Mathematical Model for Optimal ATM Refill Problem

1 Sets

I. $\mathcal{I}$: Set of ATMs, indexed by i (i.e., $i = 1, \ldots, |\mathcal{I}|$).

V. $\mathcal{V}$: Set of vehicles, indexed by v (i.e., $v = 1, \ldots, |\mathcal{V}|$).

D. $\mathcal{D}$: Set of days in the planning horizon, indexed by d (e.g., $d = 1, \ldots, 7$).

K. $\mathcal{K}$: (Optional) Set of denominations, indexed by k.

2 Parameters

- $d_{i,d}$: Cash demand at ATM i on day d.
- I_i^0: Initial cash inventory at ATM i (at start of day 1).
- C_i: Maximum cash capacity of ATM i.
- L_i: Minimum cash level required at ATM i.
- $Q_{\min}$: Minimum cash deposit per refill (e.g., 50K).
- Q_{unit}: Cash deposit unit (e.g., 10K), so that any nonzero refill is a multiple of Q_{unit}.
- cap_v: Maximum cash carrying capacity of vehicle v (security limited).
- cost_v: Cost associated with using vehicle v.
- $\text{maxVisit}_{v,d}$: Maximum number of ATMs vehicle v can service on day d (e.g., 20 on weekdays, 30 on weekends).
- S: Security gap in days. If vehicle v visits ATM i on day d, it cannot visit the same ATM again in the next $S - 1$ days.
- $r_{i,d,k}$: (Optional) Denomination requirement for ATM i on day d for denomination k (soft constraint).
- w_1, w_2, w_3: Weights for the objective function corresponding to vehicles used, ATM inventory holding cost, and number of visits, respectively.
- M: A sufficiently large constant (Big-M).

3 Decision Variables

- $x_{i,v,d} \in \{0,1\}$: Equals 1 if vehicle v refills ATM i on day d; 0 otherwise.
- $y_{i,v,d} \geq 0$: Amount of cash delivered to ATM i by vehicle v on day d.

$$y_{i,v,d} = Q_{\text{unit}} \cdot q_{i,v,d}, \quad q_{i,v,d} \in \mathbb{Z}_{\geq 0}$$

  with the additional condition that if $q_{i,v,d} > 0$ then $q_{i,v,d} \geq \frac{Q_{\min}}{Q_{\text{unit}}}$.
- $z_{v,d} \in \{0,1\}$: Equals 1 if vehicle v is used on day d; 0 otherwise.
- $I_{i,d}$: Cash inventory at ATM i at the end of day d.
- $y_{i,v,d,k} \geq 0$ (Optional): Cash delivered in denomination k for ATM i on day d.

4 Objective Function

We consider a weighted-sum objective that minimizes:

$$\min \quad Z = w_1 \sum_{d \in \mathcal{D}} \sum_{v \in \mathcal{V}} z_{v,d} \; + \; w_2 \sum_{d \in \mathcal{D}} \sum_{i \in \mathcal{I}} I_{i,d} \; + \; w_3 \sum_{d \in \mathcal{D}} \sum_{v \in \mathcal{V}} \sum_{i \in \mathcal{I}} x_{i,v,d}.$$

Explanation:

- The first term minimizes the (weighted) number of vehicles used.
- The second term minimizes the total ATM cash inventory holding cost.
- The third term minimizes the total number of ATM visits.

5 Constraints

A. ATM Inventory Balance and Capacity

(A1) Inventory Update: For each $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$I_{i,d} = \begin{cases} I_i^0 + \sum_{v \in \mathcal{V}} y_{i,v,1} - d_{i,1}, & d = 1, \\ I_{i,d-1} + \sum_{v \in \mathcal{V}} y_{i,v,d} - d_{i,d}, & d \geq 2. \end{cases}$$

(A2) Inventory Limits: For all $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$L_i \leq I_{i,d} \leq C_i.$$

B. Service (Assignment) Constraints

(B1) One Refill per ATM per Day: For every $i \in \mathcal{I}$ and $d \in \mathcal{D}$,

$$\sum_{v \in \mathcal{V}} x_{i,v,d} \le 1.$$

(B2) Linking x and y: For all $i \in \mathcal{I}$, $v \in \mathcal{V}$, and $d \in \mathcal{D}$,

$$y_{i,v,d} \ge Q_{\min} \, x_{i,v,d} \quad \text{and} \quad y_{i,v,d} \le M \, x_{i,v,d}.$$

C. Vehicle Capacity and Workload

(C1) Vehicle Loading Limit: For each $v \in \mathcal{V}$ and $d \in \mathcal{D}$,

$$\sum_{i \in \mathcal{I}} y_{i,v,d} \le \text{cap}_v \, z_{v,d}.$$

(C2) Maximum Visits per Vehicle per Day: For all $v \in \mathcal{V}$ and $d \in \mathcal{D}$,

$$\sum_{i \in \mathcal{I}} x_{i,v,d} \le \text{maxVisit}_{v,d} \, z_{v,d}.$$

D. Security and Scheduling Constraints

(D1) Spread Out ATM Visits: For all $v \in \mathcal{V}$, for each $i \in \mathcal{I}$ and for each day $d \in \mathcal{D}$ such that $d + S - 1 \le \max(\mathcal{D})$,

$$x_{i,v,d} + \sum_{d'=d+1}^{\min(d+S-1, \max(\mathcal{D}))} x_{i,v,d'} \le 1.$$

E. (Optional) Denomination Constraints (Soft)

(E1) For each $i \in \mathcal{I}$, $d \in \mathcal{D}$, and $k \in \mathcal{K}$, if modeling denomination details,

$$\sum_{v \in \mathcal{V}} y_{i,v,d,k} \ge r_{i,d,k} - \text{penalty}_{i,d,k}.$$

A penalty term may be added to the objective to account for unmet denomination preferences.

F. Deposit Size Discreteness

(F1) To ensure that if cash is delivered, it is either zero or at least $Q_{\min}$ (in multiples of Q_{unit}):

$$y_{i,v,d} = Q_{\text{unit}} \cdot q_{i,v,d}, \quad q_{i,v,d} \in \mathbb{Z}_{\ge 0}, \quad q_{i,v,d} \ge \frac{Q_{\min}}{Q_{\text{unit}}} \, x_{i,v,d}.$$

6 Assumptions

- The planning horizon is one week (7 days).
- Daily demands $d_{i,d}$ and initial inventories I_i^0 are known.
- ATMs must always have cash between the levels L_i and C_i.
- Vehicles are loaded only at the beginning of the day (no mid-day refilling).
- Vehicles' cash capacities are limited by security considerations, which may be lower than their physical capacities.
- In case of high demand, additional (rented) vehicles may be used, represented by higher cost parameters.
- A refill at an ATM is done by at most one vehicle per day.
- For security, if a vehicle visits an ATM on day d, it cannot visit the same ATM again for the next $S-1$ days.
- Refills must be in multiples of Q_{unit} (e.g., 10K), with any nonzero refill being at least Q_{min} (e.g., 50K).
- The denomination requirement is a soft constraint; unmet denomination preferences are penalized but do not prevent meeting the overall cash demand.

7 Output

The solution to this Mixed-Integer Programming (MIP) model will determine:

- Which vehicles are used on each day (via $z_{v,d}$).
- Which ATMs are refilled by which vehicle on each day (via $x_{i,v,d}$).
- The amount of cash delivered to each ATM on each day (via $y_{i,v,d}$).
- The evolution of ATM cash inventories $I_{i,d}$ over the planning horizon.