by a result of Hirsch [10]. Hence the word problem is decidable in any polycyclic group G . By a known theorem of Mal'cev [14] (for an elementary proof due to J. S. Wilson see [18]) every subgroup of a polycyclic group is closed in the profinite topology. Hence the generalized word problem (also known as the membership problem) is also decidable in polycyclic groups.

Every polycyclic group is conjugacy separable, i.e. if two elements of a polycyclic group G are not conjugate in G , they fail to be conjugate in some finite quotient. This deep result was proved by Formanek [4] and Remeslennikov [16]. So the conjugacy problem is decidable in any polycyclic group.

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It follows (see [3]) that the twisted conjugacy problem is decidable for any automorphism $\psi \in \text{Aut}(G)$ of any polycyclic group G . Indeed, the split extension $G_\psi = G \rtimes \langle \psi \rangle$ is polycyclic, and the twisted conjugacy problem for ψ reduces to the ordinary conjugacy problem for G_ψ , which is decidable from above. However we cannot apply this method when $\psi \in \text{End}(G)$ is not automorphism. Here we develop a new approach based on the free Fox calculus.

We want to determine whether or not the equation

$$(x\psi)u = vx, \quad (1)$$

where $\psi \in \text{End}(G)$, $u, v \in G$, has a solution x in an arbitrary polycyclic group G . This question can be reduced (in any group G , not only a polycyclic group) to the case where one of the elements is trivial. To do this we change ψ to $\varphi = \psi \circ \sigma_u$, where $\sigma_u \in \text{Aut}(G)$ is the inner automorphism $g \mapsto u^{-1}gu$. Hence $(x\psi)u = vx$ if and only if

$$x\varphi = wx, \quad (2)$$

where $w = u^{-1}v$.

Note that every polycyclic group is Noetherian, i.e. satisfies to the maximal condition for subgroups (equivalently, every subgroup is finitely generated). The class \mathcal{P} can be characterized as the class of all solvable matrix groups over the integers \mathbb{Z} (see [11]). Thus, by the famous Mal'cev–Kolchin theorem, any polycyclic group is virtually nilpotent-by-abelian. This last description will be used in the paper.

The main result of this paper is the following

Theorem. *Let G be any efficiently presented polycyclic group. Then there is an algorithm that decides the twisted conjugacy problem in G for any endomorphism $\varphi \in \text{End}(G)$.*

Section 2 is devoted to preliminary results and Section 3 contains a proof of the main theorem.

2 Preliminary results

Let G be any group and $u, v \in G$. We consider the question of the decidability of the equation (1), equivalently (2). The following lemma shows that the question can under some assumptions be transformed to a similar question about a subgroup H of finite index in G .

Lemma 2.1. *Let G be any group and $H \leq G$ be any $\text{End}(G)$ -invariant subgroup of finite index. Suppose that the membership problem in H is decidable in G . Then, if the twisted conjugacy problem is decidable in H , it is also decidable in G .*

Proof. Let

$$G = t_1 H \cup \dots \cup t_k H$$

be a representation of G as a union of distinct cosets. Let (2) have a solution $x = t_i h$ for some $i \in \{1, \dots, k\}$ and $h \in H$. Then the equation

$$y_i \varphi = (t_i \varphi)^{-1} w t_i y_i = w_i y_i \quad (3)$$

has the solution $y_i = h \in H$. The necessary condition that $w_i = (t_i \varphi)^{-1} w t_i \in H$ can be verified effectively by our assumption. Obviously, any solution $y_i \in H$ of (3) gives a solution $x = t_i y_i \in G$ of (2). Hence (2), and so also (1), is solvable in G if and only if (3) has a solution in H for some $i \in \{1, \dots, k\}$, and this can be decided effectively. \square

Let F_n be the free group of rank n with basis $\{f_1, \dots, f_n\}$. In [5], Fox developed the differential calculus in the free group ring $\mathbb{Z}F_n$. The partial Fox derivatives may be defined as the mappings

$$\partial/\partial f_j : \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n, \quad \text{for } 1 \leq j \leq n,$$

satisfying the following conditions for all $\alpha, \beta \in \mathbb{Z}$ and all $u, v \in F_n$:

$$\partial f_i / \partial f_j = \delta_{ij} \quad (\text{where } \delta_{ij} \text{ is the Kronecker delta});$$

$$\partial(\alpha u + \beta v) / \partial f_j = \alpha \partial u / \partial f_j + \beta \partial v / \partial f_j;$$

$$\partial(uv) / \partial f_j = \partial u / \partial f_j + u \partial v / \partial f_j.$$

