# Optimization in Industry Assignment 1
# Mathematical Formulation for Round Robin Match Scheduling

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## Sets

- $T$: Set of teams, indexed by $i, j$, where $i, j \in T$.
- $D$: Set of days in the tournament, indexed by $d$.

## Parameters

- $I_{i,j}$ $(i \neq j)$: Binary parameter, 1 if the match between team $i$ and $j$ is interesting, 0 otherwise
- $W_d$: Binary parameter, 1 if $d$ is a weekend, 0 otherwise.
- $\text{Dist}_{i,j}$: Distance (in kms) between home stadiums of teams $i$ and $j$.
- $E_{d,i}$: Binary parameter, 1 if there is a big event in the city of home stadium of team $i$ on day $d$, 0 otherwise.
- $\text{MinGap}, \text{MaxGap}$: Minimum and maximum gap between two consecutive matches for a team.
- $\text{MaxConsecHome}, \text{MaxConsecAway}$: Integer parameter, maximum number of consecutive home and away matches allowed for teams.
- MaxDiff: Integer parameter, maximum difference allowed between the number of home matches and away matches played by a team.
- MaxNightMatches, MinNightMatches: Integer parameter, maximum and minimum number of night matches to be played by each team.
- MaxSlot: Integer parameter, maximum number of matches allowed to be played simultaneouly in a single time slot (day/night).
- M is the Big - M constant for our formulation.

## Decision Variables

- $x_{i,j,d}$ $(i \neq j)$: Binary variable, 1 if match (day or night) is scheduled on day $d$ at home stadium of $i$ between team $i$ amd $j$, 0 otherwise.
- $N_{i,j,d}$ $(i \neq j)$: Binary variable, 1 if there is a night match between team $i$ and $j$ at home stadium of $i$ in day d, 0 otherwise.
- $\text{count}_{i,d}$ : Integer variable to count the number of matches played by team $i$ as of day $d$
- $\text{count\_home}_{i,d}$ : Integer variable to count the number of home matches played by team $i$ as of day $d$
- $\text{count\_away}_{i,d}$ : Integer variable to count the number of away matches played by team $i$ as of day $d$
- $L_{i,j,n}$: Binary variable, 1 if team $j$ played their nth match of the round robin in the home stadium of team $i$.
- $\text{Travel}_{i,j,k,n}(n \geq 2)$: Binary variable, 1 if team $k$ travel from home stadium of team $i$ to home stadium of team $j$ to play their nth match, i.e. $\text{Travel}_{i,j,k,n} = L_{i,k,n-1} \cdot L_{j,k,n}$.
- $\text{Match\_diff}_{i,j,d}(i \neq j)$: Continuous variable, which takes the value $|\text{count}_{i,d} - \text{count}_{j,d}|$
- $\text{Max\_match\_difference}_d$: Continuous variable, Max difference in the number of matches played by two teams as of day d.
- $\text{Home\_away\_diff}_{i,d}$: Continuous variable, which takes the value $|\text{count\_away}_{i,d} - \text{count\_home}_{i,d}|$
- $\text{Max\_home\_away\_diff}_d$: Continuous variable, Max difference between the number of home and away matches played by a team as of day d.
- MaxDist, MinDist: Continuous variable, maximum and minimum distance covered by a team.
- max_night_played, min_night_played: Integer variable, max and min number of night matches played by a team.
- $s_{i,j,d}(i \neq j)$: Binary variable, 1 if $\text{count}_{j,d} \geq \text{count}_{i,d}$, else 0.
- $t_{i,d}$: Binary variable, 1 if $\text{count\_home}_{i,d} \geq \text{count\_away}_{i,d}$, else 0.
- $\text{p\_away}_{i,d,d'}(d' > d)$: Binary variable, 1 if team $i$ has played atleast one away match in $\{d+1, d+2, \cdots, d'\}$, else 0.
- $\text{p\_home}_{i,d,d'}(d' > d)$: Binary variable, 1 if team $i$ has played atleast one home match in $\{d+1, d+2, \cdots, d'\}$, else 0.
- $\text{XN}_{i,j,d}(i \neq j)$: Binary variable, 1 if $x_{i,j,d} \cdot N_{i,j,d} = 1$, else 0.
- $\text{XC1}_{i,j,d}(i \neq j)$: Integer variable, equal to $x_{i,j,d} \cdot \text{count}_{j,d}$
- $\text{XC2}_{j,i,d}(i \neq j)$: Integer variable, equal to $x_{j,i,d} \cdot \text{count}_{j,d}$

