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IMPROVED PERCENTILE ESTIMATION FOR THE TWO-PARAMETER WEIBULL DISTRIBUTION

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ABSTRACT

This paper presents an improvement of a technique recently published to estimate the parameters of the two-parameter Weibull distribution. A simple percentile method is used to estimate the two parameters. Computer simulation is employed to compare the proposed method with the maximum likelihood estimation and graphical methods results. A set of frequently-used and newer expressions for estimating the cumulative density are examined. Comparisons are made with both complete and censored data. The primary advantage of the method is its computational simplicity. Results indicate that with respect to Mean Square Error and estimation of the characteristic value with complete data, the percentile method cannot outperform the maximum likelihood method, although differences are minor in many instances. However, with censored data, improvements over the maximum likelihood are observed. When the shape parameter is estimated, the percentile method is quite competitive with that of maximum likelihood for both complete and censored data under a variety of conditions.

1. INTRODUCTION

The two-parameter Weibull distribution is often used in life testing and reliability theory, because it models either increasing or decreasing failure rate in a simple manner. Many authors have proposed various methods in order to obtain the estimates of these two parameters--the shape parameter β and the scale parameter θ. The method of maximum likelihood is the most general method [7]. The likelihood ratio method has been discussed [5]. The graphical methods (probability plot and hazard plot) are very popular means to estimate these parameters [7]. In this paper, we modify the method of Seki-Yokoyama [8] which is a simple and robust method. Using the percentile property, we can obtain the percentile such that the shape parameter β is the optimum value. The simulation is used to compare the effects of the simple percentile methods with the other methods for complete data and censored data. We chose several popular conventions for estimating the cumulative distribution function at the time of failure. The simulation results show that the proposed method is superior to the Seki-Yokoyama method, and the graphical methods (probability plot and hazard plot) for nearly all cases and superior to the maximum likelihood estimation for estimation of the shape parameter β. The method also holds promise for use with censored data.

2. ESTIMATION OF THE TWO WEIBULL PARAMETERS

The two-parameter Weibull probability density function is

$$f(t;\theta,\beta) = \left(\beta / \theta^{\beta}\right) t^{\beta-1} \exp\left[-(t/\theta)^{\beta}\right], \quad t > 0. \tag{1}$$

The shape parameter β and the scale parameter θ are positive. θ is also called the "characteristic life". Then the Weibull cumulative distribution function is

$$F(t) = 1 - \exp\left[1 - (t/\theta)^{\beta}\right], \quad t > 0. \tag{2}$$

The 100 p th percentile of a Weibull distribution is obtained from (2) as

$$t_p = \theta\left[-\ln(1-p)\right]^{1/\beta}. \tag{3}$$

Then the 100 (1 - e^{-1}) = 63.2 th percentile is $t_{0.632} = \theta$ for any Weibull distribution. Therefore we can obtain the estimate of the shape parameter, β from (3) as follows

$$\hat{\beta} = \frac{\ln\left[-\ln(1-p)\right]}{\ln\left(\frac{t_p}{t_{0.632}}\right)} \tag{4}$$

where $0 < t_p < t_{0.632}$.

Seki and Yokoyama [8] approximated the numerator of the equation (4) as -1 then obtained p = 0.31 approximately, to yield the estimation of the β. In fact, if we restrict p to two decimals, the numerator assumes values between -4.6 and nearly 0. We conducted hundreds of thousands of simulation runs with censored and uncensored data over a wide range of "true" shape parameters and obtained values of p which minimized the bias of the estimate of β in equation 4. Our results indicated that a single, approximately optimal value is p = 0.15. The t_p value is then found using the linear interpolation equation,

$$t_p = \frac{p - F(t_r)}{F(t_{r+1}) - F(t_r)}\left(t_{r+1} - t_r\right) + t_r \tag{5}$$

where t_r is the time of the r th ordered failure and $F(t_r)$ is the proportion of the population having failed by t_r.

There are many popular estimates of $F(t_r)$, including the Herd-Johnson method , the Kaplan-Meier method, the approximate normal method and the median method. Each of these four was used by Seki and Yokoyama, but only with complete (uncensored) data.