It is easy to verify that these derivatives have another property. Every element $\beta \in \mathbb{Z}F_n$ can be uniquely written in the form

$$\beta = \beta^\varepsilon + \partial \beta / \partial f_1 (f_1 - 1) + \dots + \partial \beta / \partial f_n (f_n - 1),$$

where $\varepsilon : \mathbb{Z}F_n \rightarrow \mathbb{Z}$ is the augmentation homomorphism, defined by $f_j^\varepsilon = 1$ for $j = 1, \dots, n$. It follows from the definitions that $\partial \alpha / \partial f_j = 0$ for all $\alpha \in \mathbb{Z}$ and for each j .

The following formula which holds for every element $u \in F_n$ is called *the main identity for the Fox derivatives*:

$$\sum_{i=1}^n \partial u / \partial f_i (f_i - 1) = u - 1. \quad (4)$$

Let $G = F_n / R$ be a group presented as a quotient of F_n . For brevity we keep the notation f_1, \dots, f_n for the images in G of the generators of F_n . So we have $G = \langle f_1, \dots, f_n \rangle$. For any word w in these generators we define formal derivatives $\partial w / \partial f_i$ for $i = 1, \dots, n$ as elements of the group ring $\mathbb{Z}G$. We simply compute their values as sums of words (i.e. elements of $\mathbb{Z}F_n$) and apply the epimorphism $\mathbb{Z}F_n \rightarrow \mathbb{Z}G$. These values depend on the choice of the word w representing an element of G . But in any case property (4) is valid in $\mathbb{Z}G$.

Let C be an abelian normal subgroup of G . We regard C as a module over $\mathbb{Z}G$ via conjugation in G extended by linearity to $\mathbb{Z}G$, and also as a module over $\mathbb{Z}(G/C)$; this is possible because C acts trivially on itself by conjugation. Then for any word c representing an element $c \in C$ and linear combinations of words α representing an element $\alpha \in \mathbb{Z}G$ one has

$$\partial c^\alpha / \partial f_i = \alpha \partial c / \partial f_i, \quad \text{for } i = 1, \dots, n.$$

Our notation for conjugates and commutators is as follows: $g^f = fgf^{-1}$ and $[g, f] = gfg^{-1}f^{-1}$.

Let $\varphi \in \text{End}(G)$ be any endomorphism that acts trivially on G/C , where C is as before an abelian normal subgroup of G . Then φ is uniquely determined by the images

$$f_i \varphi = c_i f_i \quad (i = 1, \dots, n), \quad (5)$$

where c_i are some words representing elements $c_i \in C$. Then for any element $g \in G$ the image under φ can be written as

$$g\varphi = c_1^{\partial g / \partial f_1} \dots c_n^{\partial g / \partial f_n} g. \quad (6)$$

Lemma 2.2. *Let $G = F_n/R$ be presented as a quotient of F_n , and let $\varphi \in \text{End}(G)$ be an endomorphism that acts trivially on G/C , where C is an abelian normal subgroup of G . Assume that φ is determined by (5). Also assume that the derived subgroup G' acts trivially on C , and $G' \leq \text{Fix}_\varphi(G)$, where $\text{Fix}_\varphi(G) = \{g \in G \mid g\varphi = g\}$ is the subgroup of φ -invariant elements of G .*

Then $\text{Fix}_\varphi(G)$ lies in $C_G(c_1, \dots, c_n)$, the centralizer of the elements c_1, \dots, c_n in G . If the center $\zeta_1(G)$ is trivial then $\text{Fix}_\varphi(G) = C_G(c_1, \dots, c_n)$.

