Smooth Distances for Second Order Kinematic Robot Control

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I. Introduction

This is a supplementary material for the paper "Smooth Distances for Second Order Kinematic Robot Control", explaining how to compute $D^{\mathcal{A}}_{h,R}(p)$ for some 3D objects. Through this document, we will use $p = [x\ y\ z]^T$.

A. *Removing the regularization parameter*

In this document, we will explain how to compute a particular case of $D^{\mathcal{A}}_{h,R}(p)$, $\Pi^{\mathcal{A}}_{h,R}(p)$ when $R \rightarrow \infty$, henceforth denoted simply by $D^{\mathcal{A}}_{h}(p)$ and $\Pi^{\mathcal{A}}_{h}(p)$, respectively. This is because it turns out that we can compute these expressions for a generic R if we have procedures that can compute it for $R \rightarrow \infty$. Let $a_c = \mathit{Cen}(\mathcal{A})$ (as it is the geometric center of objects, this can computed easily for objects as boxes, cylinders and spheres). We start by noting the following identity:

$$\frac{\|a-a_c\|^2}{R^2} + \frac{\|p-a\|^2}{h^2} = \frac{\|\hat{p}-a\|^2}{\eta^2} + \left(\frac{\|a_c\|^2}{R^2} + \frac{\|p\|^2}{h^2} - \frac{\|\hat{p}\|^2}{\eta^2}\right) \tag{1}$$

in which $\eta \triangleq (1/h^2 + 1/R^2)^{-1/2}$ and $\hat{p} \triangleq \eta^2(p/h^2 + a_c/R^2)$. Using this formula and the definition of $D^{\mathcal{A}}_{h,R}(p)$ and $D^{\mathcal{A}}_{h}(p)$ (the latter being the case in which $R \rightarrow \infty$ and thus $W^{\mathcal{A}}_{R}(a) = 1$ and $\mathit{Vol}_R(\mathcal{A})$ is simply the volume of $\mathcal{A}$, $\mathit{Vol}(\mathcal{A})$):

$$D^{\mathcal{A}}_{h,R}(p) = h^2 \log\left(\frac{\mathit{Vol}_R(\mathcal{A})}{\mathit{Vol}(\mathcal{A})}\right) + \frac{h^2}{2}\left(\frac{\|a_c\|^2}{R^2} + \frac{\|p\|^2}{h^2} - \frac{\|\hat{p}\|^2}{\eta^2}\right) + \frac{h^2}{\eta^2} D^{\mathcal{A}}_{\eta}(\hat{p}). \tag{2}$$

Furthermore, note that $h^2 \log\left(\frac{\mathit{Vol}_R(\mathcal{A})}{\mathit{Vol}(\mathcal{A})}\right) = -\frac{h^2}{R^2} D^{\mathcal{A}}_{R}(a_c)$. So:

$$D^{\mathcal{A}}_{h,R}(p) = -\frac{h^2}{R^2} D^{\mathcal{A}}_{R}(a_c) + \frac{h^2}{2}\left(\frac{\|a_c\|^2}{R^2} + \frac{\|p\|^2}{h^2} - \frac{\|\hat{p}\|^2}{\eta^2}\right) + \frac{h^2}{\eta^2} D^{\mathcal{A}}_{\eta}(\hat{p}). \tag{3}$$

The righthand side only depends on the computations of $D^{\mathcal{A}}_{g}(u)$ for the different values of g and points u. The formula for the projection is even simpler: taking the derivative of both sides with respect to p and using the fact that $\frac{\partial}{\partial p} D^{\mathcal{A}}_{h,R}(p) = p - \Pi^{\mathcal{A}}_{h,R}(p)$, $\frac{\partial}{\partial p} D^{\mathcal{A}}_{\eta}(p) = p - \Pi^{\mathcal{A}}_{\eta}(p)$ and also that $\frac{\partial \hat{p}}{\partial p} = (\eta^2/h^2)I$ (in which I is the identity matrix):

$$p - \Pi^{\mathcal{A}}_{h,R}(p) = (p - \hat{p}) + \left(\hat{p} - \Pi^{\mathcal{A}}_{\eta}(\hat{p})\right) \ \rightarrow\ \Pi^{\mathcal{A}}_{h,R}(p) = \Pi^{\mathcal{A}}_{\eta}(\hat{p}). \tag{4}$$

Consequently, henceforth, without loss of generality, we will work with the case in which $R \rightarrow \infty$.