The percentile of the failure data by the selected methods may be obtained from the following table. That is, $F(t_r)$ is given by

Method	Complete Data	Censored Data
Herd-Johnson	$r/(n+1)$	$1-\prod\frac{n-r+1}{n-r+2}$
Kaplan-Meier	r/n	$1-\prod\frac{n-r}{n-r+1}$
Approximate Normal	$(r-3/8)/(n+1/4)$	$(r-3/8)/(n+1/4)$
Median	$(r-1/2)/n$	$(r-1/2)/n$

In computer-generated probability and hazard plots, the least squares method is used to estimate the parameters β and θ. The probability plot is obtained from the following linearization

$$\ln(t_r) = \ln\left(\frac{-\theta}{\beta}\right) + \beta \ln\left(\ln\left(\frac{1}{1-F(t_r)}\right)\right) \tag{6}$$

where $F(t)$ is the cumulative failure distribution function.

The hazard plot is obtained from the following linearization

$$\ln(t_r) = \ln(\theta) + \frac{1}{\beta}\ln(H(t_r)) \tag{7}$$

where for the r th ordered failure, the estimate of $h(t_r)$ is given by $\hat{h}(t_r) = 1$ divided by the reverse rank of the i th failure and $\hat{H}(t_r) = \sum_{i=1}^{r}\hat{h}(t_i)$.

The Maximum Likelihood Estimates for the Weibull two-parameter distribution are obtained as follows: The Weibull log likelihood for a sample of n units with r failures is

$$L = r\ln\beta - r\beta\ln\theta + (\beta-1)\sum_{i=1}^{r}\ln t_{i,f} - \frac{\sum_{i=1}^{r}\ln t_{i,f}^{\beta} + \sum_{j=1}^{n-r}\ln t_{j,s}^{\beta}}{\theta^{\beta}} \tag{8}$$

where $t_{i,f}$ is the failure time of the i th failed unit and $t_{j,s}$ is the non-failure running time of the j th unfailed unit. In order to maximize equation (8), we need to develop the first derivatives of the log likelihood equations with respect to β and θ. The first will yield a single equation in $\hat{\beta}$. That is

$$\frac{\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} \ln t_{i,f} + \sum_{j=1}^{n-r} t_{j,s}^{\hat{\beta}} \ln t_{j,s}}{\sum_{i=1}^{r} t_{i,f}^{\hat{\beta}} + \sum_{j=1}^{n-r} t_{j,s}^{\hat{\beta}}} - \frac{1}{\hat{\beta}} = \frac{1}{r} \sum_{i=1}^{r} \ln t_{i,f} \tag{9}$$

Equation (9) may be solved iteratively to get β, since the left-hand side (9) is a monotone function of β. Then one may calculate the value of θ

$$\theta = \left(\sum_{i} t_i^{\beta} / r \right)^{1/\beta}. \tag{10}$$

We have written FORTRAN code to obtain the Maximum Likelihood Estimates (MLE's) using equations (9) and (10). Our code has been validated by using selected data sets and comparing them with those published in Bain [1] and with those generated by Statgraphics® software. Our results are identical with those of Bain and Statgraphics® to three digits to the right of the decimal.

3. SIMULATION RESULTS

In this section, complete and censored data were used to compare the effects of the simple percentile estimation including the Seki-Yokoyama method to estimate β, the graphical methods (probability plot and hazard plot using least-squares estimation) and the maximum likelihood estimation procedure described in section 2. For complete data, the Weibull with θ = 100.0 and β = 0.5, 1.0, 1.5, 2.0, 3.0, 4.0 for selected samples between 5 and 100 was used to generate 10,000 random samples from IMSL subroutine DRNWIB, a FORTRAN-based procedure for generating Weibull variates. 10,000 random samples were also generated, using DRNWIB for each of three censored cases of sample size 20 with 12 failures and 8 censored units. The three censored data patterns are (1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0), (0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1) and (1 0 1 0 1 0 1 0 0 0 0 0 0 1 0 1 0 1 0 1). where 0 is a failure unit and 1 is nonfailure unit. These patterns represent just three of 20! / 12! patterns possible with 12 failures among 20 units and hence are not intended to represent this case. They do represent left, right and multiply-censored patterns, respectively. The computer simulation was implemented in a FORTRAN program and processed on the IBM 3090 Processor Complex at Arizona State University. The results of the simulation are shown in Tables 1 - 4. Table 1 reports the simulated mean bias