Proof. Recall that $G = \langle f_1, \dots, f_n \rangle$. Since φ acts trivially on G' , it maps every commutator $[f_i, f_j]$ to itself. Since φ is determined by (5) with $c_1, \dots, c_n \in C$ we can use (6). We obtain

$$[f_i, f_j] = [f_i, f_j]\varphi = [c_i f_i, c_j f_j] = c_i^{1-f_j} c_j^{f_i-1} [f_i, f_j],$$

or equivalently

$$c_i^{f_j-1} = c_j^{f_i-1}, \quad \text{for } i, j = 1, \dots, n. \quad (7)$$

So we can use the formula (6) again to conclude that $g\varphi = g$ if and only if

$$c_1^{\partial g / \partial f_1} \dots c_n^{\partial g / \partial f_n} = 1. \quad (8)$$

Here both sides of (8) are regarded as elements of the module C . Since G' acts trivially on C by hypothesis, G acts as an abelian group on C , so we can permute elements

acting on elements of C . Multiplying both sides of (8) by $f_i - 1$ for $i = 1, \dots, n$, we deduce from (7) and (4) that $g\varphi = g$ is equivalent to

$$c_i^{\sum_{j=1}^n \partial g / \partial f_j (f_j - 1)} = c_i^{g-1} = 1, \quad \text{for } i = 1, \dots, n$$

and so equivalent to $c_i^g = c_i$ for $i = 1, \dots, n$. Hence $g \in C_G(c_1, \dots, c_n)$.

Conversely, suppose that $g \in C_G(c_1, \dots, c_n)$. Suppose that

$$c_1^{\partial g / \partial f_1} \dots c_n^{\partial g / \partial f_n} = v.$$

Then using the same argument as before we obtain that

$$v^{f_i-1} = c_i^{g-1} = 1, \quad \text{for } i = 1, \dots, n.$$

Thus $v \in \zeta_1(G)$, so if the additional assumption that $\zeta_1(G) = 1$ holds then $v = 1$, hence $g \in \text{Fix}_\varphi(G)$. We conclude that if $\zeta_1(G) = 1$ then $\text{Fix}_\varphi(G) = C_G(c_1, \dots, c_n)$. \square

Corollary 2.3. *Let G be a centerless polycyclic group satisfying the assumptions of Lemma 2.2. Then there is an algorithm which finds a finite set $\{g_1, \dots, g_k\}$ of generators of $\text{Fix}_\varphi(G)$.*

Proof. By Lemma 2.2, $\text{Fix}_\varphi(G)$ is the centralizer of a finite set of elements of G . It is known (see [19] or [23]) that a finite generating set of such a centralizer can be constructed effectively. \square

The statement of Corollary 2.3 can be improved by elimination of the condition that $\zeta_1(G) = 1$.

Lemma 2.4. *Let G be a polycyclic group satisfying the assumptions of Lemma 2.2. Then there is an algorithm which finds a finite set $\{g_1, \dots, g_k\}$ of generators of $\text{Fix}_\varphi(G)$.*

Proof. It was proved in Lemma 2.2 that $G_1 = C_G(c_1, \dots, c_n) \geq \text{Fix}_\varphi(G)$. This group G_1 can be constructed effectively. Moreover, the proof of Lemma 2.2 shows that for any $g \in G_1$ one has $g\varphi = c(g)g$, with $c(g) \in \zeta_1(G)$. In particular G_1 is invariant under φ , so we may consider φ as an endomorphism of G_1 .

Define a homomorphism $\mu: G_1 \rightarrow \zeta_1(G)$ by $g \mapsto c(g)$. Obviously we have $\text{Fix}_\varphi(G) = \ker(\mu)$. Hence a set of generators of $\text{Fix}_\varphi(G)$ can be effectively constructed by the standard procedure. \square

Lemma 2.5. *Let G be any group, and let $\varphi \in \text{End}(G)$ be an endomorphism that acts trivially on G/U , where U is an abelian normal subgroup of G . Then every term $C_i = \zeta_i(G)$ of the upper central series of G is φ -invariant, so φ induces an automorphism of every quotient G/C_i .*

Proof. It is enough to prove the statement for the center $C = C_1$. Let $c \in C$. Since φ acts trivially modulo U and U is abelian, $c\varphi$ acts trivially on U . For each $g \in G$ one has $g\varphi = u(g)g$, with $u(g) \in U$. It follows that if $[g, c] = 1$ then

$$[g\varphi, c\varphi] = [u(g)g, c\varphi] = [g, c\varphi] = 1.$$

Hence $c\varphi \in C$. \square

Lemma 2.6. *Let G be a polycyclic group with nilpotent derived subgroup G' . Let $\varphi \in \text{End}(G)$ be any endomorphism. Then there is an algorithm which finds a finite set $\{g_1, \dots, g_k\}$ of generators of the subgroup $\text{Fix}_\varphi(G)$.*

