Optimization in Industry Assignment 2
Mathematical Model for Optimal ATM Refill Problem

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Sets

1. $\mathcal{I}$: Set of ATMs, indexed by i (i.e., $i = 1, \ldots, |\mathcal{I}|$).

2. $\mathcal{V}$: Set of vehicles, indexed by v (i.e., $v = 1, \ldots, |\mathcal{V}|$).

3. $\mathcal{D}$: Set of days in the planning horizon, indexed by d (e.g., $d = 1, \ldots, 7$).

4. $\mathcal{K}$: Set of denominations, indexed by k.

Parameters

1. $d_{i,d}$: Cash demand at ATM i on day d.
2. I_i^0: Initial cash inventory at ATM i (at start of day 1).
3. C_i: Maximum cash capacity of ATM i.
4. L_i: Minimum cash level required at ATM i.
5. $Q_{\min}$: Minimum cash deposit per refill (e.g., 50K).
6. Q_{unit}: Cash deposit unit (e.g., 10K), so that any nonzero refill is a multiple of Q_{unit}.
7. cap_v: Maximum cash carrying capacity of vehicle v (security limited).
8. cost_v: Cost associated with using vehicle v.
9. $\text{maxVisit}_{v,d}$: Maximum number of ATMs vehicle v can service on day d (e.g., 20 on weekdays, 30 on weekends).
10. S: Security gap in days. If vehicle v visits ATM i on day d, it cannot visit the same ATM again in the next $S - 1$ days.
11. $r_{i,d,k}$: Denomination requirement for ATM i on day d for denomination k (soft constraint).

12. w_1, w_2, w_3, w_4: Weights for the objective function corresponding to vehicles used, ATM inventory holding cost, and number of visits, respectively.

13. M: A sufficiently large constant (Big-M).

Decision Variables

1. $x_{i,v,d} \in \{0, 1\}$: Equals 1 if vehicle v refills ATM i on day d; 0 otherwise.

2. $q_{i,v,d} \in \mathbb{Z}_{\geq 0}$: Number of cash deposit units Q_{unit}, delivered to ATM i by vehicle v on day d.

3. $y_{i,v,d} \geq 0$: Amount of cash delivered to ATM i by vehicle v on day d.

$$y_{i,v,d} = Q_{\text{unit}} \cdot q_{i,v,d}, \quad q_{i,v,d} \in \mathbb{Z}_{\geq 0}$$

   with the additional condition that if $q_{i,v,d} > 0$ then $q_{i,v,d} \geq \frac{Q_{\min}}{Q_{\text{unit}}}$.

4. $z_{v,d} \in \{0, 1\}$: Equals 1 if vehicle v is used on day d; 0 otherwise.

5. $I_{i,d}$: Cash inventory at ATM i at the end of day d.

6. $y_{i,v,d,k} \geq 0$: Cash delivered in denomination k for ATM i on day d.

7. $\text{penalty}_{i,d,k} \geq 0$: Unmet demand for cash delivered in denomination k for ATM i on day d.

Objective Function

We consider a weighted-sum objective function:

$$\text{Minimize:} \quad w_1 \sum_{d \in \mathcal{D}} \sum_{v \in \mathcal{V}} z_{v,d} \; + \; w_2 \sum_{d \in \mathcal{D}} \sum_{i \in \mathcal{I}} I_{i,d} \; + \; w_3 \sum_{d \in \mathcal{D}} \sum_{v \in \mathcal{V}} \sum_{i \in \mathcal{I}} x_{i,v,d} + w_4 \sum_{i \in \mathcal{I}} \sum_{d \in \mathcal{D}} \sum_{k \in \mathcal{K}} \text{penalty}_{i,d,k}$$

Explanation:

- The first term minimizes the (weighted) number of vehicles used.
- The second term minimizes the total ATM cash inventory holding cost.
- The third term minimizes the total number of ATM visits.
- The fourth term minimizes the unmet denomination preferences at each of the ATM.

Constraints

A. ATM Inventory Balance and Capacity

(A1) Inventory Update: For each $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$I_{i,d} = \begin{cases} I_i^0 + \sum_{v \in \mathcal{V}} y_{i,v,1} - d_{i,1}, & d = 1, \\ I_{i,d-1} + \sum_{v \in \mathcal{V}} y_{i,v,d} - d_{i,d}, & d \geq 2. \end{cases}$$

(A2) Inventory Limits: For all $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$L_i \leq I_{i,d}$$

$$I_{i,d} \leq C_i$$

B. Service (Assignment) Constraints

(B1) One Refill per ATM per Day: For every $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$\sum_{v \in \mathcal{V}} x_{i,v,d} \leq 1$$