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Table 1

The Mean Bias of $\hat{\theta}$ with complete data in 10,000 simluation runs.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	Method 5	M.L.E.
	5	87.1703	4.1827	24.9297	69.8376	87.1703	29.6711
	6	77.2615	0.7774	10.8001	57.5582	77.2615	24.7037
	8	52.6489	0.5509	13.5755	41.7678	52.6489	19.3655
0.5	10	37.9578	2.3517	11.2535	30.5214	37.9578	15.5359
	20	19.3136	1.2194	5.2522	14.9918	19.3136	8.2295
	50	7.4053	0.2919	1.9943	5.8097	7.4053	3.3800
	100	3.7047	0.2638	1.1268	2.9888	3.7047	1.8878
	5	18.1111	-13.7249	-5.7659	11.4618	18.1111	3.2781
	6	16.7483	-11.6069	-6.2916	10.0992	16.7483	2.9144
	8	11.9000	-9.2249	-3.9437	7.4878	11.9000	2.3803
1.0	10	8.8674	-6.9454	-2.9921	5.5649	8.8674	2.2035
	20	4.5110	-4.0162	-1.9978	2.6046	4.5110	0.9964
	50	1.8776	-1.5399	-0.7086	1.1303	1.8776	0.5509
	100	0.6054	-1.0916	-0.6660	0.2523	0.6054	0.1351
	5	8.5239	-12.3338	-7.1193	4.1676	8.5239	0.1048
	6	7.4111	-10.6736	-6.8707	3.3395	7.4111	0.2470
	8	4.9327	-8.8403	-5.3970	2.0561	4.9327	-0.2062
1.5	10	3.6729	-6.8542	-4.2223	1.4743	3.6729	0.0761
	20	2.1051	-3.6039	-2.2330	0.8540	2.1051	0.0189
	50	0.5811	-1.6761	-1.1197	0.0981	0.5811	-0.0807
	100	0.5787	-0.5438	-0.2638	0.3447	0.5787	0.1601
	5	4.3084	-11.2482	-7.3591	1.0593	4.3084	-1.1251
	6	3.9823	-9.7196	-6.6775	0.9634	3.9823	-1.0424
	8	2.7206	-7.4831	-4.9322	0.5895	2.7206	-0.6532
2.0	10	2.0145	-5.9871	-3.9866	0.3434	2.0145	-0.5265
	20	1.1628	-3.0487	-2.0228	0.2533	1.1628	-0.2043
	50	0.3370	-1.3451	-0.9215	-0.0101	0.3370	-0.0897
	100	0.2124	-0.6350	-0.4225	0.0361	0.2124	-0.0725
	5	1.9440	-8.4712	-5.8674	-0.2313	1.9440	-1.2658
	6	1.7052	-7.3941	-5.2549	-0.2509	1.7052	-1.1666
	8	1.2970	-5.5680	-3.8517	-0.1368	1.2970	-0.8020
3.0	10	0.9955	-4.4756	-3.1078	-0.1472	0.9955	-0.5761
	20	0.3712	-2.4189	-1.7317	-0.2230	0.3712	-0.3470
	50	0.2463	-0.8970	-0.6126	0.0053	0.2463	-0.0707
	100	0.0220	-0.5560	-0.4110	-0.0982	0.0220	-0.0643
	5	1.1042	-6.9883	-4.9652	-0.5860	1.1042	-1.2808
	6	1.1215	-5.8015	-4.1431	-0.3541	1.1215	-0.9471
	8	0.7714	-4.3630	-3.0794	-0.3009	0.7714	-0.6524
4.0	10	0.5373	-3.5729	-2.5453	-0.3211	0.5373	-0.5894
	20	0.2118	-1.9056	-1.3817	-0.2365	0.2118	-0.2938
	50	0.1343	-0.7215	-0.5088	-0.0463	0.1343	-0.1268
	100	0.0590	-0.3665	-0.2598	-0.0295	0.0590	-0.0646