B. *Computing projections numerically*

In this document, we will show how to compute $D^{\mathcal{A}}_{h}(p)$ for some sets. However, we will also need to compute the *h-projections* $\Pi^{\mathcal{A}}_{h}(p)$. In that case, we note that $\Pi^{\mathcal{A}}(p) = p - \frac{\partial D^{\mathcal{A}}_{h}}{\partial p}(p)$. We suggest to compute $\frac{\partial D^{\mathcal{A}}_{h}}{\partial p}(p)$ numerically, that is:

$$\begin{aligned}
\frac{\partial D^{\mathcal{A}}_{h}}{\partial x}(p) &\approx \frac{D^{\mathcal{A}}_{h}(p+\epsilon e_x) - D^{\mathcal{A}}_{h}(p-\epsilon e_x)}{2\epsilon}\\
\frac{\partial D^{\mathcal{A}}_{h}}{\partial y}(p) &\approx \frac{D^{\mathcal{A}}_{h}(p+\epsilon e_y) - D^{\mathcal{A}}_{h}(p-\epsilon e_y)}{2\epsilon}\\
\frac{\partial D^{\mathcal{A}}_{h}}{\partial z}(p) &\approx \frac{D^{\mathcal{A}}_{h}(p+\epsilon e_z) - D^{\mathcal{A}}_{h}(p-\epsilon e_z)}{2\epsilon}
\end{aligned} \tag{5}$$

in which $e_x = [1\ 0\ 0]^T, e_y = [0\ 1\ 0]^T$, $e_z = [0\ 0\ 1]^T$ and ϵ is a small number (we suggest $\epsilon = 0.001$).

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C. Canonical objects

In this document, we will show how to compute $D_h^{\mathcal{A}}(p)$ (and thus $\Pi_h^{\mathcal{A}}(p)$, see the previous subsection) for some sets in a *canonical* pose. For example, for a box centered at $p = [0\ 0\ 0]^T$ of a reference frame with its sides aligned with the axis of this reference frame. For these objects in a general pose, other than the canonical one, we use the property derived in the paper: if $E(\cdot)$ is a rigid transformation and $E^{-1}(\cdot)$ is its inverse:

$$\begin{aligned} D_h^{E(\mathcal{A})}(p) &= D_h^{\mathcal{A}}\Big(E^{-1}(p)\Big) \\ \Pi_h^{E(\mathcal{A})}(p) &= E\Big(\Pi_h^{\mathcal{A}}\Big(E^{-1}(p)\Big)\Big). \end{aligned} \tag{6}$$

D. The Cartesian Product Property

Using the definition of $\mathcal{A}$, it is easy to see that if $\mathcal{A}_i$ are subsets of $\mathbb{R}^{n_i}$ for a $n_i \geq 1$, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times ... \times \mathcal{A}_m$, $p^i \in \mathbb{R}^{n_i}$ and $p = [(p^1)^T\ (p^2)^T\ ...\ (p^m)^T]^T$ then:

$$D_h^{\mathcal{A}}(p) = \sum_{i=1}^{m} D_h^{\mathcal{A}_i}(p^i). \tag{7}$$

We can use this to compute the h-distance function for complex sets that are build as Cartesian product of simpler sets.

E. The Error Function

The *Error Function* $\mathit{Erf}(u) \triangleq \frac{2}{\sqrt{\pi}}\int_0^u e^{-t^2}dt$ appears often in the calculations of $D^{\mathcal{A}}(p)$ for some simple objects. It has no closed form in terms of a finite number elementary functions, but there are very good approximation for it in terms of elementary functions that will be used here. Let

$$J(u) \triangleq \frac{a}{(a-1)\sqrt{\pi u^2} + \sqrt{\pi u^2 + a^2}}. \tag{8}$$

in which $a = 2.7889$. Then, $\mathit{Erf}(u) \approx \mathit{sign}(u)(1 - e^{-u^2}J(u))$. This approximation is excellent for all values of u. More details can be seen in [1].

