Aspects of Nonabelian Group Based Cryptography: A Survey and Open Problems

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Abstract

Most common public key cryptosystems and public key exchange protocols presently in use, such as the RSA algorithm, Diffie-Hellman, and elliptic curve methods are number theory based and hence depend on the structure of abelian groups. The strength of computing machinery has made these techniques theoretically susceptible to attack and hence recently there has been an active line of research to develop cryptosystems and key exchange protocols using noncommutative cryptographic platforms. This line of investigation has been given the broad title of **noncommutative algebraic cryptography**. This was initiated by two public key protocols that used the braid groups, one by Ko, Lee et.al.and one by Anshel, Anshel and Goldfeld. The study of these protocols and the group theory surrounding them has had a large effect on research

in infinite group theory. In this paper we survey these noncommutative group based methods and discuss several ideas in abstract infinite group theory that have arisen from them. We then present a set of open problems.

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1 Introduction and Nonabelian Group Based Cryptography

Traditionally **cryptography** is the science and/or art of devising and implementing secret codes or **cryptosystems**. **Cryptanalysis** is the science and/or art of breaking cryptosystems while **cryptology** refers to the whole field of cryptography plus cryptanalysis. In most modern literature cryptography is used synonomously with cryptology. Presently there is an increasing need for secure cryptosystems due to the use of internet shopping, electronic financial transfers and so on.

Most common public key cryptosystems and public key exchange protocols presently in use, such as the RSA algorithm, Diffie-Hellman, and elliptic curve methods are number theory based and hence theoretically depend on the structure of abelian groups. Although there have been no successful attacks on the standard protocols there is a feeling that the strength of computing machinery has made these techniques less secure. As a result of this there has been an active line of research to develop and analyze new cryptosystems and key exchange protocols based on noncommutative cryptographic platforms. This line of investigation has been given the broad title of **noncommutative algebraic cryptography** (see [MSU]).

Up to this point the main sources for noncommutative cryptographic platforms has been nonabelian groups. In cryptosystems based on these objects algebraic properties of the platforms are used prominently in both devising cryptosystems and in cryptanalysis. In particular the difficulty, in a complexity sense, of certain algorithmic problems in finitely presented groups, such as the conjugator search problem, has been crucial in encryption and decryption.

The main sources for nonabelian groups are combinatorial group theory and linear group theory. Braid group cryptography (see [D]), where encryption is done within the classical

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braid groups, is one prominent example. The one way functions in braid group systems are based on the difficulty of solving group theoretic decision problems such as the conjugacy problem and conjugator search problem. Although braid group cryptography had initial spectacular success, various potential attacks have been identified. Borovik, Myasnikov, Shpilrain [BMS] and others have studied the statistical aspects of these attacks and have identitifed what are termed black holes in the platform groups outside of which present cryptographic problems. Baumslag. Fine and Xu in [BFX] and [X] suggested potential cryptosystems using a combination of combinatorial group theory and linear groups and a general schema for these types of cryptosystems was given. In [BFX 2] a public key version of this schema using the classical modular group as a platform was presented. A cryptosystem using the the extended modular group $SL_2(\mathbb{Z})$ was developed by Yamamura ([Y]) but was subsequently shown to have loopholes ([BG], [S], [HGS]). In [BFX 2] attacks based on these loopholes were closed.

The study and cryptanalysis of potential platform groups has had a strong positive effect on both group theory and complexity theory. Motivated in large part by cryptography, there has been tremendous interest in **asymptotic group theory** and **generic properties** (see sections 9 and 10).

In this article we will present an overview of these combinatorial group theoretic methods. We will introduce free group cryptography and then the seminal Anshel-Anshel-Goldfeld and Ko-Lee Protocols. We will further discuss how to use combinatorial group theory in digital signatures and password verification. Then we will discuss potential platform groups and give a brief review of braid group cryptography. Next we introduce asymptotic density and its application to cryptanalysis and evaluation of cryptosystems. In particular we will look at some theoretical results concerning the generic free group property. Finally a list of open problems (in no way exhaustive) in this area will be presented.

There are several recent books on this area. The most comprehensive is by Myasnikov, Shpilrain and Ushakov [MSU]. In this article we touch on several topics that are not mentioned in that book, for example polycylic group cryptography, digital signature protocols and password authentication and the generic free group property.

2   Basics of Public Key Cryptography

In this section we describe the standard terminology used in cryptography and then introduce the two most common public key methods, the Diffie-Hellman protocol and the RSA protocol.

In general both the **plaintext message** (uncoded message) and the **ciphertext message** (coded message) are written in some N-letter alphabet which is usually the same for both plaintext and code. The method of coding or the encoding algorithm is then a transformation of the N-letters. The most common way to perform this transformation is to consider the N letters as N integers modulo N and then perform a number theoretical function on them. Therefore most encoding algorithms use modular arithmetic and hence cryptography is closely tied to number theory.

Modern cryptography is usually separated into **classical** or **symmetric key cryptography** and **public key cryptography**. In the former, both the encoding and decoding algorithms are supposedly known only to the sender and receiver, usually referred to as Bob and Alice. In the latter, the encryption method is public knowledge but only the receiver knows how to decode.

The process of putting the plaintext message into code is called **enciphering** or **encryption** while the reverse process is called **deciphering** or **decryption**. Encryption algorithms partition the plaintext and ciphertext message into **message units**. These are single letters or pairs of letters or more generally k-vectors of letters. The transformations are done on these message units and the encryption algorithm is a mapping from the set of plaintext message units to the set of ciphertext message units. Putting this into a mathematical formulation we let

$$\mathcal{P} = \{\text{set of all plaintext message units }\} \text{ and}$$

$$\mathcal{C} = \{\text{ set of all ciphertext message units }\}.$$

The encryption algorithm is then the application of an invertible function

$$f : \mathcal{P} \mapsto \mathcal{C}.$$

The function f is the **encryption map**. The inverse

$$f^{-1} : \mathcal{C} \mapsto \mathcal{P}$$

is the **decryption** or **deciphering map**. The triple $\{\mathcal{P}, \mathcal{C}, f\}$, consisting of a set of plaintext message units, a set of ciphertext message units and an encryption map is called a **cryptosystem**.

Breaking a code is called **cryptanalysis**. An attempt to break a code is called an **attack**. Most cryptanalysis starts with a statistical frequency analysis of the plaintext language used. Cryptanalysis depends also on a knowledge of the form of the code, that is, the type of cryptosystem used (see [K] or [MSU]).

Most classical cryptosystems are number theoretically derived crytosystems. In applying a cryptosystem to an N letter alphabet we consider the letters as integers mod N. The encryption algorithms then apply number theoretic functions and use modular arithmetic on these integers.

We usually do not use a single letter at a time but rather a sequence of k letters. The k letters are then a **message unit**. An encryption algorithm is then a function

$$f : \mathbb{Z}_N^k \mapsto \mathbb{Z}_N^m$$

Presently there are many instances where secure information must be sent over open communication lines. These include for example banking and financial transactions, purchasing items via credit cards over the internet and similar things. This led to the development of **public key cryptography**. Roughly, in classical cryptography only the sender and receiver

know the encoding and decoding methods. Further it is a feature of such cryptosystems that if the encrypting method is known the decrypting can be carried out.

In **public key cryptography** the encryption method is public knowledge but only the receiver knows how to decode. More precisely in a classical cryptosystem once the encrypting algorithm is known the decryption algorithm can be implemented in approximately the same order of magnitude of time. In a public the decryption algorithm is much more difficult to implement. This difficulty depends on the type of computing machinery used (much as primality testing) and as computers get more powerful, new and more secure pulic key cryptosystems become necessary.

The basic idea in a public key cryptosystem is to have a **one-way function**. That is a function which is easy to implement but very hard to invert. Hence it becomes simple to encrypt a message but very hard, unless you know the inverse, to decrypt.

The standard model for a public key cryptosystem is the following. Alice wants to send a message to Bob. The encrypting map f_A for Alice is public knowledge as well as the encrypting map f_B for Bob. On the other hand the decryption algorithms f_A^{-1} and f_B^{-1} are secret and known only to Alice and Bob respectively. Let $\mathcal{P}$ be the message Alice wants to send to Bob. She sends $f_B f_A^{-1}(\mathcal{P})$. To decode Bob applies first f_B^{-1}, which only he knows. This gives him $f_B^{-1}(f_B f_A^{-1}(\mathcal{P})) = f_A^{-1}(\mathcal{P})$. He then looks up f_A which is publically available and applies this $f_A(f_A^{-1}(\mathcal{P})) = \mathcal{P}$ to obtain the message.