The Mean Bias of $\hat{\beta}$ with complete data in 10,000 simulation runs.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	Method 5	M.L.E.
	5	0.0522	0.0807	0.0653	0.1163	0.1761	0.2277
	6	0.0200	0.0744	0.0699	0.0737	0.1057	0.1694
	8	0.0126	0.0391	0.0254	0.0498	0.0996	0.1127
0.5	10	0.0099	0.0173	0.0144	0.0484	0.0980	0.0857
	20	0.0020	0.0053	0.0053	0.0167	0.0412	0.0373
	50	-0.0004	0.0005	0.0005	0.0042	0.0158	0.0134
	100	-0.0001	0.0001	0.0001	0.0020	0.0089	0.0070
	5	0.0218	0.0467	0.0186	0.1654	0.2828	0.4363
	6	-0.0045	0.0492	0.0379	0.1223	0.1767	0.3283
	8	0.0053	0.0401	0.0216	0.0997	0.1658	0.2273
1.0	10	0.0074	0.0196	0.0133	0.0782	0.1707	0.1684
	20	-0.0021	0.0044	0.0037	0.0304	0.0703	0.0730
	50	0.0005	0.0022	0.0021	0.0093	0.0342	0.0300
	100	0.0004	0.0008	0.0008	0.0045	0.0194	0.0146
	5	-0.0759	-0.0877	-0.1041	0.1609	0.3979	0.6633
	6	-0.0604	-0.0318	-0.0400	0.1681	0.2948	0.5094
	8	-0.0272	-0.0064	-0.0190	0.1423	0.2267	0.3273
1.5	10	-0.0108	0.0000	-0.0075	0.1046	0.2169	0.2432
	20	-0.0043	0.0071	0.0047	0.0434	0.1093	0.1152
	50	0.0015	0.0048	0.0044	0.0151	0.0488	0.0439
	100	-0.0004	0.0004	0.0004	0.0054	0.0222	0.0201
	5	-0.1628	-0.1846	-0.1903	0.1213	0.4631	0.8790
	6	-0.1430	-0.1357	-0.1343	0.1499	0.3848	0.6450
	8	-0.0672	-0.0646	-0.0708	0.1993	0.3221	0.4495
2.0	10	-0.0313	-0.0257	-0.0349	0.1591	0.3200	0.3409
	20	-0.0075	0.0044	0.0018	0.0546	0.1429	0.1540
	50	-0.0005	0.0030	0.0026	0.0174	0.0691	0.0579
	100	0.0003	0.0018	0.0013	0.0079	0.0298	0.0277
	5	-0.2687	-0.2825	-0.2866	-0.0179	0.6517	******
	6	-0.2298	-0.2258	-0.2149	0.1199	0.6422	******
	8	-0.1596	-0.1675	-0.1590	0.2732	0.4543	0.6771
3.0	10	-0.0979	-0.1041	-0.1107	0.2238	0.4602	0.5116
	20	-0.0031	0.0193	0.0156	0.0950	0.2216	0.2383
	50	-0.0011	0.0053	0.0044	0.0263	0.0951	0.0801
	100	0.0009	0.0024	0.0019	0.0070	0.0595	0.0400
	5	-0.3729	-0.3768	-0.3815	-0.1462	0.8998	******
	6	-0.3017	-0.2962	-0.2828	0.0022	0.8179	******
	8	-0.2142	-0.2236	-0.2105	0.3660	0.5854	0.9124
4.0	10	-0.1744	-0.1707	-0.1741	0.2863	0.5848	0.6793
	20	-0.0153	0.0124	0.0070	0.1113	0.2872	0.3026
	50	-0.0032	0.0052	0.0042	0.0317	0.1145	0.1173
	100	-0.0009	0.0012	0.0008	0.0021	0.0692	0.0557

Note : Method 1 : Percentile Estimate using Herd-Johnson method; Method 2 : Percentile Estimate using Kapaln-Meier method; Method 3 : Percentile Estimate using Approximate Normal method; Method 4 : Percentile Estimate using Median method; Method 5 : Seki-Yokoyama method using Herd-Johnson; M.L.E. : Maximum Likelihood Estimation.