A related function that will often appear in our calculations is, for $L \geq 0$ and $v \in \mathbb{R}$:

$$\begin{aligned} &\mathit{Int}_h(v,L) \triangleq -h^2 \log\left(\frac{1}{2L}\int_{-L}^{L} e^{-\frac{(u-v)^2}{2h^2}}du\right) = \\ &-h^2 \log\left(\sqrt{\frac{\pi}{2}}\frac{h}{2L}\left(\mathit{Erf}\left(\frac{L+v}{\sqrt{2}h}\right) + \mathit{Erf}\left(\frac{L-v}{\sqrt{2}h}\right)\right)\right). \end{aligned} \tag{9}$$

Int stands for *interval* as it is $D_h^{\mathcal{A}}(p)$ for $\mathcal{A} = [-L, L]$, in which $p \in \mathbb{R}$.

Without a careful evaluation, this function can easily be problematic to be computed. When $|v| \geq 5\sqrt{2}h + L$ the sum of *Erf*'s inside the log is already very close to 0 in most naive implementations of the error function, generating $+\infty$ as a result. The approximation $\mathit{Erf}(v) \approx \mathit{sign}(v)(1 - e^{-v^2}J(v))$ allow us to solve this problem. We will consider two cases, $|v| \leq L$, in which the aforementioned problem do not happen, and when $|v| \geq L$, in which we need to rewrite the function to avoid underflows.

For $|v| \leq L$, both $u = (L+v)/(\sqrt{2}h)$ and $u = (L+v)/(\sqrt{2}h)$ are nonnegative and we can simply use $\mathit{Erf}(u) \approx 1 - e^{-u^2}J(u)$. Then we have the following approximation for $\mathit{Int}_h(v,L)$:

$$\begin{aligned} &\mathit{Int}_h(v,L) \approx -h^2 \log\left(\sqrt{\frac{\pi}{2}}\frac{h}{2L}\left(2 - e^{-\frac{(L+v)^2}{2h^2}}J\Big(\frac{v+L}{\sqrt{2}h}\Big) - e^{-\frac{(L-v)^2}{2h^2}}J\Big(\frac{v-L}{\sqrt{2}h}\Big)\right)\right) \\ &\text{for } |v| \leq L. \end{aligned} \tag{10}$$

For $|v| \geq L$, we note that we can assume, without loss of generality, that $v \geq 0$, since $\mathit{Int}_h(v,L)$ is an even function of v. Note than, in this case, $u_1 = (L+v)/(\sqrt{2}h) \geq 0$ and $u_2 = (L-v)/(\sqrt{2}h) \leq 0$. Using the approximations $\mathit{Erf}(u_1) \approx (1 - e^{-u_1^2}J(u_1))$, $\mathit{Erf}(u_2) \approx -(1 - e^{-u_2^2}J(u_2))$, and factoring out the term $-e^{-\frac{(L-v)^2}{2h^2}}$ out of the log, we can write:

$$\textit{Int}_h(v,L) \approx \frac{(v-L)^2}{2} - h^2 \log\left(\sqrt{\frac{\pi}{2}}\frac{h}{2L}\left(J\left(\frac{v-L}{\sqrt{2}h}\right) - J\left(\frac{v+L}{\sqrt{2}h}\right)e^{\frac{-2Lv}{h^2}}\right)\right)$$
$$\text{for } v \geq L. \text{ If } v \leq -L, \text{ use } -v \text{ in the formula instead.} \tag{11}$$