# Objective Function

**Maximize:**

$$\alpha_1 \cdot \sum_{i,j\in T; i\neq j} \sum_{d\in D} I_{i,j} \cdot x_{i,j,d} \cdot W_d - \alpha_2 \cdot \sum_{n=2}^{2(|T|-1)} \sum_{k\in T} \sum_{i,j\in T; i\neq j} \text{dist}_{i,j} \text{Travel}_{i,j,k,n}$$
$$- \alpha_3 \cdot \sum_{d\in D} \text{Max\_match\_difference}_d - \alpha_4 \cdot (\text{MaxDist} - \text{MinDist})$$
$$- \alpha_5 \cdot \sum_{d\in D} \text{Max\_home\_away\_diff}_d$$
$$- \alpha_6 \cdot (\text{max\_night\_played} - \text{min\_night\_played})$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ are coefficients which determine the individual weights of the individual objective functions and $\alpha_i \geq 0, \forall 1 \leq i \leq 6$.

**Choice of $\alpha_i$'s**

- $\alpha_1 = 10 \rightarrow$ Prioritizes interesting games on weekends to maximize revenue & viewership.
- $\alpha_2 = 0.008 \rightarrow$ Keeps total travel distance low without overpowering other objectives.
- $\alpha_3 = 3 \rightarrow$ Ensures fairness in scheduling so that no team has too many consecutive games or long breaks.
- $\alpha_4 = 0.01 \rightarrow$ Prevents some teams from unfairly high travel.
- $\alpha_5 = 3 \rightarrow$ Ensures fairness in home vs. away distribution (important for team performance).
- $\alpha_6 = 2 \rightarrow$ Balances night matches, ensuring no team gets too many late-night games.

# Constraints

1. **Each team plays all others twice (one home and one away):**

$$\sum_{d\in D} x_{i,j,d} = 1, \forall j \in T \quad (i \text{ plays } j \text{ exactly once at home})$$
$$\sum_{d\in D} x_{j,i,d} = 1, \forall j \in T \quad (i \text{ plays } j \text{ exactly once away})$$

2. **Each team plays at most one match per date**

$$\sum_{j\in T; i\neq j} (x_{i,j,d} + x_{j,i,d}) \leq 1, \forall i \in T \quad (i \text{ plays atmost one match each day, home or away})$$

3. **Number of matches in a single day/night slot:**

$$\text{XN}_{i,j,d} \leq x_{i,j,d} \quad \forall i,j \in T, i \neq j; \forall d \in D$$
$$\text{XN}_{i,j,d} \leq N_{i,j,d} \quad \forall i,j \in T, i \neq j; \forall d \in D$$
$$\text{XN}_{i,j,d} \geq x_{i,j,d} + N_{i,j,d} - 1 \quad \forall i,j \in T, i \neq j; \forall d \in D$$
$$\sum_{i,j\in T, i\neq j} (x_{i,j,d} - \text{XN}_{i,j,d}) \leq \text{MaxSlot}, \forall d \in D \quad (\text{Bound on number of day matches on day d})$$
$$\sum_{i,j\in T, i\neq j} \text{XN}_{i,j,d} \leq \text{MaxSlot}, \forall d \in D \quad (\text{Bound on number of night matches on day d})$$

