Econometrics Homework 2

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Question 1

We have the following ordinal regression model:

$$
\begin{gathered}
z_i = x_i'\beta + \epsilon_i \quad \forall i = 1, \cdots, n \\
\gamma_{j-1} < z_i \leq \gamma_j \implies y_i = j, \quad \forall i, j = 1, \cdots, J
\end{gathered}
$$

where (in the first equation) z_i is the latent variable for individual i, x_i is a vector of covariates, β is a $k \times 1$ vector of unknown parameters, and n denotes the number of observations. The second equation shows how z_i is related to the observed discrete response y_i, where $-\infty = \gamma_0 < \gamma_1 < \gamma_{J-1} < \gamma_J = \infty$ are the cut-points (or thresholds) and y_i is assumed to have J categories or outcomes.

(a)

Probability of success

We assume that $\epsilon_i \sim N(0,1)$, for $i = 1, 2, \cdots, n$. Therefore we have,
The probability of success,

$$
\begin{aligned}
Pr(y_i = j) &= Pr(\gamma_{j-1} < z_i \leq \gamma_j) \\
&= Pr(\gamma_{j-1} < x_i'\beta + \epsilon_i \leq \gamma_j) \\
&= Pr(\gamma_{j-1} - x_i'\beta < \epsilon_i \leq \gamma_j - x_i'\beta) \\
&= \Phi(\gamma_j - x_i'\beta) - \Phi(\gamma_{j-1} - x_i'\beta) \qquad \text{[where } \Phi(\cdot) \text{ is the cdf of } N(0,1)]
\end{aligned}
$$

Likelihood function

The likelihood function for the ordinal probit model is,

$$
\begin{aligned}
L(\beta, \gamma; y) &= \prod_{i=1}^{n} \prod_{j=1}^{J} Pr(y_i = j | \beta, \gamma)^{I(y_i = j)} \qquad \text{[where } I(\cdot) \text{ is the indicator function]} \\
&= \prod_{i=1}^{n} \prod_{j=1}^{J} (\Phi(\gamma_j - x_i'\beta) - \Phi(\gamma_{j-1} - x_i'\beta))^{I(y_i = j)}
\end{aligned}
$$

(b)

Probability of success

We assume that $\epsilon_i \sim \mathcal{L}(0,1)$, for $i = 1, 2, \cdots, n$. Therefore the probability of success,

$$
\begin{aligned}
Pr(y_i = j) &= Pr(\gamma_{j-1} < z_i \leq \gamma_j) \\
&= Pr(\gamma_{j-1} < x_i'\beta + \epsilon_i \leq \gamma_j) \\
&= Pr(\gamma_{j-1} - x_i'\beta < \epsilon_i \leq \gamma_j - x_i'\beta) \\
&= \frac{1}{1 + e^{-(\gamma_j - x_i'\beta)}} - \frac{1}{1 + e^{-(\gamma_{j-1} - x_i'\beta)}} \qquad [\text{as } \epsilon_i \sim \mathcal{L}(0,1)] \\
&= \frac{e^{-(\gamma_{j-1} - x_i'\beta)} - e^{-(\gamma_j - x_i'\beta)}}{(1 + e^{-(\gamma_j - x_i'\beta)})(1 + e^{-(\gamma_{j-1} - x_i'\beta)})} \\
&= \frac{e^{x_i'\beta}(e^{-\gamma_{j-1}} - e^{-\gamma_j})}{(1 + e^{-(\gamma_j - x_i'\beta)})(1 + e^{-(\gamma_{j-1} - x_i'\beta)})}
\end{aligned}
$$

Likelihood function

The likelihood function for the ordinal logit model is,

$$
\begin{aligned}
L(\beta, \gamma; y) &= \prod_{i=1}^{n}\prod_{j=1}^{J} Pr(y_i = j|\beta,\gamma)^{I(y_i=j)} \qquad \text{[where } I(\cdot) \text{ is the indicator function]}\\
&= \prod_{i=1}^{n}\prod_{j=1}^{J} [\frac{e^{x_i'\beta}(e^{-\gamma_{j-1}} - e^{-\gamma_j})}{(1+e^{-(\gamma_j - x_i'\beta)})(1+e^{-(\gamma_{j-1}-x_i'\beta)})}]^{I(y_i=j)}
\end{aligned}
$$

(c)

Probability remain unchanged on adding a constant c to cut-points and the mean

Adding a constant c to the cut-point γ_j and the mean $x_i'\beta$, $\forall i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, J$, the latent variable z_i becomes $x_i'\beta + c + \epsilon_i$ and the cut-point γ_j becomes $\gamma_j + c$

The probability of y_i taking the value j is,

$$
\begin{aligned}
Pr(y_i = j) &= Pr(\gamma_{j-1} + c < z_i \leq \gamma_j + c)\\
&= Pr(\gamma_{j-1} + c < x_i'\beta + c + \epsilon_i \leq \gamma_j + c)\\
&= Pr(\gamma_{j-1} - x_i'\beta < \epsilon_i \leq \gamma_j - x_i'\beta)\\
&= \Phi(\gamma_j - x_i'\beta) - \Phi(\gamma_{j-1} - x_i'\beta)
\end{aligned}
$$

which is the same as the value obtained in part(a) of Question 1. So adding a constant c to the cut-point γ_j and the mean $x_i'\beta$ does not change the outcome probability.

Identification Problem

This identification problem can be solved by fixing the value of one of $\gamma_1, \gamma_2, \cdots, \gamma_{J-1}$. In particular setting $\gamma_1 = 0$ will solve this identification problem.

(d)

Rescaling the parameters (γ_j, β) and scale of distribution does not change outcome probability

Rescaling the parameters (γ_j, β) and the scale of the distribution of ϵ_i by some constant d, the latent variable z_i becomes $x_i'd\beta + \epsilon_i$ where $\epsilon_i \sim N(0, d^2)$ and the cut-point γ_j becomes $d\gamma_j, \forall j = 1, 2, \cdots, J-1$.

The probability of y_i taking the value j is,

$$
\begin{aligned}
Pr(y_i = j) &= Pr(d\gamma_{j-1} < z_i \leq d\gamma_j)\\
&= Pr(d\gamma_{j-1} < x_i'd\beta + \epsilon_i \leq d\gamma_j)\\
&= Pr(\gamma_{j-1} - x_i'\beta < \frac{\epsilon_i}{d} \leq \gamma_j - x_i'\beta)\\
&= \Phi(\gamma_j - x_i'\beta) - \Phi(\gamma_{j-1} - x_i'\beta) \qquad [\text{as } \epsilon_i \sim N(0, d^2), \frac{\epsilon_i}{d} \sim N(0,1)]
\end{aligned}
$$

which is the same as the value obtained in part(b) of Question 1. So escaling the parameters (γ_j, β) and the scale of the distribution by some arbitrary constant d lead to same outcome probabilities.

Identification problem

This identification problem can be solved by fixing the scale of the distribution of ϵ_i. In particular we can set the scale of the distribution of ϵ_i to 1, i.e. var$(\epsilon_i) = 1$.

(e)

(i) Descriptive Summary of the data

(ii) Public opinion on extent of marijuana legalization

(iii) Covariate effects of the variables

Question 2