Here is the C code:

```c
#include <math.h>


#define PI 3.1415926
#define SQRTHALFPI 1.2533141
#define SQRT2 1.4142135
#define CONSTJA 2.7889


double fun_J(double u)
{
   return CONSTJA/((CONSTJA-1)*sqrt(PI*u*u) + sqrt(PI*u*u+CONSTJA*CONSTJA));
}

double Int(double v, double h, double L)
{
   if( abs(v) <= L)
   {
      double A1 = exp(-(L-v)*(L-v)/(2*h*h))*fun_J((v-L)/(SQRT2*h));
      double A2 = exp(-(L+v)*(L+v)/(2*h*h))*fun_J((v+L)/(SQRT2*h));
      return -h*h*log(SQRTHALFPI*(h/(2*L))*(2-A1-A2));
   }
   else
   {
      //The function is even
      v = abs(v);

      double A1 = fun_J((v-L)/(SQRT2*h));
      double A2 = exp(-2*L*v/(h*h))*fun_J((v+L)/(SQRT2*h));
      return 0.5*(v-L)*(v-L) -h*h*log(SQRTHALFPI*(h/(2*L))*(A1-A2));
   }
}
```

F. The Modified Bessel Function of the First Kind

Another function that will often appear is the *Modified Bessel Function of the First Kind* of order 0, defined as $I_0(u) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{u\cos(\theta)}d\theta$.

Let $\hat{I}_0(u) = \cosh(u)^{-1} I_0(u)$. Then we define, for $R \geq 0$ and $v \in \mathbb{R}^+$, the following function that will appear in our calculations:

$$\textit{Cir}_h(v,R) \triangleq -h^2 \log\left(\frac{1}{R^2}\int_0^R r\left(e^{-\frac{(r-v)^2}{2h^2}} + e^{-\frac{(r+v)^2}{2h^2}}\right)\hat{I}_0\left(\frac{rv}{h^2}\right)dr\right). \tag{12}$$

Cir stands for *circle* as it is related $D_h^{\mathcal{A}}(p)$ when $\mathcal{A}$ is a circle (with interior) in $\mathbb{R}^2$, centered at the origin and with radius R. More precisely, since the distance function in this case will be radially symmetric, $D_h^{\mathcal{A}}(p) = \textit{Cir}_h(\|p\|, R)$.

It is beneficial to study an scaled version of this function, in which v, r and R are scaled by $1/h$. Making the change of variables $\rho = r/h$ in the integral and considering $\nu = v/h$, $P = R/h$ we obtain that

$$\textit{Cir}_h(h\nu, hP) = -h^2 \log\left(\frac{1}{P^2}\int_0^P \rho\left(e^{-\frac{1}{2}(\rho-\nu)^2} + e^{-\frac{1}{2}(\rho+\nu)^2}\right)\hat{I}_0(\rho\nu)\,d\rho\right). \tag{13}$$

Now, if we graph the function $f(\rho,\nu) \triangleq \rho\left(e^{-\frac{1}{2}(\rho-\nu)^2} + e^{-\frac{1}{2}(\rho+\nu)^2}\right)\hat{I}_0(\rho\nu)$ on ρ for fixed values of ν ($\nu \geq 0$), we will see that the maximum of $f(\rho,\nu)$ is approximatelly at $r^*(\nu) = \sqrt{1+\nu^2}$, and it is practically zero for $r \leq r^*(\nu) - 3$ and $r \geq r^*(\nu) + 3$. Therefore, let $\underline{F}(\nu,P) \triangleq \max(0, r^*(\nu) - 3)$ and $\overline{F}(\nu,P) \triangleq \min(P, r^*(\nu) + 3)$

$$\int_0^{R/h} f(\rho,\nu)d\rho \approx \int_{\underline{F}(\nu,P)}^{\overline{F}(\nu,P)} f(\rho,\nu)d\rho. \tag{14}$$

We can integrate the integral at the right numerically using, for example, Gaussian quadrature. For that, let $\rho = \underline{F}(\nu,P) + \frac{\overline{F}(\nu,P) - \underline{F}(\nu,P)}{2}(g+1)$. Then the integral becomes:

$$\int_0^{R/h} f(\rho,\nu)d\rho \approx \left(\frac{\overline{F}-\underline{F}}{2}\right)\int_{-1}^{1} f\left(\underline{F}+\left(\frac{\overline{F}-\underline{F}}{2}\right)(g+1),\nu\right)dg. \tag{15}$$

and thus Gauss-Legendre quadrature can be applied. This integral only make sense if $\overline{F}(\nu,P) \geq \underline{F}(\nu,P)$. This holds if $\nu \leq P$, which, returning to the original variables, implies $v \leq R$.