****** infeasible with our MLE procedure due to small variance and small sample size

Table 2

The Standardized Mean Square Error (MSE/θ^2)of $\hat{\theta}$ with complete data in 10,000 simulation runs.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	Method 5	M.L.E.
	5	5.3870	1.3684	1.9090	4.1388	5.3870	1.6746
	6	3.5625	1.1704	1.4648	2.6631	3.5625	1.2752
	8	2.1367	0.9250	1.0761	1.7498	2.1367	0.8679
0.5	10	1.4724	0.7133	0.8376	1.2561	1.4724	0.6452
	20	0.5084	0.3546	0.3884	0.4719	0.5084	0.2731
	50	0.1654	0.1404	0.1461	0.1604	0.1654	0.0973
	100	0.0758	0.0698	0.0708	0.0742	0.0758	0.0463
	5	0.4036	0.2381	0.2416	0.3357	0.4036	0.2295
	6	0.3146	0.2163	0.2377	0.2740	0.3146	0.1923
	8	0.2342	0.1762	0.1738	0.2072	0.2342	0.1420
1.0	10	0.1920	0.1397	0.1435	0.1729	0.1920	0.1164
	20	0.0875	0.0755	0.0780	0.0848	0.0875	0.0555
	50	0.0336	0.0317	0.0320	0.0333	0.0336	0.0214
	100	0.0171	0.0167	0.0167	0.0169	0.0171	0.0111
	5	0.1446	0.1167	0.1086	0.1256	0.1446	0.0941
	6	0.1148	0.1022	0.1063	0.1062	0.1148	0.0797
	8	0.0915	0.0852	0.0801	0.0842	0.0915	0.0624
1.5	10	0.0771	0.0659	0.0648	0.0713	0.0771	0.0493
	20	0.0372	0.0351	0.0355	0.0367	0.0372	0.0246
	50	0.0148	0.0146	0.0146	0.0148	0.0148	0.0096
	100	0.0077	0.0076	0.0076	0.0076	0.0077	0.0050
	5	0.0755	0.0750	0.0671	0.0648	0.0755	0.0552
	6	0.0611	0.0645	0.0644	0.0587	0.0611	0.0462
	8	0.0489	0.0504	0.0464	0.0460	0.0489	0.0343
2.0	10	0.0416	0.0387	0.0372	0.0390	0.0416	0.0273
	20	0.0200	0.0199	0.0198	0.0199	0.0200	0.0136
	50	0.0083	0.0083	0.0083	0.0084	0.0083	0.0054
	100	0.0044	0.0044	0.0043	0.0043	0.0044	0.0029
	5	0.0322	0.0370	0.0322	0.0300	0.0322	0.0241
	6	0.0264	0.0313	0.0305	0.0262	0.0264	0.0206
	8	0.0206	0.0239	0.0215	0.0199	0.0206	0.0151
3.0	10	0.0182	0.0185	0.0174	0.0174	0.0182	0.0124
	20	0.0089	0.0093	0.0091	0.0090	0.0089	0.0060
	50	0.0038	0.0038	0.0038	0.0038	0.0038	0.0025
	100	0.0019	0.0019	0.0019	0.0019	0.0019	0.0013
	5	0.0179	0.0230	0.0196	0.0171	0.0179	0.0140
	6	0.0142	0.0180	0.0172	0.0143	0.0142	0.0115
	8	0.0115	0.0142	0.0126	0.0113	0.0115	0.0087
4.0	10	0.0100	0.0107	0.0099	0.0096	0.0100	0.0070
	20	0.0051	0.0053	0.0052	0.0051	0.0051	0.0035
	50	0.0021	0.0017	0.0021	0.0021	0.0021	0.0014
	100	0.0010	0.0011	0.0010	0.0010	0.0010	0.0007