For the above constraints,

- The first three constraints ensure that $\mathrm{XN}_{i,j,d} = x_{i,j,d} \cdot N_{i,j,d}$
- The last two constraints ensure that $\sum_{i,j \in T; i \neq j} x_{i,j,d}(1 - N_{i,j,d}) \leq \mathrm{MaxSlot}$ and $\sum_{i,j \in T; i \neq j} x_{i,j,d} \cdot N_{i,j,d} \leq \mathrm{MaxSlot}$

4. **Number of Day and Night matches played by a team should be balanced across teams:**

$$\sum_{d \in D} \sum_{j \in T; j \neq i} (N_{i,j,d} + N_{j,i,d}) \leq \mathrm{max\_night\_played} \quad \forall i \in T$$
$$\sum_{d \in D} \sum_{j \in T; j \neq i} (N_{i,j,d} + N_{j,i,d}) \geq \mathrm{min\_night\_played} \quad \forall i \in T$$

5. **Night Match is not possible if there is no match on that day**

$$N_{i,j,d} \leq x_{i,j,d} \quad \forall i, j \in T, d \in D \text{ and } i \neq j$$

6. **No overlapping of interesting matches:**

$$\sum_{i,j \in T, i \neq j} I_{i,j} \mathrm{XN}_{i,j,d} \leq 1 \quad \forall d \in D \quad \text{(At most one interesting night match on a weekend)}$$
$$\sum_{i,j \in T, i \neq j} I_{i,j}(x_{i,j,d} - \mathrm{XN}_{i,j,d}) \leq 1 \quad \forall d \in D \quad \text{(At most one interesting day match on a weekend)}$$

7. **Count of Home and Away matches**

$$\mathrm{count\_home}_{i,d} = \sum_{d' \leq d} \sum_{j \in T; j \neq i} x_{i,j,d'} \quad \forall i \in T, \forall d \in D$$
$$\mathrm{count\_away}_{i,d} = \sum_{d' \leq d} \sum_{j \in T; j \neq i} x_{j,i,d'} \quad \forall u \in T, \forall d \in D$$
$$\mathrm{count}_{i,d} = \mathrm{count\_home}_{i,d} + \mathrm{count\_away}_{i,d} \quad \forall i \in T, \forall d \in D$$

8. **Number of matches played by teams should be equally distributed at any point of time in the Round Robin**

$$\mathrm{Match\_diff}_{i,j,d} \geq \mathrm{count}_{i,d} - \mathrm{count}_{j,d} \quad \forall i, j \in T, i \neq j; d \in D$$
$$\mathrm{Match\_diff}_{i,j,d} \geq \mathrm{count}_{j,d} - \mathrm{count}_{i,d} \quad \forall i, j \in T, i \neq j; d \in D$$
$$\mathrm{Match\_diff}_{i,j,d} \leq \mathrm{count}_{i,d} - \mathrm{count}_{j,d} + M s_{i,j,d} \quad \forall i, j \in T, i \neq j; d \in D$$
$$\mathrm{Match\_diff}_{i,j,d} \leq \mathrm{count}_{j,d} - \mathrm{count}_{i,d} + M(1 - s_{i,j,d}) \quad \forall i, j \in T, i \neq j; d \in D$$
$$\mathrm{Match\_diff}_{i,j,d} \leq \mathrm{Max\_match\_difference}_d \quad \forall i, j \in T, i \neq j; d \in D$$

For the above constraints,

- The first four constraints ensure that $\mathrm{Match\_diff}_{i,j,d} = |\mathrm{count}_{i,d} = \mathrm{count}_{j,d}|$.
- The last constraint bounds $\mathrm{Max\_match\_difference}_d$ below by the maximum difference between the number of matches played by any two teams.