If $v \geq R$, we will integrate in the whole interval from 0 to $P = R/h$ in the Gauss Legendre quadrature rule. However, we need to be careful to avoid underflows.

Let $g_i \in [-1,1]$ be the N points in the Gauss-Legendre quadrature, in an increasing order, with associated weights w_i. Thus, $g_N \leq 1$ is the greatest of the weights and the mapped point in the interval 0 to $P = R/h$ is $\tilde{\rho} \triangleq 0 + 0.5(R/h-0)(g_N+1) = 0.5(R/h)(g_N+1)$. Define the function

$$\hat{f}(\rho,\nu,\tilde{\rho}) \triangleq e^{\frac{1}{2}(\tilde{\rho}-\nu)^2} f(\rho,\nu) = \rho\left(e^{-\frac{1}{2}(\rho-\nu)^2+\frac{1}{2}(\tilde{\rho}-\nu)^2} + e^{-\frac{1}{2}(\rho+\nu)^2+\frac{1}{2}(\tilde{\rho}-\nu)^2}\right)\hat{I}_0(\rho\nu). \tag{16}$$

Thus, for $v \geq R$, we compute

$$\mathit{Cir}_h(v,R) = \frac{(v-h\tilde{\rho})^2}{2} - h^2\log\left(\frac{h^2}{R^2}\int_0^{R/h}\hat{f}(\rho,vh,\tilde{\rho})d\rho\right) \tag{17}$$

in which the integral inside is approximated using Gauss-Legendre quadrature in the interval $[0,R/h]$.

Note that we need to compute the values of the Bessel Function. If it is not readily available, we can use excellent approximation given by (see [2]):

$$I_0(u) \approx \frac{\cosh(u)}{(1+0.25u^2)^{1/4}}\frac{1+0.24273u^2}{1+0.43023u^2}. \tag{18}$$

And thus:

$$\hat{I}_0(u) \approx \frac{1}{(1+0.25u^2)^{1/4}}\frac{1+0.24273u^2}{1+0.43023u^2}. \tag{19}$$

Here we provide the codes in C. Note that we use Gauss-Legendre quadrature of 7^{th} order, which seems good enough, but the code is easily modifiable if one wants to use higher-order quadratures.

```
#include <math.h>


#define PI 3.1415926;
#define SQRTHALFPI 1.2533141;
#define SQRT2 1.4142135;


double fun_I0_hat(double u)
{
    return pow(1+0.25*u*u,-0.25)*(1 + 0.24273*u*u)/(1 + 0.43023*u*u);
}


double fun_f(double nu, double rho)
{
    double A1 = exp(-0.5*(rho-nu)*(rho-nu));
    double A2 = exp(-0.5*(rho+nu)*(rho+nu));
    return rho*(A1+A2)*fun_I0_hat(rho*nu);
}

double fun_f_hat(double nu, double rho, double rhobar)
{
    double A1 = exp(-0.5*(rho-nu)*(rho-nu) + 0.5*(rhobar-nu)*(rhobar-nu));
    double A2 = exp(-0.5*(rho+nu)*(rho+nu) + 0.5*(rhobar-nu)*(rhobar-nu));
    return rho*(A1+A2)*fun_I0_hat(rho*nu);
}

double max(double a, double b)
{
    if (a >= b)
    {
        return a;
    }
    else
    {
```

```
        return  b;
    }
}

double min(double a, double b)
{
    if (a >= b)
    {
        return  b;
    }
    else
    {
        return  a;
    }
}

double Cir(double v, double h, double R)
{
    //The function should be called only for v >= 0
    v = abs(v);

    //Change here the Gauss-Legendre quadrature
    int N=7;
    double node[N] = {-0.94910, -0.74153, -0.40584, 0,   0.40584, 0.74153, 0.94910};
    double weight[N]= {0.12948,0.27970,0.38183,0.4179,0.38183,0.27970,0.12948};
    //end

    double F_low,F_up,delta,rhobar,y;


    if( v <= R)
    {
       F_low = max(0,sqrt((v/h)*(v/h)+1)-3);
       F_up  = min(R/h,sqrt((v/h)*(v/h)+1)+3);
       delta  = 0.5*(F_up-F_low);

       y=0;
       for(int i=0; i<N; i++)
       {
          y = y + weight[i]*fun_f(v/h,F_low + delta*(node[i]+1));
       }
       y = delta*y;
       return  -h*h*log(y*(h/R)*(h/R));
    }
    else
    {
       F_low = 0;
       F_up  = R/h;
       delta  = 0.5*(F_up-F_low);
       rhobar = F_low + delta*(node[N-1]+1);

       y=0;
       for(int i=0; i<N; i++)
       {
          y = y + weight[i]*fun_f_hat(v/h,F_low + delta*(node[i]+1),rhobar);
       }
       y = delta*y;
       return  0.5*(v-h*rhobar)*(v-h*rhobar)-h*h*log(y*(h/R)*(h/R));
    }
}
```