The Standardized Mean Square Error (MSE/β^2) of $\hat{\beta}$ with complete data in 10,000 simulation runs.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	Method 5	M.L.E.
	5	0.1924	0.2656	0.2096	0.4397	1.9063	0.9608
	6	0.1037	0.2371	0.2232	0.2326	1.1828	0.5006
	8	0.0691	0.1148	0.0849	0.1299	0.7548	0.2413
0.5	10	0.0481	0.0555	0.0516	0.0920	0.5800	0.1547
	20	0.0149	0.0167	0.0173	0.0232	0.1796	0.0484
	50	0.0038	0.0037	0.0038	0.0045	0.0610	0.0143
	100	0.0017	0.0015	0.0016	0.0019	0.0280	0.0066
	5	0.0709	0.0842	0.0691	0.2233	1.7394	0.8241
	6	0.0580	0.0781	0.0724	0.1738	0.9359	0.4751
	8	0.0498	0.0660	0.0525	0.1374	0.6152	0.2485
1.0	10	0.0404	0.0443	0.0407	0.0816	0.6050	0.1502
	20	0.0134	0.0144	0.0151	0.0230	0.1743	0.0470
	50	0.0041	0.0040	0.0041	0.0048	0.0595	0.0148
	100	0.0016	0.0015	0.0015	0.0019	0.0278	0.0068
	5	0.0379	0.0392	0.0377	0.1185	2.2093	0.8515
	6	0.0356	0.0343	0.0330	0.1332	1.0880	0.4750
	8	0.0330	0.0364	0.0307	0.1203	0.5793	0.2386
1.5	10	0.0300	0.0310	0.0280	0.0765	0.5014	0.1422
	20	0.0126	0.0143	0.0146	0.0215	0.1835	0.0477
	50	0.0040	0.0039	0.0040	0.0047	0.0588	0.0147
	100	0.0018	0.0017	0.0017	0.0021	0.0273	0.0067
	5	0.0320	0.0366	0.0363	0.0645	1.0955	0.8222
	6	0.0293	0.0267	0.0263	0.0874	1.1046	0.4720
	8	0.0231	0.0219	0.0195	0.1370	0.5968	0.2382
2.0	10	0.0252	0.0232	0.0214	0.0981	0.5413	0.1501
	20	0.0122	0.0128	0.0133	0.0210	0.1728	0.0489
	50	0.0038	0.0036	0.0038	0.0045	0.0604	0.0150
	100	0.0017	0.0016	0.0016	0.0020	0.0283	0.0067
	5	0.0321	0.0379	0.0368	0.0183	1.1476	******
	6	0.0266	0.0269	0.0260	0.0468	1.1789	******
	8	0.0171	0.0178	0.0165	0.1145	0.5806	0.2405
3.0	10	0.0153	0.0133	0.0130	0.0850	0.6080	0.1473
	20	0.0132	0.0144	0.0148	0.0227	0.1643	0.0488
	50	0.0041	0.0039	0.0041	0.0047	0.0617	0.0146
	100	0.0017	0.0016	0.0016	0.0020	0.0287	0.0067
	5	0.0346	0.0388	0.0377	0.0127	1.2851	******
	6	0.0264	0.0268	0.0257	0.0210	1.1468	******
	8	0.0163	0.0177	0.0165	0.1095	0.5400	0.2298
4.0	10	0.0133	0.0127	0.0127	0.0777	0.5713	0.1524
	20	0.0117	0.0127	0.0131	0.0198	0.1690	0.0485
	50	0.0038	0.0036	0.0038	0.0043	0.0606	0.0149
	100	0.0018	0.0017	0.0017	0.0021	0.0291	0.0068

Note : Method 1 : Percentile Estimate using Herd-Johnson method; Method 2 : Percentile Estimate using Kapaln-Meier method; Method 3 : Percentile Estimate using Approximate Normal method; Method 4 : Percentile Estimate using Median method; Method 5 : Seki-Yokoyama method using Herd-Johnson; M.L.E. : Maximum Likelihood Estimation.

****** infeasible with our MLE procedure due to small variance and small sample size

Table 3

The Mean Bias of $\hat{\theta}$ in 10,000 simulation runs for three censored data patterns

Shape	Sample	Method 1	Method 2	Method 3	Method 4	M.L.E.
0.5	Case 1	168.7957	123.6509	210.5755	164.9002	214.3620
	Case 2	10.1586	-2.3986	198.1846	52.4107	247.6297
	Case 3	168.1970	96.1424	192.0659	128.3629	184.1965
1.0	Case 1	57.5013	43.6099	71.5863	58.4212	72.6282
	Case 2	1.9159	-5.0933	40.4565	13.1866	80.2096
	Case 3	56.1965	34.4825	64.0852	46.0178	64.0827
1.5	Case 1	34.0047	26.1119	42.2365	34.9222	42.8246
	Case 2	0.7868	-4.1537	19.9911	6.3923	46.8041
	Case 3	32.8465	20.6794	37.8945	27.6769	38.0974
2.0	Case 1	24.0403	18.5048	30.0214	24.9358	30.4196
	Case 2	0.6015	-3.2767	13.5536	4.2583	32.7086
	Case 3	23.2264	14.6939	27.0902	19.9450	27.1773
3.0	Case 1	15.0191	11.5740	18.8113	15.6869	19.0722
	Case 2	0.1454	-2.5117	7.9757	2.1785	20.3327
	Case 3	14.4559	9.1686	16.9110	12.4930	17.0942
4.0	Case 1	11.1158	8.5560	13.8360	11.5870	14.0138
	Case 2	0.0365	-1.9827	5.7882	1.5269	14.7890
	Case 3	10.6860	0.7621	12.4464	9.2683	12.5910

