Optimization in Industry Assignment 2
Mathematical Model for Optimal ATM Refill Problem

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Sets

1. $\mathcal{I}$: Set of ATMs, indexed by i (i.e., $i = 1, \ldots, |\mathcal{I}|$).
2. $\mathcal{V}$: Set of vehicles, indexed by v (i.e., $v = 1, \ldots, |\mathcal{V}|$).
3. $\mathcal{D}$: Set of days in the planning horizon, indexed by d (e.g., $d = 1, \ldots, 7$).
4. $\mathcal{K}$: Set of denominations, indexed by k.

Parameters

1. $\text{Demand}_{i,d}$: Cash demand at ATM i on day d.
2. I_i^0: Initial cash inventory at ATM i (at start of day 1).
3. C_i: Maximum cash capacity of ATM i.
4. L_i: Minimum cash level required at ATM i.
5. Q_{min}: Minimum cash deposit per refill (e.g., 50K).
6. Q_{unit}: Cash deposit unit (e.g., 10K), so that any nonzero refill is a multiple of Q_{unit}.
7. $Q_{\text{unit},k}$: Cash deposit unit (e.g., 10K) for denomination k, so that any nonzero refill is a multiple of $Q_{\text{unit},k}$.
8. cap_v: Maximum cash carrying capacity of vehicle v (security limited).
9. cost_v: Cost associated with using vehicle v for a day.
10. $\text{maxVisit}_{v,d}$: Maximum number of ATMs vehicle v can service on day d (e.g., 20 on weekdays, 30 on weekends).
11. S: Security gap in days. If vehicle v visits ATM i on day d, it cannot visit the same ATM again in the next $S - 1$ days.

12. Δ_x: Maximum allowed difference in the number of ATMs visited by any two vehicles in the planning horizon.
13. Δ_y: Maximum allowed difference in the cash deposited by any two vehicles in the planning horizon.
14. $r_{i,d,k}$: Denomination requirement for ATM i on day d for denomination k (soft constraint).
15. w_1, w_2, w_3, w_4, w_5: Weights for the objective function corresponding to vehicles used, ATM inventory holding cost, and number of visits, respectively.
16. $M\left(= 1 + \max_{v \in \mathcal{V}} \mathrm{cap}_v\right)$: A sufficiently large constant (Big-M constant).

Decision Variables

1. $x_{i,v,d} \in \{0, 1\}$: Equals 1 if vehicle v refills ATM i on day d; 0 otherwise.
2. $q_{i,v,d} \in \mathbb{Z}_{\geq 0}$: Number of cash deposit units Q_{unit}, delivered to ATM i by vehicle v on day d.
3. $y_{i,v,d} \geq 0$: Amount of cash delivered to ATM i by vehicle v on day d.

$$y_{i,v,d} = Q_{\text{unit}} \cdot q_{i,v,d}, \quad q_{i,v,d} \in \mathbb{Z}_{\geq 0}$$

   with the additional condition that if $q_{i,v,d} > 0$ then $q_{i,v,d} \geq \frac{Q_{\min}}{Q_{\text{unit}}}$.

4. $z_{v,d} \in \{0, 1\}$: Equals 1 if vehicle v is used on day d; 0 otherwise.
5. $I_{i,d}$: Cash inventory at ATM i at the end of day d.
6. $q_{i,v,d,k} \in \mathbb{Z}_{\geq 0}$: Number of cash deposit units $Q_{\text{unit},k}$ of currency k, delivered to ATM i by vehicle v on day d.
7. $y_{i,v,d,k} \geq 0$: Cash delivered in denomination k for ATM i by vehicle v on day d.

$$y_{i,v,d,k} = Q_{\text{unit},k} \cdot q_{i,v,d,k}, \quad q_{i,v,d,k} \in \mathbb{Z}_{\geq 0}$$

8. $\text{penalty}_{i,d,k} \geq 0$: Unmet demand for cash delivered in denomination k for ATM i on day d.