Alice sends $f_B f_A^{-1}(\mathcal{P})$ rather than just $f_B(\mathcal{P})$ for **authentication**, that is being certain from Bob's point of view that the message really came from Alice. Suppose $\mathcal{P}$ is Alice's verification; signature, social security number etc.. If Bob receives $f_B(\mathcal{P})$ it could be sent by anyone since f_B is public. On the other hand since only Alice supposedly knows f_A^{-1} getting a reasonable message from $f_A(f_B^{-1} f_B f_A^{-1}(\mathcal{P}))$ would verify that it is from Alice. Applying f_B^{-1} alone should result in nonsense.

Getting a reasonable one way function can be a formidable task. The most widely used (at present) public key systems are based on the difficulty of inverting certain number theoretic functions. The first real public key protocol was developed in 1976 by Diffie and Hellman using the difficulty of the **discrete log problem**.

In modular arithmetic it is easy to raise an element to a power but difficult to determine, given an element, if it is a power of another element. Specifically if G is a finite group, such as the cyclic multiplicative group of $\mathbb{Z}_p$ where p is a prime, and $h = g^k$ for some k then the **discrete log** of h to the base g is any integer t with $h = g^t$. The rough form of the Diffie-Helman public key system is as follows.

Bob and Alice will use a classical cryptosystem based on a key k with $1 < k < q-1$ where q is a prime. It is the key k that Alice must send to Bob. Let g be a multiplicative generator of $\mathbb{Z}_q^\star$. Alice chooses an $a \in \mathbb{Z}_q$ with $1 < a < q-1$. She makes public g^a. Bob chooses a $b \in \mathbb{Z}_q^\star$ and makes public g^b. The secret key is g^{ab}. Both Bob and Alice, but presumably noone else, can discover this key. Alice knows her secret power a and the value g^b is public from Bob. Hence she can compute the key $g^{ab} = (g^b)^a$. The analogous situation holds for Bob. An attacker however only knows g, g^a and g^b. Unless the attacker can solve the discrete log problem, that is finding a or b the key exchange is secure.

Notice that this depends upon

$$(g^a)^b = (g^b)^a$$

As we will see, the Ko-Lee protocol exactly mimics the Diffie-Hellman protocol within a nonabelian group by interpreting powers as conjugation.

In 1997 it became known that the ideas of public key cryptography were developed by British Intelligience Services prior to Diffie and Hellman.

In 1977 Rivest, Adelman and Shamir developed the **RSA Algorithm** which is presently the most widely used public key cryptosystems. It is based on the difficulty of factoring large integers and in particular on the fact that it is easier to test for primality than to factor. In basic outline at the simplest level it works as follows.

Alice chooses two large primes p_A, q_A and an integer e_A relatively prime to $\Phi(p_A q_A) = (p_A - 1)(q_A - 1)$. It is assumed that these integers are chosen randomly to minimize attack. The primes she chooses should be quite large. Originally RSA used primes of approximately 100 decimal digits, but as computing and attack have become more sophisticated, larger primes have had to be utilized. Once Alice has obtained p_A, q_A, e_A she lets $n_A = p_A q_A$ and computes d_A , the multiplicative inverse of e_A modulo $\Phi(n_A)$. That is d_A satisfies $e_A d_A \equiv 1 \mod (p_A - 1)(q_A - 1)$. She makes public the enciphering key $K_A = (n_A, e_A)$ and the encryption algorithm known to all is

$$f_A(\mathcal{P}) = \mathcal{P}^{e_A} \mod n_A$$

where $\mathcal{P} \in \mathbb{Z}_{n_A}$ is a message unit.

It can be shown that if $(e_A, (p_A - 1)(q_A - 1)) = 1$ and $e_A d_A \equiv 1 \mod (p_A - 1)(q_A - 1)$ then $\mathcal{P}^{e_A d_A} \equiv \mathcal{P} \mod n_A$ Therefore the decryption algorithm is

$$f_A^{-1}(\mathcal{C}) = \mathcal{C}^{d_A} \mod n_A.$$

Notice then that $f_A^{-1}(f_A(\mathcal{P})) = \mathcal{P}^{e_A d_A} \equiv \mathcal{P} \mod n_A$ so it is the inverse.

Now Bob makes the same type of choices to obtain p_B, q_B, e_B. He lets $n_B = p_B q_B$ and makes public his key $K_B = (n_B, e_B)$.

If Alice wants to send a message to Bob that can be authenticated to be from Alice she sends $f_B(f_A^{-1}(\mathcal{P}))$. An attack then requires factoring n_A or n_B which is much more difficult than obtaining the primes p_A, q_A, p_B, q_B.

Again notice the use of commutativity.

There have been many extensions and enhancements of these basic public key protocols. Elliptic curve cryptography uses the discrete log problem within the group of an elliptic curve. This group is a finite abelian group and has certain advantages over the cyclic groups used in the standard Diffie-Hellman protocol. The book by Koblitz [K] has a thorough description of elliptic curve methods.

The El-Gamal cryptosystem is a technique to use the Diffie-Hellman key exchange method to do encryption. The method works as follows. Suppose that Bob and Alice want to

communicate openly. They have exchanged a secret key k that supposedly only they know. Let f_k be an encryption function or encryption algorithm based on the key k. Alice wants to send the message m to Bob and m is given as a binary bit string. Alice sends to Bob

$$f_k(m) \oplus k$$

where k is a bit string for the key k and $\oplus$ is addition modulo 2.

Bob knows the key k and hence can compute it as a bibary string. He now computes

$$f_k(m) \oplus k \oplus k$$

Since addition modulo 2 has order 2 we have

$$f_k(m) \oplus k \oplus k = f_k(m).$$

Bob now applies the decryption algorithm f_K^{-1} to decode the message. In practice a **hash function** is usually applied to k (see [B]).

3   The Fundamentals of Free Group Cryptography

The extension of all these ideas to noncommutative platforms is the subject of **noncommutative algebraic cryptography**. This involves the following ideas,

(1) General Algebraic Techniques for Developing Cryptosystems

(2) Potential Algebraic Platforms (Specific Groups, Rings, Etc.) for implementing the Techniques

(3) Cryptanalysis and Security Analysis of the Resulting Systems

The main source for noncommutative platforms are nonabelian groups and the main method for handling nonabelian groups in cryptography is combinatorial group theory. This refers to the branch of group theory that studies groups by using group presentations, that is sets of generators and relations between them. The basic idea in using combinatorial group theory for cryptography is that elements of groups can be expressed as words in some alphabet. If there is an easy method to rewrite group elements in terms of these words and further the technique used in this rewriting process can be supplied by a secret key then a cryptosystem can be created. The simplest example is perhaps a **free group cryptosystem**. This can be described in the following manner.

Consider a free group F on free generators $x_1, ..., x_r$. Then each element g in F has a unique expression as a word $W(x_1, ..., x_r)$. Let $W_1, ..., W_k$ with $W_i = W_i(x_1, ..., x_r)$ be a set of words in the generators $x_1, ..., x_r$ of the free group F. At the most basic level, to construct a cryptosystem, suppose that we have a plaintext alphabet $\mathcal{A}$. For example suppose that $\mathcal{A} = \{a, b, ...\}$ are the symbols needed to construct meaningful messages in English. To encrypt, use a substitution ciphertext

$$\mathcal{A} \mapsto \{W_1, ..., W_k\}.$$

That is
$$a \mapsto W_1, b \mapsto W_2, ...$$

Then given an word $W(a, b, ...)$ in the plaintext alphabet form the free group word $W(W_1, W_2, ...)$. This represents an element g in F. Send out g as the secret message.

In order to implement this scheme we need a concrete representation of g and then for decryption a way to rewrite g back in terms of $W_1, ..., W_k$. This concrete representation is the idea behind **homomorphic cryptosystems**.

The decryption algorithm in a free group cryptosystem then depends on the **Reidemeister-Schreier rewriting process**. This is a method to rewrite elements of a subgroup of a free group in terms of the generators of that subgroup. We refer to [MKS] or [GB 1] for a complete description of the technique. Roughly it works as follows. Assume that $W_1, ..., W_k$ are free generators for some subgroup H of a free group F on $\{x_1, ..., x_n\}$. Each W_i is then a reduced word in the generators $\{x_1, ..., x_n\}$. A **Schreier transversal** for H is a set $\{h_1, ..., h_t, ...\}$ of (left) coset representatives for H in F of a special form (see [MKS]). Any subgroup of a free group has a Schreier transversal. The Reidemeister-Schreier process allows one to construct a set of generators $W_1, ..., W_k$ for H by using a Schreier transversal. Further given the Schreier transversal from which the set of generators for H was constructed, the **Reidemeister-Schreier Rewriting Process** allows us to algorithmically rewrite an element of H. Given such an element expressed as a word $W = W(x_1, ..., x_r)$ in the generators of F this algorithm rewrites W as a word $W^\star(W_1, ..., W_k)$ in the generators of H.