(B2) Linking x and y: For all $i \in \mathcal{I}$, $v \in \mathcal{V}$, and $d \in \mathcal{D}$,

$$y_{i,v,d} \geq Q_{\min} \, x_{i,v,d}$$

$$y_{i,v,d} \leq M \cdot x_{i,v,d}$$

C. Vehicle Capacity and Workload

(C1) Vehicle Loading Limit: For each $v \in \mathcal{V}$ and $d \in \mathcal{D}$,

$$\sum_{i \in \mathcal{I}} y_{i,v,d} \leq \mathrm{cap}_v \cdot z_{v,d}$$

(C2) Maximum Visits per Vehicle per Day: For all $v \in \mathcal{V}$ and $d \in \mathcal{D}$,

$$\sum_{i \in \mathcal{I}} x_{i,v,d} \leq \mathrm{maxVisit}_{v,d} \cdot z_{v,d}$$

D. Security and Scheduling Constraints

(D1) Spread Out ATM Visits: For all $v \in \mathcal{V}$, for each $i \in \mathcal{I}$ and for each day $d \in \mathcal{D}$ such that $d + S - 1 \leq \max(\mathcal{D})$,

$$x_{i,v,d} + \sum_{d'=d+1}^{\min(d+S-1,\max(\mathcal{D}))} x_{i,v,d'} \leq 1.$$

E. Workload Balance Across Vehicles

(E1) Number of ATMs visited: For each $v \in \mathcal{V}$ and $d \in \mathcal{D}$, define the total number of visits:

$$W_{v,d} = \sum_{i \in \mathcal{I}} x_{i,v,d}.$$

Introduce auxiliary variables $W_d^{\max}$ and $W_d^{\min}$ for each day $d \in \mathcal{D}$ and enforce:

$$W_{v,d} \leq W_d^{\max} \quad \forall v \in \mathcal{V}, \quad W_{v,d} \geq W_d^{\min} \quad \forall v \in \mathcal{V},$$

$$W_d^{\max} - W_d^{\min} \leq \Delta_x \quad \forall d \in \mathcal{D},$$

where Δ_x is a parameter specifying the maximum allowed difference in the number of ATMs visited by any two vehicles on day d.

(E2) Cash Deposited in ATMS: For each $v \in \mathcal{V}$ and $d \in \mathcal{D}$, define the total cash deposited:

$$C_{v,d} = \sum_{i \in \mathcal{I}} y_{i,v,d}.$$

Introduce auxiliary variables $C_d^{\max}$ and $C_d^{\min}$ for each day $d \in \mathcal{D}$ and impose:

$$C_{v,d} \leq C_d^{\max} \quad \forall v \in \mathcal{V}, \quad C_{v,d} \geq C_d^{\min} \quad \forall v \in \mathcal{V},$$

$$C_d^{\max} - C_d^{\min} \leq \Delta_y \quad \forall d \in \mathcal{D},$$

where Δ_y is a parameter specifying the maximum allowed difference in cash deposited by any two vehicles on day d.

F. Denomination Constraints (Soft)

(F1) For each $i \in \mathcal{I}$, $d \in \mathcal{D}$, and $k \in \mathcal{K}$, if modeling denomination details,

$$\sum_{v \in \mathcal{V}} y_{i,v,d,k} \geq r_{i,d,k} - \text{penalty}_{i,d,k}$$

G. Deposit Size Discreteness

(G1) To ensure that if cash is delivered, it is either zero or at least $Q_{\min}$ (in multiples of Q_{unit}):

$$y_{i,v,d} = Q_{\text{unit}} \cdot q_{i,v,d}$$

$$q_{i,v,d} \geq \frac{Q_{\min}}{Q_{\text{unit}}} \cdot x_{i,v,d}$$

Assumptions

- The planning horizon is one week (7 days).
- Daily demands $d_{i,d}$ and initial inventories I_i^0 are known.
- ATMs must always have cash between the levels L_i and C_i.
- Vehicles are loaded only at the beginning of the day (no mid-day refilling).
- Vehicles' cash capacities are limited by security considerations, which may be lower than their physical capacities.
- In case of high demand, additional (rented) vehicles may be used, represented by higher cost parameters.
- A refill at an ATM is done by at most one vehicle per day.
- For security, if a vehicle visits an ATM on day d, it cannot visit the same ATM again for the next $S - 1$ days.
- Refills must be in multiples of Q_{unit} (e.g., 10K), with any nonzero refill being at least $Q_{\min}$ (e.g., 50K).
- The denomination requirement is a soft constraint; unmet denomination preferences are penalized but do not prevent meeting the overall cash demand.

Output

The solution to this Mixed-Integer Programming (MIP) model will determine:

- Which vehicles are used on each day (via $z_{v,d}$).
- Which ATMs are refilled by which vehicle on each day (via $x_{i,v,d}$).
- The amount of cash delivered to each ATM on each day (via $y_{i,v,d}$).
- The evolution of ATM cash inventories $I_{i,d}$ over the planning horizon.