Econometrics, Semester II, 2024-25
Homework III (100 points)

Instructor: M.A. Rahman

Deadline: 7:00 pm, April 8, 2025.

Please read the instructions carefully and follow them while writing answers.

- *Solutions to homework should be typed in LaTeXor written in A4 size loose sheets.*
- *Questions should be answered in order as they appear in the homework. Every new question should begin in a new page. Please number all the pages of your homework solution.*
- *Please leave a margin of one inch from top and one inch from left. Staple the sheets on top-left.*
- *Matlab assignments (if any) and written answers should be together and in order.*
- *Please write your name and names of your group members on the first page of your answer script.*

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1. **(5+10 = 15 points)** Let θ be an unknown parameter satisfying $0 \leq \theta \leq 1$. Consider the following two experiments involving θ. $\mathcal{E}_1 = \{Y_1, \theta, f_1(y_1|\theta)\}$ is a binomial experiment in which a coin is flipped T_1 times, where T_1 is predetermined and Y_1 is the number of "heads" obtained in the T_1 flips. $\mathcal{E}_2 = \{Y_2, \theta, f_2(y_2|\theta)\}$ is a negative binomial experiment in which a coin is flipped until m "tails" are obtained, where $m > 0$ is predetermined, and Y_2 is defined to be the number of "heads" obtained in the process.

   (a) Write down the *p.m.f.* for Y_1 and Y_2.

   (b) Suppose $T_1 = 12$ and $m = 3$, and that the two experiments yield the following results: $y_1 = y_2 = 9$. Based on this information, write down the likelihood for the two experiments. What can you say based on the Likelihood Principle?

2. **(5+10+10 = 25 points)** Consider the uniform distribution with density function $f(y_i|\theta) = 1/\theta, 0 \leq y_i \leq \theta$ and θ unknown.

   (a) Show that the Pareto distribution,

$$\pi(\theta) = a\ k^a\ \theta^{-(a+1)}, \qquad \theta \geq k,\ a > 0,$$

and 0, otherwise, is a conjugate prior distribution for the uniform distribution.

(b) Show that $\hat{\theta} = \max(y_1, \ldots, y_n)$ is the MLE of θ, where the y_i are a random sample from $f(y_i|\theta)$.

(c) Find the posterior distribution of θ and its expected value.

3. **(25 points)** Consider the following two sets of data obtained after tossing a die 100 and 1,000 times, respectively:

n	1	2	3	4	5	6
100	19	12	17	18	20	14
1000	190	120	170	180	200	140

Suppose you are interested in θ_1, the probability of obtaining a one spot. Assume your prior for all the probabilities is a Dirichlet distribution, where each $\alpha_i = 2$. Compute the posterior distribution for θ_1 for each of the sample sizes in the table. Plot the resulting distribution and compare the results. Comment on the effect of having a larger sample.

4. **(5+5+5 = 15 points)**. Suppose Y_i $(i = 1, \cdots, n)$ are *i.i.d.* EXP(θ), where the pdf of y is given by,

$$f(y_i|\theta) = \frac{1}{\theta} \exp\Big(- \frac{y_i}{\theta}\Big), \qquad \text{for } i = 1, \cdots, n.$$

(a) Derive Jeffrey's prior for θ.

(b) Derive Jeffrey's prior for $\alpha = \theta^{-1}$. What do you observe?

(c) Find the posterior density of θ corresponding to the prior density in (a). Be specific in noting the family to which it belongs.

5. **(10+10 = 20 points)** Consider the multiple linear regression model $y_i = x_i'\beta + \epsilon_i$ where $\epsilon_i|x_i \sim N(0, \sigma^2)$ for all $i = 1, \cdots, n$. Further, assume that the priors on (β, σ^2) are independent and $\pi(\beta, \sigma^2) = \pi(\beta)\pi(\sigma^2) = N_k(\beta_0, B_0)\, IG(\alpha_0/2, \delta_0/2)$. Based on the given setting answer the following without skipping any steps.

(a) Derive the conditional posterior distribution of β and show that $\pi(\beta|\sigma^2, y) \sim N(\bar{\beta}, B_1)$, where $B_1 = \big[\sigma^{-2}X'X + B_0^{-1}\big]^{-1}$ and $\bar{\beta} = B_1\big[\sigma^{-2}X'y + B_0^{-1}\beta_0\big]$.

(b) Derive the conditional posterior distribution of σ^2 and show that $\pi(\sigma^2|\beta, y) \sim IG(\alpha_1/2, \delta_1/2)$, where $\alpha_1 = \alpha_0 + n$ and $\delta_1 = \delta_0 + (y - X\beta)'(y - X\beta)$.