Proof. In the case when G' is abelian the statement was proved in [22]. So we can assume that φ acts trivially on G modulo the second derived subgroup G'' . Clearly

$$\text{Fix}_\varphi(G) \leq \text{Fix}_{\varphi, G''}(G) = \{g \in G : g\varphi \equiv g \pmod{G''}\}.$$

Obviously $\text{Fix}_{\varphi, G''}(G)$ is the full preimage of $\text{Fix}_\varphi(G/G'')$ and so can be constructed effectively.

Suppose by induction that we have found the subgroup

$$G(i) = \text{Fix}_{\varphi, \gamma_i(G')}(G) = \{g \in G : g\varphi \equiv g \pmod{\gamma_i(G')}\},$$

for some $i \geq 2$. Consider the quotient $G_{i+1} = G(i)/\gamma_{i+1}(G')$ and let

$$C_{i+1} = \gamma_i(G(i)')/\gamma_{i+1}(G')$$

be an abelian $\text{End}(G_{i+1})$ -invariant subgroup of G_{i+1} . Since C_{i+1} is central in G'_{i+1} the derived subgroup G'_{i+1} acts trivially on C_{i+1} . Moreover $\varphi \in \text{End}(G_{i+1})$ acts trivially on G_{i+1}/C_{i+1} and fixes every element $g \in G'_{i+1}$. Then by Lemma 2.4 we can find effectively a finite set of generators of $G(i+1) = \text{Fix}_{\varphi, \gamma_{i+1}(G')}(G)$. Continuing this process we will eventually obtain a finite set of generators of $\text{Fix}_\varphi(G)$. \square

Lemma 2.7. *Suppose that the twisted conjugacy problem is decidable for any endomorphism of any polycyclic group H with nilpotent derived subgroup H' . Then there is an algorithm which finds the subgroup $\text{Fix}_\varphi(G)$ in any polycyclic group G for any endomorphism $\varphi \in \text{End}(G)$.*

Proof. It is known (see for example [11]) that each polycyclic group G contains an $\text{End}(G)$ -invariant subgroup H such that H' is nilpotent. Such a subgroup H can be constructed effectively. Let

$$G = t_1 H \cup \dots \cup t_k H$$

be a representation of G as a union of cosets with $t_1 = 1$. If a coset $t_i H$ intersects $\text{Fix}_\varphi(G)$ one has

$$y\varphi = (t_i\varphi)^{-1}t_i y, \quad (9)$$

for some $y = h \in H$.

Since φ induces an endomorphism of H and we can effectively verify the inclusion $(t_i\varphi)^{-1}t_i \in H$ which is necessary for the solvability of (9), we can check this by our assumption. If (9) has a solution $y = h$ we change t_i to $t_i h$. If a coset $t_i H$ does not intersect $\text{Fix}_\varphi(G)$, we do not need to consider it further. In this way we obtain representatives $t_{i_1}, \dots, t_{i_q} \in \text{Fix}_\varphi(G)$ of all cosets with non-empty intersection with $\text{Fix}_\varphi(G)$. From this we obtain by Lemma 2.7 the subgroup $\text{Fix}_\varphi(H)$, and we conclude that

$$\text{Fix}_\varphi(G) = t_{i_1} \text{Fix}_\varphi(H) \cup \dots \cup t_{i_q} \text{Fix}_\varphi(H). \quad \square$$

3 Proof of the theorem

Now G is a polycyclic group, and $\varphi \in \text{End}(G)$ is an arbitrary endomorphism. We must decide whether or not the equation (2) has a solution $x \in G$.

As a first step we reduce the problem to the case when G is nilpotent-by-abelian. It is well known (see [11] or [23]) that every polycyclic group admits a characteristic series

$$1 \leq N \leq A \leq G,$$

where N is nilpotent, A/N is abelian, and G/A is finite. By Lemma 2.1 it is enough to find an algorithm deciding the twisted conjugacy problem in this subgroup A . To simplify notation we may therefore replace G by A and assume that G itself is polycyclic group with nilpotent derived group $N = G'$.