The Mean Bias of $\hat{\beta}$ in 10,000 simulation runs for three censored data patterns.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	M.L.E.
0.5	Case 1	0.2963	0.3836	0.3816	0.5230	0.5235
	Case 2	0.0127	0.0104	-0.0519	-0.0164	-0.1509
	Case 3	0.0176	0.0400	0.0203	0.0626	0.0667
1.0	Case 1	0.6014	0.7640	0.7630	1.0438	1.0463
	Case 2	0.0114	0.0111	-0.1075	-0.0361	-0.3046
	Case 3	0.0276	0.0667	0.0396	0.1236	0.1319
1.5	Case 1	0.9171	1.1447	1.1509	1.5704	1.5738
	Case 2	0.0113	0.0150	-0.1533	-0.0448	-0.4486
	Case 3	0.0443	0.0966	0.0637	0.1909	0.2029
2.0	Case 1	1.2438	1.5612	1.5325	2.1019	2.0940
	Case 2	0.0088	0.0126	-0.2088	-0.0677	-0.5991
	Case 3	0.0613	0.1351	0.0817	0.2499	0.2711
3.0	Case 1	1.8734	2.3552	2.3071	3.0594	3.1527
	Case 2	0.0159	0.0280	-0.2972	-0.0784	-0.8897
	Case 3	0.0917	0.2005	0.1387	0.3966	0.4128
4.0	Case 1	2.4333	3.1118	3.0483	4.1634	4.1783
	Case 2	0.0042	0.0213	-0.4178	-0.1286	-1.1999
	Case 3	0.1142	0.2595	0.1616	0.5006	0.5381

Note : Method 1 : Percentile Estimation using Herd-Johnson method; Method 2 :Percentile Estimation using Kaplan-Meier method; Method 3 : Probability Plot with Herd-Johnson Method; Method 4 : Hazard Plot; M.L.E. : Maximum Likelihood Estimation.

of the scale parameter estimate, $\hat{\theta}$ and the shape parameter estimate, $\hat{\beta}$ for complete data. Table 2 reports simulation results for the standardized mean square error (MSE $/\theta^2$ and MSE $/\beta^2$) of the θ and β estimates for complete data. Table 3 reports the simulated mean bias of the scale parameter estimate, $\hat{\theta}$ and the shape parameter estimate, $\hat{\beta}$ for censored data. The bias measure is absolute and for any estimator , ϕ, it is given by Bias($\hat{\phi}$) = E($\hat{\phi}$)- ϕ. Table 4 reports simulation results for the standardized mean square error (MSE $/\theta^2$ and MSE $/\beta^2$) of the θ and β estimates for censored data.

Table 4

The Standardized Mean Square Error (MSE/θ^2) of $\hat{\theta}$ for three censored data patterns in 10,000 simulation runs.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	M.L.E.
	Case 1	5.0821	3.0722	6.4664	4.2220	6.6611
0.5	Case 2	0.5726	0.3981	346.0350	2.9509	27.3243
	Case 3	4.8884	2.1216	6.9808	3.0480	5.0680
	Case 1	0.5091	0.3389	0.6620	0.4702	0.6780
1.0	Case 2	0.1200	0.0912	2.0855	0.2735	0.8586
	Case 3	0.4799	0.2541	0.6045	0.3442	0.5465
	Case 1	0.1729	0.1188	0.2245	0.1641	0.2295
1.5	Case 2	0.0534	0.0425	0.2524	0.0921	0.2898
	Case 3	0.1606	0.0911	0.2043	0.1223	0.1883
	Case 1	0.0847	0.0587	0.1117	0.0823	0.1141
2.0	Case 2	0.0306	0.0245	0.1081	0.0468	0.1462
	Case 3	0.0790	0.0456	0.1021	0.0623	0.0942
	Case 1	0.0330	0.0232	0.0434	0.0323	0.0444
3.0	Case 2	0.0138	0.0113	0.0385	0.0190	0.0581
	Case 3	0.0306	0.0181	0.0393	0.0244	0.0370
	Case 1	0.0179	0.0126	0.0233	0.0175	0.0238
4.0	Case 2	0.0078	0.0065	0.0203	0.0104	0.0325
	Case 3	0.0165	0.0099	0.0209	0.0132	0.0199

The Standardized Mean Square Error (MSE/β^2) of $\hat{\beta}$ for three censored data patterns in 10,000 simulation runs.