Objective Function

We consider a weighted-sum objective function:
Minimize:

$$
\begin{aligned}
Z = w_1 \cdot \sum_{d\in\mathcal{D}} \sum_{v\in\mathcal{V}} z_{v,d} \;+\; w_2 \cdot \sum_{d\in\mathcal{D}} \sum_{i\in\mathcal{I}} I_{i,d} \;+\; w_3 \cdot \sum_{d\in\mathcal{D}} \sum_{v\in\mathcal{V}} \sum_{i\in\mathcal{I}} x_{i,v,d} \\
+\; w_4 \cdot \sum_{i\in\mathcal{I}} \sum_{d\in\mathcal{D}} \sum_{k\in\mathcal{K}} \text{penalty}_{i,d,k} \;+\; w_5 \cdot \sum_{v\in\mathcal{V}} \sum_{d\in\mathcal{D}} \text{cost}_v \cdot z_{v,d}
\end{aligned}
$$

Explanation:

- The first term represents the number of vehicles used.
- The second term represents the total ATM cash inventory holding cost.
- The third term represents the total number of ATM visits.
- The fourth term represents the unmet denomination preferences at each of the ATM.
- The fifth term represents the cost using the vehicles during the planning horizon.

Constraints

A. ATM Inventory Balance and Capacity

(A1) Inventory Update: For each $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$
I_{i,d} = \begin{cases} I_i^0 + \sum_{v\in\mathcal{V}} y_{i,v,1} - \text{Demand}_{i,1}, & d = 1, \\ I_{i,d-1} + \sum_{v\in\mathcal{V}} y_{i,v,d} - \text{Demand}_{i,d}, & d \geq 2. \end{cases}
$$

(A2) Inventory Limits: For all $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$
L_i \leq I_{i,d}
$$

$$
I_{i,d} \leq C_i
$$

Explanation:

- (A1) Keeps track of the cash in the ATM across the planning horizon.
- (A2) Ensures the cash in the inventory does not go beyond the lower and upper limits.

B. Service (Assignment) Constraints

(B1) One Refill per ATM per Day: For every $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$\sum_{v \in \mathcal{V}} x_{i,v,d} \leq 1$$

(B2) Linking x and y: For all $i \in \mathcal{I}$, $v \in \mathcal{V}$, and $d \in \mathcal{D}$,

$$y_{i,v,d} \geq Q_{\min} x_{i,v,d}$$

$$y_{i,v,d} \leq M \cdot x_{i,v,d}$$

Explanation:

- (B1) Ensures that each ATM is filled atmost once in a day.
- (B2) Ensures that cash is either not deposited or if deposited atleast $Q_{\min}$ of cash is deposited.

C. Vehicle Capacity and Workload

(C1) Vehicle Loading Limit: For each $v \in \mathcal{V}$ and $d \in \mathcal{D}$,

$$\sum_{i \in \mathcal{I}} y_{i,v,d} \leq \mathrm{cap}_v \cdot z_{v,d}$$

(C2) Maximum Visits per Vehicle per Day: For all $v \in \mathcal{V}$ and $d \in \mathcal{D}$,

$$\sum_{i \in \mathcal{I}} x_{i,v,d} \leq \mathrm{maxVisit}_{v,d} \cdot z_{v,d}$$

Explanation:

- (C1) Ensures that total cash to be delivered by vehicle v on a day d does not exceed its capacity.
- (C2) Ensures a vehicle v does not visit more than $\mathrm{maxVisit}_{v,d}$ ATMs on day d.

D. Security and Scheduling Constraints

(D1) Spread Out ATM Visits: For all $v \in \mathcal{V}$, for each $i \in \mathcal{I}$ and for each day $d \in \mathcal{D}$ such that $d + S - 1 \leq \max(\mathcal{D})$,

$$x_{i,v,d} + \sum_{d'=d+1}^{\min(d+S-1, \max(\mathcal{D}))} x_{i,v,d'} \leq 1.$$

Explanation:

- (D1) Ensures if vehicle v visits ATM i on day d, it cannot visit the same ATM again in the next $S-1$ days.