The knowledge of a Schreier transversal and the use of Reidemeister-Schreier rewriting facilitates the decoding process in the free group case but is not essential. Given a known set of generators for a subgroup the Stallings Folding Method to develop a subgroup graph can also be utilized to rewrite in terms of the given generators. The paper by Kapovich and Myasnikov [KM] is now a standard reference for this method in free groups. At present there is an ongoing study of the complexity of Reidemeister-Schreier being done by Baumslag, Brukhov, Fine and Troeger.

Pure free group cryptosystems are subject to various attacks and can be broken easily. However a public key free group cryptosystem using a free group representation in the Modular group was developed by Baumslag, Fine and Xu [BFX 1 2]. The most successful attacks on free group cyrpotsystems are called **length based attacks**. Here an attacker multiplies a word in ciphertext by a generator to get a shorter word which could possibly be decoded.

Baumslag, Fine and Xu in [BFX 1] described the following general encryption scheme using free group cryptography. A further enhancement was discussed in the paper [BFX 2].

We start with a finitely presented group
$$G = < X | R >$$

where $X = \{x_1, ..., x_n\}$ and a faithful representation
$$\rho : G \mapsto \overline{G}.$$

$\overline{G}$ can be any one of several different kinds of objects - linear group, permutation group, power series ring etc.

We assume that there is an algorithm to re-express an element of $\rho(G)$ in $\overline{G}$ in terms of the generators of G. That is if $g = W(x_1, ..., x_n, ...) \in G$ where W is a word in the these generators and we are given $\rho(g) \in \overline{G}$ we can algorithmically find g and its expression as the word $W(x_1, ..., x_n)$.

Once we have G we assume that we have two free subgroups K, H with

$$H \subset K \subset G.$$

We assume that we have fixed Schreier transversals for K in G and for H in K both of which are held in secret by the communicating parties Bob and Alice. Now based on the fixed Schreier transversals we have sets of Schreier generators constructed from the Reidemeister-Schreier process for K and for H.

$$k_1, ...k_m, ... \qquad \text{for } K$$

and

$$h_1, ..., h_t, ... \qquad \text{for } H.$$

Notice that the generators for K will be given as words in $x_1, ..., x_n$, the generators of G while the generators for H will be given as words in the generators $k_1, k_2, ...$ for K. We note further that H and K may coincide and that H and K need not in general be free but only have a unique set of normal forms so that the representation of an element in terms of the given Schreier generators is unique.

We will encode within H, or more precisely within $\rho(H)$. We assume that the number of generators for H is larger than the set of characters within our plaintext alphabet. Let $\mathcal{A} = \{a, b, c...\}$ be our plaintext alphabet. At the simplest level we choose a starting point i, within the generators of H, and enclode

$$a \mapsto h_i, b \mapsto h_{i+1}, ... \text{ etc.}$$

Suppose that Bob wants to communicate the message $W(a, b, c...)$ to Alice where W is a word in the plaintext alphabet. Recall that both Bob and Alice know the various Schreier transversals which are kept secret between them. Bob then encodes $W(h_i, h_{i+1}...)$ and computes in $\overline{G}$ the element $W(\rho(h_i), \rho(h_{i+1}), ..)$ which he sends to Alice. This is sent as a matrix if $\overline{G}$ is a linear group or as a permutation if $\overline{G}$ is a permutation group and so on.

Alice uses the algorithm for $\overline{G}$ relative to G to rewrite $W(\rho(h_i), \rho(h_{i+1}), ..)$ as a word $W^\star(x_1, ...x_n)$ in the generators of G. She then uses the Schreier transversal for K in G to rewrite, using the Reidemeister-Schreier process, $W^\star$ as a word $W^{\star\star}(k_1, ..., k_s..)$ in the generators of K. Since K is free or has unique normal forms this expression for the element of K is unique. Once she has the word written in the generators of K she uses the transversal for H in K to rewrite again, using the Reidemeister-Schreier process, in terms of the generators for H. She then has a word $W^{\star\star\star}(h_i, h_{i+1}, ...)$ and using $h_i \mapsto a, h_{i+1} \mapsto b, ...$ decodes the message.

In actual implementation an additional *random noise factor* is added.

In [FBX 1,2] an inplementation of this process was presented that used for the base group G the classical modular group $M = PSL(2, \mathbb{Z})$. Further it was a polyalphabetic cipher which was secure.

The system in the modular group M was presented as follows. A list of finitely generated free subgroups $H_1, ..., H_m$ of M is public and presented by their systems of generators (presented as matrices). In a full practical implementation it is assumed that m is large. For each H_i we have a Schreier transversal

$$h_{1,i}, ..., h_{t(i),i}$$

and a corresponding ordered set of generators

$$W_{1,i}, ..., W_{m(i),i}$$

constructed from the Schreier transversal by the Reidemeister-Schreier process. It is assumed that each $m(i) >> l$ where l is the size of the plaintext alphabet, that is each subgroup has many more generators than the size of the plaintext alphabet. Although Bob and Alice know these subgroups in terms of free group generators, what is made public are generating systems given in terms of matrices.

The subgroups on this list and their corresponding Schreier transversals can be chosen in a variety of ways. For example the commutator subgroup of the Modular group is free of rank 2 and some of the subgroups H_i can be determined from homomorphisms of this subgroup onto a set of finite groups.

Suppose that Bob wants to send a message to Alice. Bob first chooses three integers (m, q, t) where

$$m = \text{ choice of the subgroup } H_m$$

$$q = \text{ starting point among the generators of } H_m$$

$$\text{for the substitution of the plaintext alphabet}$$

$$t = \text{ size of the message unit .}$$

We clarify the meanings of q and t. Once Bob chooses m, to further clarify the meaning of q, he makes the substitution

$$a \mapsto W_{m,q}, b \mapsto W_{m,q+1}, ...$$

Again the assumption is that $m(i) >> l$ so that starting almost anywhere in the sequence of generators of H_m will allow this substitution. The message unit size t is the number of coded letters that Bob will place into each coded integral matrix.

Once Bob has made the choices (m, q, t) he takes his plaintext message $W(a, b, ...)$ and groups blocks of t letters. He then makes the given substitution above to form the corresponding matrices in the Modular group;

$$T_1, ..., T_s.$$

We now introduce a *random noise factor*. After forming $T_1, ..., T_s$ Bob then multiplies on the right each T_i by a random matrix in M, say R_{T_i} (different for each T_i). The only restriction on this random matrix R_{T_i} is that there is no free cancellation in forming the product $T_i R_{T_i}$. This can be easily checked and ensures that the freely reduced form for $T_i R_{T_i}$ is just the concatenation of the expressions for T_i and R_{T_i}. Next he sends Alice the integral key (m, q, t) by some public key method (RSA, Anshel-Goldfeld etc.). He then sends the message as s random matrices

$$T_1 R_{T_1}, T_2 R_{T_2}, ..., T_s R_{T_s}.$$

Hence what is actually being sent out are not elements of the chosen subgroup H_m but rather elements of random right cosets of H_m in M. The purpose of sending coset elements is two-fold. The first is to hinder any geometric attack by masking the subgroup. The second is that it makes the resulting words in the Modular Group generators longer, effectively hindering a brute force attack.

To decode the message Alice first uses public key decryption to obtain the integral keys (m, q, t). She then knows the subgroup H_m, the ciphertext substitution from the generators of H_m and how many letters t each matrix encodes. She next uses the algorithms described above to express each $T_i R_{T_i}$ in terms of the generators of M. She has knowledge of the Schreier transversal, which is held secretly by Bob and Alice, so now uses the Reidemeister-Schreier rewriting process to start expressing this freely reduced word in terms of the generators of H_m. The Reidemeister-Schreier rewriting is done letter by letter from left to right (see [MKS]). Hence when she reaches t of the free generators she stops. Notice that the string that she is rewriting is longer than what she needs to rewrite in order to decode as a result of the random matrix R_{T_i}. This is due to the fact that she is actually rewriting not an element of the subgroup but an element in a right coset. This presents a further difficulty to an attacker. Since these are random right cosets it makes it difficult to pick up statistical patterns in the generators even if more than one message is intercepted. In practice the subgroups should be changed with each message.

The initial key (m, q, t) is changed frequently. Hence as mentioned above this method becomes a type of polyalphabetic cipher. Polyalphabetic ciphers have historically been very difficult to decode.

A further variation of this method using the Magnus representation in a formal power series ring in noncommuting variables over a field was described in [BBFR].

