Econometrics Homework 2

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Question 1

We have the following ordinal regression model:

$$z_i = x_i'\beta + \epsilon_i \quad \forall i = 1, \cdots, n$$
$$\gamma_{j-1} < z_i \leq \gamma_j \implies y_i = j, \quad \forall i, j = 1, \cdots, J$$

where (in the first equation) z_i is the latent variable for individual i, x_i is a vector of covariates, β is a $k \times 1$ vector of unknown parameters, and n denotes the number of observations. The second equation shows how z_i is related to the observed discrete response y_i, where $-\infty = \gamma_0 < \gamma_1 < \gamma_{J-1} < \gamma_J = \infty$ are the cut-points (or thresholds) and y_i is assumed to have J categories or outcomes.

(a)

We assume that $\epsilon_i \sim N(0, 1)$, for $i = 1, 2, \cdots, n$. Therefore we have,
The probability of success,

$$\begin{aligned} Pr(y_i = j) &= Pr(\gamma_{j-1} < z_i \leq \gamma_j) \\ &= Pr(\gamma_{j-1} < x_i'\beta + \epsilon_i \leq \gamma_j) \\ &= Pr(\gamma_{j-1} - x_i'\beta < \epsilon_i \leq \gamma_j - x_i'\beta) \\ &= \Phi(\gamma_j - x_i'\beta) - \Phi(\gamma_{j-1} - x_i'\beta) \qquad &\text{[where } \Phi(\cdot) \text{ is the cdf of } N(0,1)] \end{aligned}$$

The likelihood function for the ordinal probit model is,

$$\begin{aligned} L(\beta, \gamma; y) &= \prod_{i=1}^{n} \prod_{j=1}^{J} Pr(y_i = j|\beta, \gamma)^{I(y_i=j)} \qquad &\text{[where } I(\cdot) \text{ is the indicator function]} \\ &= \prod_{i=1}^{n} \prod_{j=1}^{J} (\Phi(\gamma_j - x_i'\beta) - \Phi(\gamma_{j-1} - x_i'\beta))^{I(y_i=j)} \end{aligned}$$

(b)

We assume that $\epsilon_i \sim \mathcal{L}(0, 1)$, for $i = 1, 2, \cdots, n$. Therefore the probability of success,

$$\begin{aligned} Pr(y_i = j) &= Pr(\gamma_{j-1} < z_i \leq \gamma_j) \\ &= Pr(\gamma_{j-1} < x_i'\beta + \epsilon_i \leq \gamma_j) \\ &= Pr(\gamma_{j-1} - x_i'\beta < \epsilon_i \leq \gamma_j - x_i'\beta) \\ &= \frac{1}{1 + e^{-(\gamma_j - x_i'\beta)}} - \frac{1}{1 + e^{-(\gamma_{j-1} - x_i'\beta)}} \qquad &[\text{as } \epsilon_i \sim \mathcal{L}(0,1)] \\ &= \frac{e^{-(\gamma_{j-1} - x_i'\beta)} - e^{-(\gamma_j - x_i'\beta)}}{(1 + e^{-(\gamma_j - x_i'\beta)})(1 + e^{-(\gamma_{j-1} - x_i'\beta)})} \\ &= \frac{e^{x_i'\beta}(e^{-\gamma_{j-1}} - e^{-\gamma_j})}{(1 + e^{-(\gamma_j - x_i'\beta)})(1 + e^{-(\gamma_{j-1} - x_i'\beta)})} \end{aligned}$$

The likelihood function for the ordinal logit model is,

$$\begin{aligned} L(\beta, \gamma; y) &= \prod_{i=1}^{n} \prod_{j=1}^{J} Pr(y_i = j|\beta, \gamma)^{I(y_i=j)} \\ &= \prod_{i=1}^{n} \prod_{j=1}^{J} [\frac{e^{x_i'\beta}(e^{-\gamma_{j-1}} - e^{-\gamma_j})}{(1 + e^{-(\gamma_j - x_i'\beta)})(1 + e^{-(\gamma_{j-1} - x_i'\beta)})}]^{I(y_i=j)} \end{aligned}$$