Maximum A Posteriori Reconstruction Algorithms

Key Points:

- The Prior term creates complications that require approximations or additional steps.
- The algorithm can be applied to only certain forms of the prior.
- MAP reconstructions are smoother than their ML counterparts.
- Convergence can be controlled by the weighting parameter β (Lalush and Tsui, 1992)
- As the value of β is decreased, the degree of smoothing is reduced, and vice versa.
- Type of Priors:
    - Quadratic priors (Fessler and Rogers, 1996)
    - Linearly increasing priors (Green, 1990; Hebert and Leahy, 1992; Lalush and Tsui, 1992, 1993; Mumcuoglu et al., 1994)

Approaches:

1. One Step Late (OSL) Approach
2. Generalized EM (GEM) Approach
3. PWLS algorithm Approach
4. MAP conjugate gradient (MAP-CG) Approach
5. Poisson-based Pre-conditioned Conjugate Gradient (PCG) Approach

1. One Step Late (OSL) Approach

References: (Green 1990)

- The brief logic is to evaluate the derivative term at the previous image estimate
- Results have been shown to converge to the MAP solution for only certain forms of the prior (Lange, 1990)

Ref:
"Implementation and evaluation of a 3D one-step late reconstruction algorithm for 3D positron emission tomography brain studies using median root prior - V. Bettinardi, E. Pagani, M.C. Gilardi, S. Alenius, K. Thielemans, M. Teras, F. Fazio"

Algorithm implementation

Notation is as follows:

λ^n	Image at the nth iteration
b, b'	Voxel index, $1 \le b \le B$
d	Detector pair index, $1 \le d \le D$
p_{db}	Probability matrix: probability that annihilation photon pairs emitted from voxel b are detected in the detector pair d
n_d	Measured counts at detector pair d
λ_d	Mean counts at detector pair d
λ_b	Mean counts at voxel b
LOR	Line of response: tube which connects a pair of opposite detectors in coincidence

The OSEM algorithm is:

$$\lambda_b^{n+1} = \frac{\lambda_b^n}{\sum\limits_{d \in D_k} p_{db}} \sum_{d \in Dk} \frac{n_d p_{db}}{\sum\limits_{b' \in B} \lambda_{b'}^n p_{db'}} \tag{1}$$

where D_k represents a partition of the projection space into m subsets, $n = 0,1,2,...$ is the sub-iteration number and $k = n \bmod m$. A cycle of m sub-iterations constitutes a complete iteration.

Consider the OSL algorithm in its general form:

$$\lambda_b^{n+1} = \frac{\lambda_b^n}{\sum\limits_{d \in D} p_{db} + \beta \frac{\partial}{\partial \lambda_b} U(\lambda, b)\,|_{\lambda=\lambda^n}} \sum_{d \in D} \frac{n_d p_{db}}{\sum\limits_{b' \in B} \lambda_{b'}^n p_{db'}} \tag{2}$$

where $U(\lambda, b)$ is a potential function while β is the weight factor.

The MRP algorithm can be formulated using the median as a penalty reference:

$$\beta \frac{\partial}{\partial \lambda_b} U(\lambda, b)\,|_{\lambda=\lambda^n} = \beta \frac{\lambda_b^n - M_b(\lambda^n)}{M_b(\lambda^n)} \tag{3}$$

where M_b is the median in a 3×3×3 mask width of neighbourhood voxels centered at voxel b. The penalty is set if the voxel λ_b is different from M_b.

2. Generalized EM (GEM) Approach

References: (Hebert and Gopal, 1992)

- Brief Logic:
  - Sequentially update pixels
  - Verify that each update increases the posterior density
  - Make sure that the convergence is to a maximum of the Posterior density

3. PWLS algorithm Approach

References: (Fessler, 1994)

- Brief Logic:
  - Utilize a Co-ordinate Descent Approach
  - Uses a quadratic prior to solving explicitly for the optimal step size t
- The algorithm is efficient (Usually requires approximately 10–15 iterations).
- But requires a matrix-based approach and restricts priors to only quadratic forms.

4. MAP conjugate gradient (MAP-CG) Approach

References: (Lalush and Tsui, 1995)

- More general priors and projector-based models can be accommodated.
- The algorithm also requires approximately 10–15 iterations.
- Complex to code as needed to perform a local linear fit of the prior term in the step size calculation and has no nonnegativity constraint.

5. Poisson-based Pre-conditioned Conjugate Gradient (PCG) Approach

References: (Mumcuoglu et al., 1994)

- Brief Logic:
  - Line searching to optimize the step size.
- Does not have the convergence problems of the OSL method.