It was proved in [22] that the twisted conjugacy problem is decidable for any endomorphism in any metabelian polycyclic group. So we shall apply induction on the nilpotency class c of N , and we may assume that $c \geq 2$.

Let C be the last non-trivial member of the lower central series of N . We consider the equation induced by (2) in the quotient $G_1 = G/C$. By the inductive assumption we have an algorithm that determines whether or not the equation (2) has a solution in G_1 . If this equation is not solvable in G_1 it is not solvable in G . So suppose that there is a solution of (2) in G_1 . Then there is an element $x_1 \in G$ for which

$$x_1\varphi = wx_1c^{-1},$$

where $c \in C$. Then (2) is solvable in G if and only if the equation

$$z\varphi = cz \quad (10)$$

is solvable in G . Indeed, if $x \in G$ is a solution of (2) then $z = x_1^{-1}x$ is a solution of (10), and vice versa.

Any possible solution of (10) belongs to the subgroup $\text{Fix}_{\varphi, C}(G)$ which is the full preimage of $\text{Fix}_{\varphi}(G_1)$. Thus it can be found effectively by Lemma 2.6. For simplicity we assume that $G = \text{Fix}_{\varphi, C}(G)$. Then φ acts trivially on G' and on G/C . Moreover, we can assume that the center $\zeta_1(G)$ is trivial. By Lemma 2.5 every member of the upper central series of G is φ -admissible. Since G is polycyclic and so Noetherian this series is finite. Suppose that (10) is solvable in the quotient $G/\zeta_1(G)$. Then there is an element $z_1 \in G$ for which

$$z_1\varphi = cz_1d^{-1},$$

where $d \in \zeta_1(G)$. Using an argument similar to the one above we conclude that (10) is solvable if and only if the equation

$$y\varphi = dy \tag{11}$$

is solvable. We replace G by $\text{Fix}_{\varphi, \zeta_1(G)}(G)$ and look for a solution of (11) in this new group. Now for every element $g \in \text{Fix}_{\varphi, \zeta_1(G)}(G)$ one has

$$g\varphi = d(g)g,$$

where $d(g) \in \zeta_1(G)$. The equation (11) has a solution if and only if d belongs to the image of $\text{Fix}_{\varphi, \zeta_1(G)}(G)$ under the homomorphism $g \mapsto d(g)$. This is a homomorphism of abelian groups and hence we can answer this question by the standard procedure. Continuing this process we reduce the question to the case of a centerless group.

Let φ be defined by (5). Then for any $z \in G$ we have

$$z\varphi = c_1^{\partial z/\partial f_1} \dots c_n^{\partial z/\partial f_n} z.$$

If z is a solution of (10) then

$$c_1^{\partial z/\partial f_1} \dots c_n^{\partial z/\partial f_n} = c.$$

Regarding C as a module and multiplying both sides of this equation by $f_i - 1$ for $i = 1, \dots, n$, and using (7) and (4) we obtain the set of equations

$$c_i^z = c^{f_i-1} c_i, \quad \text{for } i = 1, \dots, n. \tag{12}$$

Since the conjugacy problem is decidable in any polycyclic group we first find a solution of (12) for c_1 . We can assume that such solution z_1 exists, or (2) has no solutions in G . Conjugating by z_1 we reduce the problem to a similar question for $C_G(c_1)$. This subgroup can be constructed effectively. We continue this process with the new group $C_G(c_1)$ replacing G and $n - 1$ equations arising from (12). Suppose

that (12) is solvable for some z . Let

$$z\varphi z^{-1} = c_1^{\partial z/\partial f_1} \dots c_n^{\partial z/\partial f_n} = v.$$

Multiplying both sides by $f_i - 1$ for $i = 1, \dots, n$, and applying the same argument as above we obtain that

$$(cv^{-1})^{f_i-1} = 1, \quad \text{for } i = 1, \dots, n.$$

Since $\zeta_1(G) = 1$, we conclude that $c = v$, and the theorem is proved. \square

Corollary 3.1. *Let G be a polycyclic group and φ any endomorphism of G . Then there is an algorithm that produces generators of the fixed-point group $\text{Fix}_\varphi(G)$ of all φ -invariant elements of G .*

Proof. The statement follows from the Theorem and Lemma 2.7. \square

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