E. Workload Balance Across Vehicles

(E1) Number of ATMs visited by vehicles: For each $v, v' \in \mathcal{V}$,

$$\sum_{i\in\mathcal{I}} \sum_{d\in\mathcal{D}} \left(x_{i,v,d} - x_{i,v',d}\right) \leq \Delta_x$$

(E2) Total Cash Deposited in ATMs by vehicles: For each $v, v' \in \mathcal{V}$,

$$\sum_{i\in\mathcal{I}} \sum_{d\in\mathcal{D}} \left(\frac{y_{i,v,d}}{\text{cap}_v} - \frac{y_{i,v',d}}{\text{cap}_{v'}}\right) \leq \Delta_y$$

Explanation:

- (E1) Ensures the difference between the number of ATM visits by any two vehicles v and v' across the planning horizon does not exceed Δ_x.
- (E2) Ensures the weighted difference between the total cash deposited by any two vehicles v and v' across the planning horizon does not exceed Δ_y. Here we have taken the Weights to be the inverse of the vehicle capacity.

F. Denomination Constraints (Soft)

(F1) For each $i \in \mathcal{I}$, $d \in \mathcal{D}$, and $k \in \mathcal{K}$,

$$\sum_{v\in\mathcal{V}} y_{i,v,d,k} + \text{penalty}_{i,d,k} \geq r_{i,d,k}$$

(F2) For each $i \in \mathcal{I}$, $d \in \mathcal{D}$

$$\sum_{k\in\mathcal{K}} y_{i,v,d,k} = y_{i,v,d}$$

Explanation:

- (F1) Penalizes unmet denomination demands for denomination k on day d at ATM i.
- (F2) Ensures total cash deposited is the sum of the total cash deposited in each denomination.

G. Deposit Size Discreteness

(G1) To ensure that if cash is either not delivered or at least Q_{min} (in multiples of Q_{unit}):

$$y_{i,v,d} = Q_{\text{unit}} \cdot q_{i,v,d}$$

$$q_{i,v,d} \geq \frac{Q_{\text{min}}}{Q_{\text{unit}}} \cdot x_{i,v,d}$$

(G2) To ensure that cash in denomination k is delivered, is in multiples of $Q_{\text{unit},k}$:

$$y_{i,v,d,k} = Q_{\text{unit},k} \cdot q_{i,v,d,k}$$

Explanation:

- (G1) Ensures that for ATM i, cash is either not deposited by a vehicle v on day d or atleast Q_{min} is deposited in multiples of Q_{unit}
- (G2) Ensures that cash in each denomination k is deposited in non-zero multiples of $Q_{\text{min},k}$.

Assumptions

- The planning horizon is one week (7 days).
- Daily demands $\text{Demand}_{i,d}$ and initial inventories I_i^0 are known.
- ATMs must always have cash between the levels L_i and C_i.
- Vehicles are loaded only at the beginning of the day (no mid-day refilling).
- Vehicles' cash capacities are limited by security considerations, which may be lower than their physical capacities.
- A refill at an ATM is done by at most one vehicle per day.
- For security, if a vehicle visits an ATM on day d, it cannot visit the same ATM again for the next $S - 1$ days.
- Refills must be in multiples of Q_{unit} (e.g., 10K), with any nonzero refill being at least Q_{min} (e.g., 50K).
- The denomination requirement is a soft constraint; unmet denomination preferences are penalized but do not prevent meeting the overall cash demand.

Output

The solution to this Mixed-Integer Programming (MIP) model will determine:

- Which vehicles are used on each day (via $z_{v,d}$).
- Which ATMs are refilled by which vehicle on each day (via $x_{i,v,d}$).
- The amount of cash delivered to each ATM on each day (via $y_{i,v,d}$).

- The amount of cash in a denomination delivered to each ATM on each day (via $y_{i,v,d,k}$).
- The evolution of ATM cash inventories $I_{i,d}$ over the planning horizon.