4   Public Key Exchange Using Nonabelian Groups

Among the first attempts to use nonabelian groups in cryptography were the schemes of Anshel-Anshel-Goldfeld[AAG] and Ko-Lee et.al.[KoL]. Both sets of authors, at about the same time, proposed using nonabelian groups and combinatorial group theory for public key exchange. The security of these systems depended on the difficulty of solving certain "hard" group theoretic problems.

The methods of both Anshel-Anshel-Goldfeld and Ko-Lee can be considered as group the-oretic analogs of the number theory based Diffie-Hellman method. The basic underlying idea is the following. If G is a group and $g, h \in G$ we let g^h denote the conjugate of g by h, that is $g^h = h^{=1}gh$. The simple observation is that this behaves like ordinary exponen-tiation in that $(g^{h_1})^{h_2} = g^{h_1 h_2}$. From this straightforward idea one can exactly mimic the Diffie-Hellman protocol within a nonabelian group.

Both the Anshel-Anshel-Goldfeld protocol and the Ko-Lee protocol start with a platform group G given by a group presentation. A major assumption in both protocols is that the elements of G have nice unique normal forms that are easy to compute for given group elements. However it is further assumed that given normal forms for $x, y \in G$, the normal form for the product xy, does not reveal x or y.

We describe the Anshel-Anshel-Goldfeld public key exchange protocol first. Let G be the platform group given by a finite prsentation and with the assumptions on normal forms as described above.

Alice and Bob want to communicate a shared secret. First, Alice and Bob choose random finitely generated subgroups of G by giving a set of generators for each.

$$A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_m\}$$

and make them public. The subgroup A is Alice's subgroup while the subgroup B is Bob's subgroup.

Alice chooses a secret group word $a = W(a_1, ..., a_n)$ in her subgroup while Bob chooses a secret group word $b = V(b_1, ..., b_m)$ in his subgroup. For an element $g \in G$ we let $NF(g)$ denote the normal form for g. Alice knows her secret word a and knows the generators b_i of Bob's subgroup. She makes public the normal forms of the conjugates

$$NF(b_i^a), i = 1, ..., m.$$

Bob knows his secret word b and the geerators a_i of Alice's subgroup and makes public the normal forms of the conjugates

$$NF(a_j^b), j = 1, ..., n).$$

The common shared secret is the commutator

$$[a, b] = a^{-1}b^{-1}ab = a^{-1}a^b = (b^a)^{-1}b$$

Notice that Alice knows a^b since she knows a in terms of generators a_i of her subgroup and she knows the conjugates by b since Bob has made the conjugates of the generators of A by b public. Since Alice knows a^b she knows $[a, b] = a^{-1}a^b$.

In an analogous manner Bob knows $[a, b] = (b^a)^{-1}b$. An attacker would have to know the corresponding **conjugator**, that is the element that conjugates each of the generators. Given elements g, h in a group G where it is known that $g^k = k^{-1}gk = h$ the **conjugator search problem** is to determine the conjugator k. It is known that this problem is undecidable in

general, that is there are groups where the conjugator cannot be determined algorithmically. On the other hand there are groups where the conjugator search problem is solvable but "difficult", that is the complexity of solving the conjugator search problem is hard. Such groups become the ideal platform groups for the Anshel-Anshel-Goldfeld protocol.

The security in this system is then in the difficulty of the **conjugator search problem**. Anshel, Anshel, Goldfeld suggested the Braid Groups as potential platforms and use for example B_{80} with 12 or more generators in the subgroups. Their suggestion and that of Ko and Lee led to development of **braid group cryptography**. There have been various attacks on the Braid group system. However some have been handled by changing the parameters. In general the ideas remain valid despite the attacks. We will discuss this further in section 7.

Ko, Lee et. al. [KoL] developed a similar system that is a direct translation of the Diffie-Hellman protocol to a nonabelian group theoretic setting. Its security is based on the difficulty of the **conjugacy problem**. We again assume that the platform group has nice unique normal forms that are easy to compute for a given group element but hard to recover the group element. Recall again that g^h means the conjugate of g by h

In the Ko-Lee protocol, Alice and Bob choose commuting subgroups A and B of the platform group G. A is Alice's subgroup while Bob's subgroup is B and these are secret. Now they completely mimic the classical Diffie-Hellman technique. There is a public element $g \in G$, Alice chooses a random secret element $a \in A$ and makes public g^a. Bob chooses a random secret element $b \in B$ and makes public g^b. The secret shared key is g^{ab}. Notice that $ab = ba$ since the subgroups commute. It follows then that $(g^a)^b = g^{ab} = g^{ba} = (g^b)^a$ just as if these were exponents. Hence both Bob and Alice can determine the common secret. The difficulty is in the difficulty of the conjugacy problem.

The **conjugacy problem** for a group G, or more precisely for a group presentation for G, is given $g, h \in G$ to determine algorithmically if they are conjugates. As with the conjugator search problem it is known that the conjugacy is undecidable in general but there are groups where it is but hard. These groups then become the target platform groups for the Ko-Lee protocol. As with the Anshel-Anshel-Goldfeld protocol, Ko and Lee suggest the use of the Braid groups.

The conjugacy problem and the conjugator search problem are only two of the group theoretic search and decision problems that have been employed to construct one way functions in a cryptogrpahic setting. We recall several other important such problems and then reference their use in encryption and public key exchange .

Definition 4.1 (Word Problem). *Given a finitely presented group G, does there exist an algorithm to decide whether or not a word in the generators is the trivial word?*

Definition 4.2 (Decision Conjugacy Problem). *Given a group G with a finite presentation, does there exist an algorithm to decide whether or not an arbitrary pair of words u and v in the generators of G are conjugate? That is, is there an $x \in G$ such that $x^{-1}ux = v$?*

Definition 4.3 (Decomposition Problem). *Let G be a finitely presented group with subgroups $A, B \leq G$. Given two elements u and v of G, is there an algorithm to find two elements $a \in A$ and $b \in B$ such that $aub = v$?*

Definition 4.4 (Simultaneous Search Conjugacy problem)**.** *Let G be a finitely presented group. Given $u_1, \cdots, u_k, v_1, \cdots, v_k \in G$ with $x^{-1} u_i x = v_i$ for each $i \in \{1, 2, \cdots, k\}$, is there an algorithm to find $z \in G$ satisfying $z^{-1} u_i z = v_i$ for each $i \in \{1, 2, ..., k\}$?*

We now reference some uses for these problems. Wagner, Birget, Magliaveras and Sramka developed key exchange protocols based on the word problem. Kurt developed a protocol based on the decomposition problem. Shpilrain and Ushakov developed a protocol based on the twisted conjugacy problem while Shpilrain and Zapata developed several encryption protocols based on various decision problems. Complete descriptions of these protocols can be found in [MSU]. More recently Anshel and Kahrobaei developed a noncommutative analog of the Cramer-Shoup key exchange method [AK].

We close this section by describing a noncommutative analog of the El Gamal public key exchange system based on the search conjugacy problem. It was proposed by Kahrobaei and Khan [KKh]. As with the Ko-Lee and Anshel-Anshel-Goldfeld protocols we start with a finitely presented platform group G given by a group presentation. As before the major assumptions are that the elements of G have nice unique normal forms that are easy to compute for given group elements. However it is further assumed that given normal forms for $x, y \in G$ the normal form for the product xy does not reveal x or y. Further G contains two commuting finitely generated proper subgroups S and T. The cryptographic goal is for Alice and Bob to establish a session key over an unsecured network.

Bob chooses a secret element $s \in S$ and an arbitrary element $b \in G$. Bob publishes b and $c = b^s$. Suppose Alice wants to send $x \in G$ as a session key to Bob. Then,

1. Alice chooses a random $t \in T$ and sends $E = x^{(c^t)}$ to Bob along with the header $h = b^t$.

2. Bob then calculates $(b^t)^s = (b^s)^t = c^t$.

3. Now, Bob may calculate $E' = (c^t)^{-1}$, allowing him to decrypt the session key since $(x^{(c^t)})^{E'} = (x^{(c^t)})^{(c^t)^{-1}} = x$.

The feasibility of this scheme relies on the assumption that products and inverses in G can be computed efficiently. Determining Bob's private key s entails solving the search conjugacy problem for G. That is given c, b, and $c = b^s$, determine s. Hence, the security of this scheme is based on the assumption that there is no practical algorithm for solving the search conjugacy problem for G.

A second El Gamal analog based on the search power conjugay problem was also proposed by Kahrobaei and Khan (see [KKh]).