II. Formulaes for objects

A. Sphere

For a sphere of radius R centered at $p = [0\ 0\ 0]^T$ (see Figure 1), clearly $D_h^{\mathcal{A}}(p)$ is radially symmetric, that is, $D_h^{\mathcal{A}}(p)$ depends only on $\|p\|$. Then, without loss of generality, we can assume that $p = [0\ 0\ \|p\|]^T$.

Using spherical coordinates, $a_x = r\cos(\phi)\sin(\theta)$, $a_y = r\sin(\phi)\sin(\theta)$ and $a_z = r\cos(\theta)$, with $dV = r^2\sin(\theta) r d\theta dr d\phi$. Now, since we have that $\|p - a\|^2 = r^2 - 2r\|p\|cos(\theta) + \|p\|^2$ we can conclude that

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3}{4\pi R^3}\int_0^{2\pi}\int_0^R\int_0^\pi e^{-\frac{r^2-2r\|p\|cos(\theta)+\|p\|^2}{2h^2}} r^2\sin(\theta) r d\theta dr d\phi\right). \tag{20}$$

This can be rewritten as:

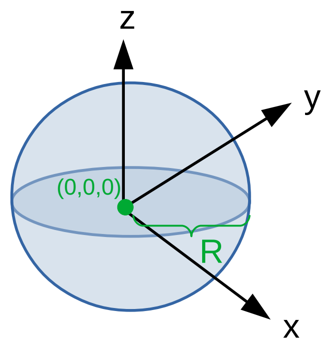

Fig. 1. Sphere in the canonical pose.

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3}{4\pi R^3}\int_0^{2\pi}\int_0^R r^2 e^{-\frac{r^2+\|p\|^2}{2h^2}}\left(\int_0^{\pi} e^{\frac{r\|p\|cos(\theta)}{h^2}}\sin(\theta)d\theta\right)drd\phi\right). \tag{21}$$

The inner integral can be easily computed with the change of variables $v = r\|p\|\cos(\theta)/h^2$, resulting in:

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3h^2}{4\pi R^3\|p\|}\int_0^{2\pi}\int_0^R r e^{-\frac{r^2+\|p\|^2}{2h^2}}\left(e^{r\|p\|/h^2} - e^{-r\|p\|/h^2}\right)drd\phi\right). \tag{22}$$

Using the fact that $e^{-\frac{r^2+\|p\|^2}{2h^2}}e^{r\|p\|/h^2} = e^{-\frac{(r-\|p\|)^2}{2h^2}}$, $e^{-\frac{r^2+\|p\|^2}{2h^2}}e^{-r\|p\|/h^2} = e^{-\frac{(r+\|p\|)^2}{2h^2}}$ and the fact that the integrand does not depend on ϕ, we can obtain

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3h^2}{2R^3\|p\|}\int_0^R r\left(e^{-\frac{(r-\|p\|)^2}{2h^2}} - e^{-\frac{(r+\|p\|)^2}{2h^2}}\right)dr\right). \tag{23}$$

Thus, if we define:

$$\mathit{Sph}_h(v,R) \triangleq -h^2 \log\left(\frac{3h^2}{2R^3 v}\int_0^R r\left(e^{-\frac{(r-v)^2}{2h^2}} - e^{-\frac{(r+v)^2}{2h^2}}\right)dr\right) \tag{24}$$

then $D_h^{\mathcal{A}}(p) = \mathit{Sph}_h(\|p\|, R)$. *Sph* stands for *Sphere*.