9. **Bounds on gaps between consecutive matches of a team:**

$$\begin{aligned}
\text{count}_{i,d'} - \text{count}_{i,d} &= 0 \quad \forall i \in T \text{ and } \forall d, d' \in D \text{ such that } d' \geq d \text{ and } d' - d \leq \text{MinGap} - 1 \\
\text{count}_{i,d'} - \text{count}_{i,d} &\geq 0 \quad \forall i \in T \text{ and } \forall d, d' \in D \text{ such that } d' \geq d \text{ and } d' - d \geq \text{MinGap} \\
\text{count}_{i,d+\text{MaxGap}} - \text{count}_{i,d} &\geq 1 \quad \forall i \in T \text{ and } \forall d \in D \setminus \{d : d > \text{end_date} - \text{MaxGap}\}
\end{aligned}$$

10. **Bound on difference of number of home and away matches played by teams:**

$$\begin{aligned}
\text{count_home}_{i,d} - \text{count_away}_{i,d} &\leq \text{MaxDiff} \quad \forall i \in T, \forall d \in D \\
\text{count_away}_{i,d} - \text{count_home}_{i,d} &\leq \text{MaxDiff} \quad \forall i \in T, \forall d \in D
\end{aligned}$$

11. **Bound on the number of consecutive home and away matches:**

$$\sum_{j \in T; j \neq i} \sum_{\mathcal{D}=d+1}^{d'} x_{j,i,\mathcal{D}} \leq \text{p\_away}_{i,d,d'} \cdot M \quad \forall i \in T; \forall d, d' \in D; d < d'$$

$$\sum_{j \in T; j \neq i} \sum_{\mathcal{D}=d+1}^{d'} x_{j,i,\mathcal{D}} \geq \text{p\_away}_{i,d,d'} \quad \forall i \in T; \forall d, d' \in D; d < d'$$

$$\sum_{j \in T; j \neq i} \sum_{\mathcal{D}=d+1}^{d'} x_{i,j,\mathcal{D}} \leq \text{p\_home}_{i,d,d'} \cdot M \quad \forall i \in T; \forall d, d' \in D; d < d'$$

$$\sum_{j \in T; j \neq i} \sum_{\mathcal{D}=d+1}^{d'} x_{i,j,\mathcal{D}} \geq \text{p\_home}_{i,d,d'} \quad \forall i \in T; \forall d, d' \in D; d < d'$$

$$\text{count\_home}_{i,d'} - \text{count\_home}_{i,d} \leq \text{MaxConsecHome} + \text{p\_away}_{i,d,d'} \cdot M \quad \forall i \in T; \forall d, d' \in D; d < d'$$

$$\text{count\_away}_{i,d'} - \text{count\_away}_{i,d} \leq \text{MaxConsecAway} + \text{p\_home}_{i,d,d'} \cdot M \quad \forall i \in T; \forall d, d' \in D; d < d'$$

For the above Constraints,

- The first two constraints ensure that $\text{p\_away}_{i,d,d'} = 1$ if team $i$ has played atleast one away match in $\{d+1, d+2, \cdots, d'\}$, else 0.
- The next two constraints ensure that $\text{p\_home}_{i,d,d'} = 1$ if team $i$ has played atleast one home match in $\{d+1, d+2, \cdots, d'\}$, else 0.
- The last two constraints put a bound on the number of consecutive home and away matches for each team.

12. **Home and away matches played by a team should be distributed in time:**

$$\begin{aligned}
&\text{Home_away_diff}_{i,d} \geq \text{count_away}_{i,d} - \text{count_home}_{i,d} \quad \forall i \in T, \forall d \in D \\
&\text{Home_away_diff}_{i,d} \geq \text{count_home}_{i,d} - \text{count_away}_{i,d} \quad \forall i \in T, \forall d \in D \\
&\text{Home_away_diff}_{i,d} \leq \text{count_away}_{i,d} - \text{count_home}_{i,d} + M t_{i,d} \quad \forall i \in T, \forall d \in D \\
&\text{Home_away_diff}_{i,d} \leq \text{count_home}_{i,d} - \text{count_away}_{i,d} + M(1 - t_{i,d}) \quad \forall i \in T, \forall d \in D \\
&\text{Home_away_diff}_{i,d} \leq \text{Max_home_away_diff}_{d} \quad \forall i \in T; \forall d \in D
\end{aligned}$$