5   The Shamir Three Pass and Key Transport Protocols

A **key transport protocol** is a method that allows the sending of a key (telling for example what encryption system to use) from one user to another over a public airway. A group

theoretic key transport protocol based on the Diffie-Hellman scheme can be developed in the following manner. Suppose that we have a finitely presented group G with the same assumptions made as in the Anshel-Anhsel-Goldfeld and Ko-Lee protocols. That is, G is given by a presentation and the elements of G have nice normal forms. Further it is assumed that G has two large subgroups A_1, A_2 that commute elementwise. Alternatively we could use one large abelian subgroup A of G. The meaning of large is of course hazy but here means that within G it is difficult to determine when an arbitrary element is in A (or A_1, A_2) and further A (or A_1, A_2) is large enough so that random choices can be made from them.

Now suppose that Bob wants to communicate with Alice via an open airway. The secret key telling them which encryption system to use is encoded within the finitely generated group G with the properties given above. The two subgroups A_1, A_2 which commute elementwise are kept secret by Bob and Alice. A_1 is the subgroup for Bob and A_2 the subgroup for Alice. Bob wants to send the key $W \in G$ to Alice. He chooses two random elements $B_1, B_2 \in A_1$ and sends Alice the message (in encrypted form) $B_1 W B_2$. Alice now chooses two random elements $C_1, C_2 \in A_2$ and sends $C_1 B_1 W B_2 C_2$ back to Bob. These messages appear in the representation of G and hence for example as matrices or as reduced words in the generators so they don't appear as solely concatenation of letters. Since A_1 commutes elementwise with A_2 we have

$$C_1 B_1 W B_2 C_2 = B_1 C_1 W C_2 B_2.$$

Further since Bob knows his chosen elements B_1 and B_2 he can multiply by their inverses to obtain $C_1 W C_2$ which he then sends back to Alice. Since Alice knows her chosen elements C_1, C_2 she can multiply by their inverses to obtain the key W. It is assumed that for each message Bob and Alice would choose different pairs of random elements from either A_1 or A_2. This method is known as a **Shamir Three-Pass** which was introduced by Shamir for general algebraic objects.

Notice that although this is roughly based on the Diffie-Hellman method it is not symmetric in the communicating parties. In the present scheme the secret key is completely determined by Bob, who then communicates it to Alice. The scheme then falls into the class of key transport protocols rather than public key exchange protocols. Key transport protocols are in most cases designed assuming that an underlying encryption system (and usually also a signature verification system) is in place. The security of the key transport protocol will rely on the security of these auxillary schemes. In the group theoretic proposal the encryption scheme is suggested to be done within the same group as the key transport protocol alhtough this is not essential. In the group theortic key transport protocol an attacker has knowledge of the overall group G and a view of encrypted messages. The security lies in the difficulty of determining the elementwise commuting subgroups A_1, A_2, which are kept secret by Bob and Alice, and in the security of the actual encryption scheme.

A group G is a candidate platform group for this type of key transport protocol if it has either a nice finite presentation $G = < X, R >$ with workable normal forms and has either a large abelian subgroup A or two large subgroups A_1, A_2 that commute elementwise. Although the word large here is ambiguous we mean large enough so that random choices can be made from them. In particular, for example, cyclic subgroups are inappropriate. There also should be some tie between the group used for the key exchange and the encryption method although

this is not essential. The standard braid groups, that will be described in the section 7 have several possibilitiess for normal forms and have large commuting subgroups. Hence they are excellent candidates for this method. In [BCFRX] several additional potential platform groups were suggested. These include the full automorphism group of a finitely generated free group, the matrix group $SL(4, \mathbb{Z})$ and the surface braid groups. Shpilrain and Ushakov [SU] used this method employing Thompson's group F as a platform. A length based attack on their system was attempted by Tsaban (see [MSU]). Further work on this method in the surface braid groups was done by Camps[C].

6 Digital Signatures, Authentication and Password Security

Authentication is the process of determining that a message, supposedly from a given person, both does come from that person and has not been tampered with. Authentication plays a major role in transmitting encrypted messages. Often this takes the form of a **digital signature**. A signature scheme provides a way for each user to sign messages so that the signatures can later be verified by anyone else. More specifically, each user can create a matched pair of private and public signature for the message (using the signer's public key). The verifier can convince himself that the message contents have not been altered since the message was signed. Also the signer cannot later repudiate having signed the message, since no one but the signer possesses his private key. By analogy with the paper world, where one might sign a letter and seal it in an envelope, one can sign an electronic message using one's private key, and then seal the result by encrypting it with the recipient's public key. The recipient can perform the inverse operations of opening the letter and verifying the signature to electronic mail are quite widespread today already (see [GoB])

A digital signature scheme within the public key framework, is defined as a triple of algorithms (A, σ, V) such that

- Key generation algorithm A is a probabilistic, polynomial-time algorithm which on input a security parameter 1^k, produces pairs (P, S) where P is called a public key and S a secret key. (We use the notation $(P, S) \in G(1^k)$ to indicate that the pair (P, S) is produced by the algorithm A.)

- Signing algorithm σ is a probabilistic polynomial time algorithm which is given a security parameter 1^k, a secret key S in range $A(1^k)$, and a message $m \in \{0, 1\}^k$ and produces as output string s which we call the signature of m. (We use notation $s \in \sigma(1^k, S, m)$ if the signing algorithm is probabilistic and $s = \sigma(1^k, S, m)$ otherwise. As a shorthand when the context is clear, the secret key may be omitted and we will write $s \in \sigma(S, m)$ to mean that s is the signature of message m.)

- Verification algorithm V is a probabilistic polynomial time algorithm which given a public key P, a digital signature s, and a message m, returns 1 (i.e "true") or 0 (i.e "false") to indicate whether or not the signature is valid. We require that

$V(P, s, m) = 1$ if $s \in \sigma(m)$ and 0 otherwise. (We may omit the public key and abbreviate $V(P, s, m)$ as $V(s, m)$ to indicate verifying signature s of message m when the context is clear.)

- The final characteristic of a digital signature system is its security against a probabilistic polynomial time forger. We delay this definition until later.

We present a digital signature procedure based on nonabelian groups developed by Ko, Lee et al(see [KCCL]. Here is the scheme:

Let G be a non-abelian group in which the search conjugacy problem is infeasible and the decision conjugacy problem is solvable. Let $h : \{0, 1\}^* \to G$ be a hash function.

Key Generation: Alice wants to sign and send a message, m, to Bob. Alice begins by choosing two conjugate elements $u, v \in G$ with conjugator a. The conjugate pair (u, v) is public information while the conjugator a is Alice's secret key.

Signature Generation: Alice chooses arbitrary $b \in G$, and computes $\alpha = u^b$ and $y = h(m\alpha)$. Then a signature σ on the message m is the triple (α, β, γ) where $\beta = y^b$ and $\gamma = y^{a^{-1}b}$. She sends this to Bob for verification and acceptance.

Verification: Upon receiving the signature, Bob checks whether or not the following hold:

1. $\exists c_1 \in G$ such that $u = \alpha^{c_1}$.

2. $\exists c_2, c_3 \in G$ such that $\gamma = \beta^{c_2}$ and $y = \gamma^{c_3}$.

3. $\exists c_4 \in G$ such that $uy = (\alpha\beta)^{c_4}$.

4. $\exists c_5 \in G$ such that $vy = (\alpha\gamma)^{c_5}$.

Bob accepts the signature if and only if 1-4 hold.

The security of this scheme lies in the assumption that given a pair of conjugate elements $u, v \in G$ finding elements α, β, γ such that 1-4 above hold is infeasible. If the conjugator a could be found then, then $(\alpha, \beta, \gamma) = (u^b, y^b, y^{a^{-1}b})$ satisfy properties 1-4 for any $b \in G$. Hence, the conjugacy search problem need be infeasible.

We mention that there is digital signature scheme proposed by Anjaneyulu et. al [AVRR] that uses a noncommutative platform but outside of group theory. In this proposal the basic cryptographic platform is a commutative division semiring and uses what is teremed the polynomial syemmtrical decompisiton problem as the one way function. The reader can refer to the paper [AVVR] for details.

Closely related to digital signatures is the problem of **secure password verification**. With the increased use of bank cards and internet credit card transactions there is at present

more than ever a need for secure password identification. For many online purchases this is being carried out by a **challenge response system** (see [W]) accompanying the password. In the simplest systems this takes the form of secondary password questions such as the user's mother's maiden name or place of birth. There are inherent difficulties with these types of challenge response systems. First of all there is the trivial problem of the users remembering their responses. More critical is the problem that this type of information for many people is readily available and easily found or guessed by would-be attackers or eavesdroppers. Challenge response systems are also subject to middleman attacks and replay attacks (see [CRW]. There have been several attempts to alleviate these problems, including zero-knowledge password proofs and challenged responses somewhat based on RSA as well as timed out responses (see CRAM-MD5, Password Authenticated Key Agreement, [CRRSA] and [W]).