Now, note that:

$$\begin{aligned}
&\int_0^R r e^{-\frac{(r+v)^2}{2h^2}}dr = \int_0^R (r+v-v)e^{-\frac{(r+v)^2}{2h^2}}dr = \\
&\int_0^R (r+v)e^{-\frac{(r+v)^2}{2h^2}}dr - v\int_0^R e^{-\frac{(r+v)^2}{2h^2}}dr = \\
&h^2\left(e^{-\frac{v^2}{2h^2}} - e^{-\frac{(R+v)^2}{2h^2}}\right) - v\sqrt{\frac{\pi}{2}}h\left(\mathit{Erf}\left(\frac{R+v}{\sqrt{2}h}\right) - \mathit{Erf}\left(\frac{v}{\sqrt{2}h}\right)\right).
\end{aligned}$$

Analogously:

$$\begin{aligned}
&\int_0^R r e^{-\frac{(r-v)^2}{2h^2}}dr = \\
&h^2\left(e^{-\frac{v^2}{2h^2}} - e^{-\frac{(R-v)^2}{2h^2}}\right) + v\sqrt{\frac{\pi}{2}}h\left(\mathit{Erf}\left(\frac{R-v}{\sqrt{2}h}\right) + \mathit{Erf}\left(\frac{v}{\sqrt{2}h}\right)\right).
\end{aligned}$$

Then:

$$D_h^{\mathcal{A}}(p) = -h^2 \log\left(\frac{3h^2}{2R^3}\left(h^2\left(\frac{e^{-\frac{(R+v)^2}{2h^2}} - e^{-\frac{(R-v)^2}{2h^2}}}{v}\right) + 2Re^{-\mathit{Int}_h(v,R)/h^2}\right)\right). \tag{25}$$

This formula provides no problems if $v \leq R$ if we use the approximation for $Int_h(v,L)$ shown in Subsection I-E. However, for $v \geq R$ there can be numerical issues. In this case, we factor out $e^{-\frac{(R-v)^2}{2h^2}}$ to rewrite it as:

$$\frac{(v-R)^2}{2} - h^2 \log\left(\frac{3h^2}{2R^3}\left(h^2\left(\frac{e^{-\frac{2Rv}{h^2}}-1}{v}\right)+2Re^{-\widehat{Int}_h(v,R)/h^2}\right)\right) \tag{26}$$

in which $\widehat{Int}_h(v,L) \triangleq Int_h(v,L) - \frac{(v-L)^2}{2}$. Note that, when $v=0$, we need the limit

$$\lim_{v\to 0}\left(\frac{e^{-\frac{(R+v)^2}{2h^2}}-e^{-\frac{(R-v)^2}{2h^2}}}{v}\right) = -\frac{2R}{h^2}e^{-\frac{R^2}{2h^2}}. \tag{27}$$

Here is the C code:

```
double Sph(double v, double h, double R)
{

  //The function should be called only for v >= 0
   v = abs(v);

   double C = 3*(h*h)/(2*R*R*R);
   double A1, A2;
    if( v <= R)
   {
     if(v==0)
     {
        return -h*h*log(C*(-2*R*exp(-(R*R)/(2*h*h)) + 2*R*exp(-Int(0,h,R)/(h*h))));
  }
  else
  {
       A1 = exp(-((R+v)*(R+v)/(2*h*h)));
       A2 = exp(-((R-v)*(R-v)/(2*h*h)));
        return -h*h*log(C*(h*h*(A1-A2)/v + 2*R*exp(-Int(v,h,R)/(h*h))));
  }
   }
   else
   {
       A1 = exp(-(2*R*v/(h*h)));
       A2 = 1;
        return 0.5*(v-R)*(v-R)-h*h*log(C*(h*h*(A1-A2)/v + 2*R*exp((0.5*(v-R)*(v-R)-Int(v,h,R))/(h*h))));
   }
}
```

B. Box

For a box centered at $p = [0\ 0\ 0]^T$ with sides ℓ_x, ℓ_y and ℓ_z aligned with the x, y and z axis, respectively (see Figure 2), we have that $\mathcal{A} = [-\frac{\ell_x}{2}, \frac{\ell_x}{2}] \times [-\frac{\ell_y}{2}, \frac{\ell_y}{2}] \times [-\frac{\ell_z}{2}, \frac{\ell_z}{2}]$.