Shape	Sample	Method 1	Method 2	Method 3	Method 4	M.L.E.
	Case 1	0.7860	1.2695	0.8112	1.3925	1.4334
0.5	Case 2	0.2380	0.0212	0.1148	0.1218	0.1895
	Case 3	0.0184	0.0378	0.0610	0.0859	0.0719
	Case 1	0.8261	1.2387	0.8102	1.3856	1.4274
1.0	Case 2	0.0213	0.0187	0.1124	0.1181	0.1911
	Case 3	0.0141	0.0314	0.0583	0.0824	0.0718
	Case 1	0.8650	1.2453	0.8111	1.3857	1.4258
1.5	Case 2	0.0213	0.0192	0.1167	0.1248	0.1438
	Case 3	0.0144	0.0300	0.0590	0.0848	0.0716
	Case 1	0.8866	1.3091	0.8045	1.3923	1.4125
2.0	Case 2	0.0198	0.0173	0.1117	0.1178	0.1441
	Case 3	0.0153	0.0327	0.0606	0.0850	0.0703
	Case 1	0.8676	1.2915	0.8140	1.3969	1.4354
3.0	Case 2	0.0219	0.0189	0.1162	0.1241	0.1427
	Case 3	0.0155	0.0323	0.0626	0.0890	0.0727
	Case 1	0.8246	1.2778	0.7997	1.3647	1.4142
4.0	Case 2	0.0201	0.0171	0.1166	0.1256	0.1444
	Case 3	0.0146	0.0310	0.0605	0.0859	0.0707

Note : Method 1 : Percentile Estimation using Herd-Johnson method; Method 2 :Percentile Estimation using Kaplan-Meier method; Method 3 : Probability Plot with Herd-Johnson Method; Method 4 : Hazard Plot; M.L.E. : Maximum Likelihood Estimation.

Table 1, which reports mean bias with complete data, suggests that with respect to $\hat{\theta}$, the maximum likelihood estimator (MLE) is best. However, with respect to $\hat{\beta}$, any percentile procedure is better than the MLE. The improved procedure (optimal percentile for $\hat{\beta}$, Methods 1-4) is superior to the Seki-Yokoyama approximate percentile estimate for β.

Table 2, which reports standardized MSE with complete data, produces results similar to Table 1. That is, with respect to $\hat{\theta}$, the maximum likelihood estimator (MLE) is best. However, with respect to $\hat{\beta}$, the improved procedure (optimal percentile for $\hat{\beta}$, Methods 1-4) is superior to the Seki-Yokoyama approximate percentile estimate for β and superior to the MLE procedure.

Table 3 reports mean bias with three censored data sets. In estimating θ, Table 3 indicates that the improved percentile procedure (Methods 1 and 2) offers promise as the procedure of choice over both MLE and least-squares approaches (Methods 3 and 4). In estimating β, the results are nearly the same. Our enthusiasm must be tempered by the fact that we examined only three censored failure patterns with one degree of censoring (12 of 20 failures in each case). There is a need to study how the method performs as censoring increases.

Table 4 which reports standardized MSE with three censored data sets produces results similar to Table 3. The improved percentile procedure (Methods 1 and 2) seems to outperform the others under the three censoring patterns studied. In estimating θ, the Kaplan-Meier estimate of $F(t_r)$ used with the improved percentile method has a slight advantage over the Herd-Johnson estimate while the reverse is true if β is to be estimated.

Another generalization may be made from Tables 3 and 4. The improved percentile methods applied to the left-censoring pattern , Case 2, appeared to yield better estimation results than the right and multiply-censored patterns, Cases 1 and 3, respectively.

4. CONCLUSION

In this paper an improved percentile estimation method for the shape parameter of the two-parameter Weibull distribution is proposed. The method is based on selection of a percentile which minimizes the bias of the estimator of the shape parameter. A p value of 0.15 is approximately optimal over a wide range of conditions. Based on simulation studies, it is concluded that the proposed method is useful and has many good properties for estimation of Weibull parameters under complete and censored data conditions. Further research with a variety of other patterns of censoring is indicated.

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