In [BBFT] an alternative method for challenge response password verification using combinatorial group theory was developed which is provably secure against cipher-text only, man in the middle and replay attacks. In particular this method depends upon the difficulty of solving the word problem within a given finitely presented group without knowing the presentation and the difficulty of solving systems of equations within free groups. This latter problem has been proved to be NP-hard. The method uses the **group randomizer system** which is a computer program that is a subset of MAGNUS a much larger computer algebra system designed to handle algorithm problems in combinatorial group theory, MAGNUS was developed at CAISS, the Center for Algorithms and Interactive Scientific Software, a research laboratory housed at City College of the City University of New York and under the direction of the first author. The group randomizer system can be placed on a simple hand held computer device presently under development at CAISS. The system can also be used from computer to computer (see [BBFT]).

These group theoretic techniques have several major advantages over other challenge response systems. We will call the password presenter the **prover** and the presentee the **verifier**. The methods we present can be used for **two-way authentication**. That is the same method can both authenticate the prover to the verifier and authenticate the verifer to the prover. To each prover in conjunction with a standard password there will be assigned a finitely presented group with a solvable word problem. This is the **challenge group**. This will be done randomly by the group randomizer system and will be held in secret by the prover and the verifier. Cryptographically we assume the adversary can steal the encrypted form of the group theoretic responses. Probabilistically this does not present a problem. Each challenge response set of questions forms a virtual one time key pad. Therefore the adversary must steal three things - the original password, the challenge group and the group randomizer. Hence there is almost total security in the challenge response system. Further there is an infinite supply of finitely presented groups to use as challenge groups and an infinite supply of challenge response questions that never have to be duplicated.

The theoretical security of the system is provided by several results in asymptotic group theory. In particular a result of Lysenok [L] implies that stealing the challenge group is NP-hard while a result of Jitsukawa [J] says that the asymptotic density of using homomorphisms to attack the group randomizer protocol is zero (see sections 9 and 10)

In brief outline the system works as follows. We assume that each prover has a group randomizer system. At the most basic level the group randomizer system has the ability to do the following things:

(1) Recognize a finite presentation of a finitely presented group with a solvable word problem and manipulate arbitrary words in the alphabet of generators according to the rewriting rules of the presentation. In particular if the group is automatic the group randomizer can rewrite an arbitrary word in the generators in terms of its group normal form.

(2) Given a finite presentation of a group with a solvable word problem recognize whether two free group words have the same value in the given group when considered in terms of the given generators of the group

(3) Randomly generate free group words on an alphabet of any finite size

(4) Recognize and store sets of free group words $W_1, ..., W_k$ on an alphabet $x_1, ..., x_n$ and rewrite words $W(W_1, ..., W_k)$ as the corresponding word in $x_1, ..., x_n$.

(5) Given a free group of finite rank on $x_1, ..., x_n$ and a set of words $W_1, ..., W_k$ on an alphabet $x_1, .., x_n$ solve the membership problem in F relative to $H = <W_1, ..., W_k>$, the subgroup of F generated by $W_1, ..., W_k$.

(6) Given a stored finitely presented group or a stored set of free group words the randomizer can accept a random free group word and rewrite it as a normal form in the finitely presented group in the former case or as a word in the ambient free group in the latter case.

Each prover further has a standard password. Suppose that F is a free group on $\{x_1, ..., x_n\}$. The prover's password is linked to a finitely generated subgroup of a free group given as words in the generators - that is the prover's password is linked to $W_1, ..., W_k$ where each W_i is a word in $x_1, ..., x_n$. The group $G = <W_1, ..., W_k>$ is called the **challenge group**. In general $k \neq n$. The prover doesn't need to know the generators. The randomizer can randomly choose words from this subgroup and then freely reduce them. The verifier has the challenge group or subgroup also stored in its randomizer. From the viewpoint of cryptology this is a symmetric key protocol with both prover and verifier having a common shared secret, (P, G), where P is a standard password and G is the challenge group. The shared secret is set at initialization of the protocol by some direct communication. This is the most common model for password security.

The prover submits his or her standard password to the verifier. This activates the verifier's randomizer to the prover's set of words. The verifier now submits a random free group word on $y_1, ..., y_k$ to the prover's randomizer say $W(y_1, ..., y_k)$. The prover's randomizer treats this as $W(W_1, ..., W_k)$ and then reduces it in terms of the free group generators $x_1, ..., x_n$ and rewrites it as $W^\star(x_1, ..., x_n)$. The verifier checks that this is correct - that is $W(W_1, ..., W_k) = W^\star(x_1, ..., x_n)$ on the free group on $x_1, ..., x_n$. If it is the verifier continues and does this three (or some other finite number) of times. There is one proviso. The verifier submits a word to the prover only once so that a submitted word can never be reused. The prover's randomizer will recognize if it has (this is a verification to the prover of the verifier).

To verify that the verifier is legitimate the process is repeated from the prover's randomizer to the verifier.

In [BBFT] several variations of this basic outline using more general finitely presented groups were presented. Using some results from asymptotic group theory it was shown that this method is provably secure.

7 Braid Group Cryptography and Platform Groups

As platform groups for their respective protocols, both Ko-Lee and Anshel-Anshel-Goldfeld suggested the braid groups B_n (see [Bi]). The groups in this class of groups possess the desired properties for the key exchange and key transport protocols; they have nice presentations with solvable word problems and conjugacy problems; the solution to the conjugacy and conjugator search problem is "hard"; there are several possibilities for normal forms for element and they have many choices for large commuting subgroups. Initially the braid groups were considered so ideal as platforms that many other cryptographic applications were framed within the braid group setting. These included **authentication** and **digital signatures**. There was so much enthusiasm about using these groups that the whole area of study was named **braid group cryptography**. A comprehensive and well-written article by Dehornoy [D] provides a detailed overview of the subject and we refer the reader to that for technical details.

After the initial successes with braid group cryptographic schemes there were some surprisingly effective attacks. There were essentially three types of attacks; an attack using solutions to the conjugacy and conjugator search problems, an attack using heuristic probability within B_n and an attack based on the fact that there are faithful linear representations of each B_n (see [D]). What is most surprising is that the Anshel-Anshel-Goldfeld method was susceptible to a length based attack. In the Anshel-Anshel-Goldfeld method the **parameters** are the specific braid group B_n and the rank of the secret subgroups for Bob and Alice. A length based attack essentially broke the method for the initial parameters suggested by Anshel, Anshel and Goldfeld in [AAG]. The parameters were then made larger and attacks by this method were less successful. However this led to research on why these attacks on the conjugator search problem within B_n were successful, What was discovered was that **generically** (see sections 9 and 10) a random subgroup of B_n is a free group and hence length based attacks are essentially attacks on free group cryptography and therefore successful. What this indicated was that although randomness is important in cryptography in using the braid groups as platforms subgroups cannot be chosen purely randomly.

Braid groups arise in several different areas of mathematics and have several equivalent formulations. What we do in the remainder of this section is describe the braid groups. A complete topological and algebraic description can be found in the book of Joan Birman [Bi].

A **braid** on n strings is obtained by starting with n parallel strings and intertwining them. We number the strings at each vertical position and keep track of where each individual string begins and ends. We say that two braids are equivalent if it is possible to move the strings of one of the braids in space without moving the endpoints or moving through a string and obtain the other braid. A braid with no crossings is called a **trivial braid**. We

form a product of braids in the following manner. If u is the first braid and v is the second braid then uv is the braid formed by placing the starting points for the strings in v at the endpoints of the strings in u. The inverse of a braid is the mirror image in the horizontal plane. It is clear that if we form the product of a braid and its mirror image we get a braid equivalent to the trivial braid. With these definitions, the set of all equivalence classes of braids on n strings forms a group B_n. We let σ_i denote the braid that has a single crossing from string i over string $i+1$. Since a general braid is just a series of crossings it follows that B_n is generated by the set $\sigma_i; i = 1, ..., n-1$.