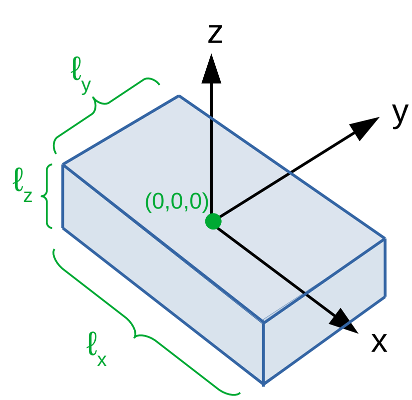

Fig. 2. Box in the canonical pose.

Thus, using the Cartesian product property (Subsection I-D) and the fact that for $\mathcal{A}_i = [-\frac{L_i}{2}, \frac{L_i}{2}]$ and $p^i \in \mathbb{R}$, $D_h^{\mathcal{A}_i}(p^i) = Int_h\left(p^i, \frac{L_i}{2}\right)$, we have that

$$D_h^{\mathcal{A}}(p) = \textit{Int}_h\left(x, \frac{\ell_x}{2}\right) + \textit{Int}_h\left(y, \frac{\ell_y}{2}\right) + \textit{Int}_h\left(z, \frac{\ell_z}{2}\right). \tag{28}$$

We can use the approximation for $\textit{Int}_h(v, L)$ shown in Subsection I-E.

C. Cylinder

For a cylinder centered at $p = [0\ 0\ 0]^T$ with radius R and height H (see Figure Figure 3), we use the fact that $\mathcal{A} = \mathcal{C}(R) \times [-H/2, H/2]$, in which $\mathcal{C}(R)$ is a circle centered at the origin of $\mathbb{R}^2$ with radius R.

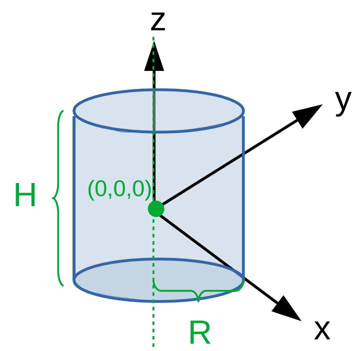

Fig. 3. Cylinder in the canonical pose.

We first compute $D_h^{\mathcal{C}(R)}(p_{xy})$, in which $p_{xy} = [x\ y]^T$. We can exploit the fact that the distance function for $\mathcal{C}(R)$ is radially symmetric in the variables p_{xy}, that is, the distance depends only on $\sqrt{x^2+y^2}$. Thus, without loss of generality, we can assume $p_{xy} = [\sqrt{x^2+y^2}\ \ 0]^T$. Plugging this into the integral definition for $D_h^{\mathcal{C}(R)}(p_{xy})$, using polar coordinates, the definition of the modified Bessel function of the first kind of order 0 and the results in Subsection I-D, we can see that $D_h^{\mathcal{C}(R)}(p_{xy}) = \textit{Cir}_h(\sqrt{x^2+y^2}, R)$.

Thus, using the Euclidean product property (Subsection I-D), we have that:

$$D_h^{\mathcal{A}}(p) = \textit{Cir}_h(\sqrt{x^2+y^2}, R) + \textit{Int}_h\left(z, \frac{H}{2}\right). \tag{29}$$

We can then use the approximation for $\textit{Int}_h(v, L)$ and $\textit{Cir}_h(v, R)$ shown in Subsections I-E and I-D, respectively.

References

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[2] J. Olivares, P. Martin, and E. Valero, "A simple approximation for the modified bessel function of zero order i0(x)," vol. 1043, p. 012003, jun 2018.