For the above constraints,

- The first four constraints ensure that $\text{Home\_away\_diff}_{i,j,d} = |\text{count\_away}_{i,d} = \text{count\_home}_{i,d}|$.
- The last constraint bounds $\text{Max\_home\_away\_diff}_{d}$ below by the maximum difference between the number of home and away matches played by a team.

13. **Avoid matches on restricted days:**

$$x_{i,j,d} \leq 1 - E_{d,i} \quad \forall d \in D, \forall i, j \in T, i \neq j$$

14. **Match reallocation day for incase of rain/other disruptions:**

$$\sum_{k \in T; k \neq i} (x_{i,k,d+1} + x_{k,i,d+1}) \leq 1 - x_{i,j,d} \quad \forall d \in D \setminus \{\text{end\_date}\}, \forall i, j \in T, i \neq j$$
$$\sum_{k \in T; k \neq j} (x_{j,k,d+1} + x_{k,j,d+1}) \leq 1 - x_{i,j,d} \quad \forall d \in D \setminus \{\text{end\_date}\}, \forall i, j \in T, i \neq j$$

15. **Bound on number of Night matches:**

$$\sum_{d \in D} \sum_{j \in T: j \neq i} (N_{i,j,d} + N_{j,i,d}) \leq \text{MaxNightMatches} \quad \forall i \in T$$
$$\sum_{d \in D} \sum_{j \in T: j \neq i} (N_{i,j,d} + N_{j,i,d}) \geq \text{MinNightMatches} \quad \forall i \in T$$

16. **Match order Constraints**

$$\text{XC1}_{i,j,d} \leq \text{count}_{j,d} + M(1 - x_{i,j,d}) \quad \forall i, j \in T, i \neq j; \forall d \in D$$
$$\text{XC1}_{i,j,d} \geq \text{count}_{j,d} - M(1 - x_{i,j,d}) \quad \forall i, j \in T, i \neq j; \forall d \in D$$
$$\text{XC1}_{i,j,d} \leq M x_{i,j,d} \quad \forall i, j \in T, i \neq j; \forall d \in D$$
$$\text{XC2}_{j,i,d} \leq \text{count}_{j,d} + M(1 - x_{j,i,d}) \quad \forall i, j \in T, i \neq j; \forall d \in D$$
$$\text{XC2}_{j,i,d} \geq \text{count}_{j,d} - M(1 - x_{j,i,d}) \quad \forall i, j \in T, i \neq j; \forall d \in D$$
$$\text{XC2}_{j,i,d} \leq M x_{j,i,d} \quad \forall i, j \in T, i \neq j; \forall d \in D$$
$$\sum_{d \in D} (\text{XC1}_{i,j,d} - (n-1) x_{i,j,d}) \leq 1 + M(1 - L_{i,j,n}) \quad \forall i, j \in T, i \neq j, \forall 1 \leq n \leq |T| - 1$$
$$\sum_{d \in D} (\text{XC1}_{i,j,d} - (n-1) x_{i,j,d}) \geq 1 - M(1 - L_{i,j,n}) \quad \forall i, j \in T, i \neq j, \forall 1 \leq n \leq |T| - 1$$
$$\sum_{i \in T; i \neq j} \sum_{d \in D} (\text{XC2}_{j,i,d} - (n-1) x_{j,i,d}) \leq 1 + M(1 - L_{j,j,n}) \quad \forall j \in T, \forall 1 \leq n \leq |T| - 1$$
$$\sum_{i \in T; i \neq j} \sum_{d \in D} (\text{XC2}_{j,i,d} - (n-1) x_{j,i,d}) \geq 1 - M(1 - L_{j,j,n}) \quad \forall j \in T, \forall 1 \leq n \leq |T| - 1$$