There is an equivalent algebraic formulation of the braid group B_n. Let F_n be a free group on the n generators $x_1, ..., x_n$ with $n > 2$. Let $\sigma_i, i = 1, ..., n-1$ be the automorphism of F_n given by

$$\sigma_i : x_i \mapsto x_{i+1}, x_{i+1} \mapsto x_{i+1}^{-1} x_i x_{i+1}$$

$$\sigma_i; x_j \mapsto x_j, j \neq i, i+1.$$

Then each σ_i corresponds precisely to the basic crossings in B_n. Therefore B_n can be considered as the subgroup of $Aut(F_n)$ generated by the automorphisms σ_i, Artin proved [A] (see also [MKS]) that a finite presentation for B_n is given by

$$B_n = < \sigma_1, ..., \sigma_{n-1}; [\sigma_i, \sigma_j] = 1 \text{ if } |i - j| > 1, x_{i+1} x_i x_{i+1} = x_i x_{i+1} x_i, i = 1, ..., n-1 > .$$

This is now called the Artin presentation. The fact that B_n is contained in $Aut(F_n)$ provides an elementary solution to the word problem in B_n since one can determine easily if an automorphism of F_n is trivial on all the generators. We note that although the braid groups B_n are linear (the Lawrence-Krammer representation is faithful (see[D]) it is known that $Aut(F_n)$ is not linear (see [F]).

There are several possibilities for normal forms for elements of B_n. The two most commonly used are the **Garside normal form** and the **Dehornoy normal form**. These are described in [D] and [MSU].

From the commuting relations in the Artin presentation it is clear that each B_n has the requisite collection of commuting subgroups.

The conjugacy problem for B_n was originally solved by Garside and it was assumed that it was hard in the complexity sense. Recently there has been significant research on the complexity of the solution to the conjugacy problem (see [MSU] and [D]).

In general, platform groups for the non-commutative protocols that we have discussed require certain properties. Most are present in the braid groups. The first is the existence of a normal form for elements in the group. Normal forms provide an effective method of disguising elements. Without this, one can determine a secret key simply by inspection of group elements. The existence of a normal form in a group implies that the group has solvable word problem, which is essential for these protocols. For purposes of practicality, the group also needs an efficiently computable normal form, which ensures an efficiently solvable word problem.

In addition to the platform group having normal form, ideally, it would also exhibit exponential growth. That is, the growth function for G, $\gamma : \mathbb{N} \to \mathbb{R}$ defined by $\gamma(n) =$

$\#\{w \in G : l(w) \leq n\}$, has an exponential growth rate. Exponential growth is a necessity since this ensures that the group will provide a large key space, making a brute force search for the secret key an infeasible algorithm.

The other property which is necessary in most of the proposed cryptosystems is the conjugator search problem for the platform group ideally should have exponential time complexity.

In addition to these, for the Ko-Lee type protocols, we need large commuting subgroups wihtin the platform group.

Currently, there are many potential platform groups that have been suggested. The following are some of the proposed platform groups:

- Braid groups(Ko-Lee, Anshel-Anshel-Goldfeld)

- Thompson Groups (Shpilrain-Ushakov) [SU]

- Polycyclic Groups (Eick-Kahrobaei) [EK]

- Linear Groups (Baumslag-Fine-Xu) [BFX 1,2]

- Free metabelian Groups (Shpilrain-Zapata) [SZ]

- Grigorchuk Groups (Petrides) [P]

- Groups of Matrices (Grigoriev-Ponomarenko) [GP]

- Surface Braid Groups (Camps) [C]

Many of these are discussed in [MSU].

8   Cryptography With Polycyclic Groups

In this section we briefly discuss polycyclic group cryptography which has not been extensively studied but has many of the essential features for ideal platform groups. A cryptosystem using polycyclic groups was developed in [EK].

A group G is called polycyclic if it has a series $G = G_{n+1} \geq G_n \geq \cdots \geq G_2 \geq G_1 = 1$ in which each G_i is a normal subgroup of G_{i+1} and G_{i+1}/G_i is cyclic for $i = 1, 2, \cdots, n$. A series of this type is called a polycyclic series. Polycyclic groups are a natural non-commutative generalization of cyclic groups. The book of Holt et,al. [HEO] is a good reference for information about polycyclic groups.

Every polycyclic group G has a finite presentation of the form:

$$\left\langle a_1, \cdots, a_n | a_i^{a_j} = w_{ij}, a_i^{a_j^{-1}} = v_{ij}, a_k^{r_k} = u_{kk} \right\rangle$$

for $1 \leq j < i \leq n$ where $r_i \in \mathbb{N} \cup \infty$, $r_i < \infty$ if $i \in I \subseteq \{1, 2, \cdots, n\}$ and w_{ij}, v_{ij}, u_{jj} are words in the generators $a_{j+1}, \cdots, a_n$. If $r_i = [G_{i+1} : G_i]$ for each $i \in \{1, 2, \cdots, n\}$

then this presentation is called a consistent polycyclic presentation. Every element in the group defined by this consistent polycyclic presentation may be written uniquely in the form $a_1^{e_1} \cdots a_n^{e_n}$ with $e_i \in \mathbb{Z}$ and $0 \le e_i < r_i$ if $i \in I$. This unique representation of each element $g \in G$ is called the normal form of G. It is known that every polycyclic group exhibits a consistent polycyclic presentation. Hence, every polycyclic group has a normal form. This is used as a basis for computations with polycyclic groups.

The word problem can be solved effectively using the collection algorithm in a group G given by a consistent polycyclic presentation. The collection algorithm computes the unique normal form for an element, g, in the group given by a word in the generators. This is done by repeatedly applying the power and conjugacy relations given in the presentation to subwords of g, transforming g to an equivalent word. The nature of the relations ensures that this process must terminate, producing the unique normal form for g. The collection algorithm is known to be a practical and effective method for solving the word problem in consistent polycyclic presentations.

Every polycyclic group can be embedded in $GL(n, \mathbb{Z})$, which reveals important properties about polycyclic groups. Since matrix multiplication is solvable in polynomial time, group multiplication in polycyclic groups is efficient. As polycyclic groups have a normal form, efficiently solvable group multiplication implies that the word problem is also efficiently solvable. It has been proven that the search conjugacy problem in any subgroup of a general linear group is solvable. Because every polycyclic group can be embedded as a subgroup of $GL(n, \mathbb{Z})$, the search conjugacy problem in polycyclic groups is solvable. The complexity of the search conjugacy problem in polycyclic groups is unknown, but widely conjectured to be exponential time.

Recall that the Anshel-Anshel-Goldfeld key exchange can be broken if the simultaneous search conjugacy problem is solvable. In polycyclic groups, the simultaneous search conjugacy problem reduces to the search conjugacy problem.

9   Generic Complexity and Asymptotic Density

In cryptanalysis involving group theoretic decision problems what is important is not just the solution to the problem but the computational complexity, polynomial or exponential for example, of the algorithm to solve the problem. The problem may be hard or even undecidable on some inputs but actually easy on most inputs. This is in reality a problem in many braid group schemes. The conjugator search problem is hard on some inputs but actually easy for many chosen subgroups. A problem may be very hard on some inputs. This is called **worst case complexity**. More important for cryptanalysis is **average case complexity**, that is the complexity on average over all inputs. **Generic complexity** refers to the complexity of the solution to a particular algorithm over most inputs. Generic and average case complexity and their uses in cryptanalysis are discussed in detail in [MSU]. There they show that generic complexity is a more useful tool in most cryptographic applications. There they show that in many cases if an algorithm is easy on average it is also easy generically. They also show that the opposite is not true and provide examples of algorithms that are exponential on average but polynomial time generically.

The problem with some braid group cryptographic algorithms is that random subgroups of B_n are generically free. We now describe what this means.

Asymptotic density is a general method to compute densities and/or probabilities on infinite discrete sets where each individual outcome is tacitly assumed to be equally likely. The method can also be used where some probability distribution is assumed on the elements. It has been effectively applied to determining densities within infinite discrete finitely generated groups where random elements are considered as being generated from random walks on the Cayley graph of the group. The paper by Borovik, Myasnikov and Shpilrain [BMS] provides a good general description of this method in group theory. Let $\mathcal{P}$ be a group property and let G be a finitely generated group. We want to determine the measure of the set of elements which satisfy $\mathcal{P}$. For each positive integer n let B_n denote the n-ball in G. Let $|B_n|$ denote the actual size of B_n (which is an integer since G is finitely generated) or the measure of $|B_n|$ if a distribution has been placed on the elements of G. Let S be the set of elements in G satisfying $\mathcal{P}$. The asymptotic density of S is then

$$\lim_{n \to \infty} \frac{|S \cap B_n|}{|B_n|}$$

provided this limit exists. We say that the property $\mathcal{P}$ is **generic** in G if the asymptotic density of the set S of elements satisfying $\mathcal{P}$ is one, $\mathcal{P}$ is called an **asymptotic visible** property, if the corresponding asymptotic density is strictly between 0 and 1. If the asymptotic density is 0, then $\mathcal{P}$ is called **negligible**.

This concept can be easily extended to properties of finitely generated subgroups, We consider the asymptotic density of finite sets of elements that generate subgroups that have a considered property. For example to say that a group has the generic free group property we mean that

$$\lim_{m,n \to \infty} \frac{|S_m \cap B_{m,n}|}{|B_{m,n}|} = 1$$

where S_m is the collection of finite sets of elements of size m that generate a free subgroup and $B_{m,n}$ is the collection of m element subsets within the n-ball.

If $\mathcal{P}$ is a group property and G is a group then we say that subgroups of G are *generically* $\mathcal{P}$ if a generic randomly chosen subgroup H of G has property $\mathcal{P}$. Equivalently this means that the asymptotic density of subgroups H of G that have property $\mathcal{P}$ is one.

10   The Generic Free Group Property

In general we say that a group G has the **generic free group property** if a finitely generated subgroup is generically a free group. A result of Epstein [E] shows that the group $GL(n, \mathbb{R})$ satisfies the generic free group property. Further G has the **strong generic free group property** if given randomly chosen elements $g_1, ..., g_n$ in G then generically they are a free basis for the free subgroup they generate. Jitsukawa [J] showed that finitely generated nonabelian free groups have the strong generic free group property while Gilman, Myasnikov and Osin [GMO] showed that torsion-free hyperbolic groups also have the generic

free group property. Myasnikov and Ushakov [MU] showed that pure braid groups P_n with $n \geq 3$ also have the strong generic free group property. A recent result of Carstensen, Fine and Rosenberger [CFR] shows that all Fuchsian groups of finite co-volume and all braid groups B_n with $n \geq 3$ have the strong generic free group property,

This result of Myasnikov and Ushakov on the pure braid groups has applications to the cryptanalysis of both the Ko-Lee cryptosystem and the Anshel-Anshel-Goldfeld cryptosystem (see [SU] and [MU]). Both cryptosystems were usually suceptible to length based attacks if the parameters chosen in the braid groups B_n were small. The reason for this is that random choices of subgroups within the braid groups are actually free groups. This does not disqualify the braid groups as platforms but rather says that subgroups cannot be chosen entirely randomly.

Extremely useful in proving that a group has the generic or strong generic free group property is the following.

Theorem 10.1. *Let G be a group and N a normal subgroup. If the quotient G/N satisfies the strong generic free group property then G also satisfies the strong generic free group property.*

In [FMR] it was shown that many group amalgams - free products, free products with amalgamtion and HNN groups, satisfy the strong generic free group property. In particular the most general result is the following.

Theorem 10.2. *Let A and B be arbitrary finitely generated infinite groups and let $G = A \star B$ be their free product. Let $\{x_1, ..., x_n\}$ be n randomly chosen elements from G. Then generically these elements are a free basis for the subgroup they generate, that is G satisfies the strong generic free group property.*

This can be extended to more general amalgams in many ways (see [FMR])

Theorem 10.3. *Let A and B be arbitrary finitely generated infinite groups and let $G = A \underset{H}{\star} B$ be their amalgamated free product with amalgamated subgroup H. Let H_1 and H_2 be the copy of H in A and B respectively. Suppose that $A/N(H_1)$ is infinite and $B/N(H_2)$ is infinite where $N(H_i)$ is the normal closure of H_i in the respective factors. Then G satisfies the strong generic subgroup property.*

Recall that a **cyclically pinched one-relator group** is a amalgamated free product of the form
$$G = F_1 \underset{\{U=V\}}{\star} F_2$$
where F_1, F_2 are finitely generated free groups and U, V are nontrivial words in the respective free groups. If U is not a power of a primitive element in F_1 and V is not a power of a primitive element in F_2 then the quotient of F_1 and F_2 by the normal closure of U and V respectively is a nontrivial, infinite one-relator group.

Corollary 10.1. *Let G be a cyclically pinched one-relator group as above. Assume that U and V are not a power of a primitive element in F_1 and F_2 respectively. Then G satisfies the strong generic subgroup property.*

In particular any orientable surface group of genus $g \geq 2$ falls into the class of cyclically pinched one-relator groups.

Corollary 10.2. *Any orientable surface group of genus $g \geq 2$ and any nonorientable surface group of genus $g \geq 4$ satisfies the strong generic subgroup property.*

The situation with HNN groups becomes even more complicated but some things can be proved as consequences of the amalgam result above. Notice first however that any HNN group with free part of rank ≥ 2 must have a free quotient of rank ≥ 2 and hence satisfy the strong generic subgroup property. Therefore only the case where the free part has rank 1 must be considered.

Theorem 10.4. *Let G be an HNN extension of the group B with a presentation*

$$G = < t, B; rel(B), t^{-1}Ut = V >$$

with U, V nontrivial isomorphic subgroups of B. Let $N_B(< U, V >)$ be the normal closure of the subgroup $< U, V >$ in B. Then if $B/N_B(< U, V >)$ is infinite, G satisfies the strong generic subgroup property.

Extensions of centralizers play a large role in the study of the elementary theory of free groups. Recall that if B is a group and $U \in B$ then a **rank one extension of centralizers** of B is a group with a presentation

$$G = < t, B; rel(B), t^{-1}UT = U > .$$

Theorem 10.5. *Let G be a rank one extension of centralizers of the group B. Suppose G has a presentation*
$$G = < t, B; rel(B), t^{-1}Ut = U >$$
wihere U is a nontrivial element of B. If $B/N_B(U)$ is infinite, where $N_B(U)$ is the normal closure of U in B, then G satisfies the strong generic subgroup property .

In the situation where the factors are finite we must be careful even for free products. The infinite dihedral group $\mathbb{Z}_2 \star \mathbb{Z}_2$ is solvable so cannot satisfy the strong generic free group property. However if at least one factor has order greater than 2, an analysis based on Kurosh bases yields the weaker generic free group property.

Theorem 10.6. *Let $G = A \star B$ be a nontrivial free product. If at least one factor has order greater than 2 then G satisfies the generic free group property.*

In [CFR] it was shown that a finitley generated group satisfies the strong generic free group property if and only if subgroups of fintie index do also. We call a group property $\mathcal{P}$ **suitable** for a finitely generated group G if it is preserved under isomorphisms and its asymptotic density is independent of finite generating systems. From a result in [MSU] the strong generic free group property is suitable in any group that has a nonabelian free quotient.

The main result in [CFR] is the following called the **inheritance theorem**.

Theorem 10.7. *Let G be a finitely generated group and $H < G$ a subgroup of finite index $[G : H] = n < \infty$. Let $\mathcal{P}$ be the strong generic free group property. Then:*

1. *If $\mathcal{P}$ is a suitable and generic property in H then it is also suitable and generic in G.*

2. *If $\mathcal{P}$ is a suitable and generic property in G then it is also suitable and generic in H.*

An interesting consequence of this is that all braid groups have the strong generic free group property.

11 Open Problems

We now give a nonexhaustive list of problems related to the rest of this article.

General:

1. What is the most appropriate platform group for non-commutative cryptography?

2. Should the group the be finite or infinite?

3. How can we show a group is provably secure for the new non-commutative schemes such as public key exchanges, digital signatures and authentication?

4. Can we design more public keys based on other search and decision problems in combinatorial group theory?

5. Can we analyze the security of this protocols?

6. what should be the measure of the security? (practicality, complexity, average case complexity, generic complexity?)

7. So far there are three known non-commutative digital signatures have been designed, can we design more non-commutative digital signatures?

8. What about the authentication schemes?

9. What is the appropriate choice of commuting subgroups in polycyclic groups which makes the described schemes secure?

Complexity Analysis and Security:

1. What is the complexity of the search conjugacy problem in polycyclic groups?

2. What is the complexity of the decision conjugacy problem, for example in Eick-Ostheimer algorithm [EO]?

3. What is the complexity of the collection algorithm to find the normal form of elements in polycyclic groups?

4. What is the complexity of the twisted search conjugacy problem in polycyclic groups?

5. What is the complexity of the Reidemeister-Schreier rewriting algorithm in free groups?

6. What is the generic case complexity of the search conjugacy problem in polycyclic groups?

Quantum Algorithms and Quantum Complexity:

Another problem to think in this direction is quantum computational approaches to these cryptosystems. Quantum algorithms for finite solvable groups (which are polycyclic) has been studied, particularly by J. Watrous (2001) [Wa]. He found a quantum algorithm to compute the order of a finite solvable group in polynomial time. Algorithm works in the setting of black-box groups none of them have polynomial-time classical algorithms. Can we design quantum algorithms for solving other decision problems in polycyclic groups (both for finite and infinite ones); especially the ones we use in cryptography.

1. Is there any quantum algorithm for solving the search conjugacy problem for polycyclic group that reduces the complexity of the algorithm?

Implementation:

1. How can we implement the proposed cryptosystems?

2. Can the computer algebra system GAP, could we use this for practical and secure cryptography?

12   Acknowledgment

13   References

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