For the above constraints,

- The first six constraints ensure $XC1_{i,j,d} = x_{i,j,d} \cdot \text{count}_{j,d}$ and $XC2_{j,i,d} = x_{j,i,d} \cdot \text{count}_{j,d}$.
- The next two constraints ensure that $\sum_{d \in D} x_{i,j,d} \cdot (\text{count}_{j,d} - n + 1) = 1$, only if $L_{i,j,n} = 1$.
- The last two constraints ensure that $\sum_{i \in T; i \neq j} \sum_{d \in D} x_{j,i,d} \cdot (\text{count}_{j,d} - n + 1) = 1$, only if $L_{j,j,n} = 1$.

17. **Travel Constraints**

$$\text{Travel}_{i,j,k,n} \leq L_{i,k,n-1} \quad \forall i, j, k \in T, 2 \leq n \leq 2(|T| - 1)$$
$$\text{Travel}_{i,j,k,n} \leq L_{j,k,n} \quad \forall i, j, k \in T, 2 \leq n \leq 2(|T| - 1)$$
$$\text{Travel}_{i,j,k,n} \geq L_{i,k,n-1} + L_{j,k,n} - 1 \quad \forall i, j, k \in T, 2 \leq n \leq 2(|T| - 1)$$

The above constraints ensures that $\text{Travel}_{i,j,k,n} = L_{i,k,n-1} \cdot L_{j,k,n}$

18. **Distance travelled by all teams should be balance for all teams**

$$\sum_{n=2}^{2(|T|-1)} \sum_{i,j \in T; i \neq j} \text{dist}_{i,j} \text{Travel}_{i,j,k,n} \leq \text{MaxDist} \quad \forall k \in T$$

$$\sum_{n=2}^{2(|T|-1)} \sum_{i,j \in T; i \neq j} \text{dist}_{i,j} \text{Travel}_{i,j,k,n} \geq \text{MinDist} \quad \forall k \in T$$

# Output

The $\{(x_{i,j,d}, N_{i,j,d}) : i, j \in T, i \neq j, d \in D\}$ obtained by solving the above formulation gives when and where should a match between 2 teams be scheduled.

# Assumptions

- Interesting matches are known in advanced and depends on which two teams are playing only.
- Stadiums are uniformly distributed across zones.
- Travel distances between stadiums are known and constant.
- Events in a stadium/city are all known in advanced, i.e. $\text{E}_{d,i}$ are all known in advanced for all days and all stadiums.
- Each team has only one home stadium.
- MaxSlot is set in advanced based on league regulations.
- MinGap and MaxGap values are set based on league regulations.
- MaxConsecHome and MaxConsecAway are set based on league regulations.
- MaxDiff is set in advanced based on league regulations.
- Day and night slots for matches of a day do not overlap with each other.
- MaxNightMatches and MinNightMatches are set in advanced based on league regulations.
- The days from the start and end date of the round robin are coded as positive integers i.e. $D = \{1, 2, 3, \cdots\}$.
- $\forall d \in D, d+1,$ is the immediate next day and so on.
- end_date is the last date in $D$.
- The Big-M constant, M is sufficiently large enough.
- $\text{dist}_{i,i} = 0 \quad \forall i \in T$.
- After playing a match at a venue, the teams directly travel to the venue where their next match is scheduled, or don't travel if there is next match is in the same venue. This assumption ensures that $\text{Travel}_{i,j,k,n} = L_{i,k,n-1} \cdot L_{j,k,n}$
- All teams are in the venue where their first match is scheduled at the start of the round robin. The total distance considered is the distance travelled after the first match till the end of the round robin.