
Journal of Forest Research

ISSN: 1341-6979 (Print) 1610-7403 (Online) Journal homepage: www.tandfonline.com/journals/tjfr20

# Fitting irregular diameter distributions of forest stands by Weibull, modified Weibull, and mixture Weibull models

Lianjun Zhang & Chuangmin Liu

J For Res (2006) 11:369–372
DOI 10.1007/s10310-006-0218-7


SHORT COMMUNICATION

**Lianjun Zhang · Chuangmin Liu**

# Fitting irregular diameter distributions of forest stands by Weibull, modified Weibull, and mixture Weibull models

Received: March 20, 2006 / Accepted: March 28, 2006

**Abstract** Irregular diameter frequency distributions of forest stands include multimodal structure of mixed-species stands, highly skewed and highly irregular shapes of uneven-aged stands, and rotated sigmoid form of old-growth stands. In this study, a traditional two-parameter Weibull model, a modified two-parameter Weibull model, and a finite mixture of two-parameter Weibull models were used to fit four artificial example plots. The model fitting and comparison results indicate that the mixture Weibull model is more flexible to fit various irregular diameter distributions, while the traditional Weibull model fails in every case to adequately describe these frequency distributions. The modified Weibull model is a good choice for fitting the "rotated-sigmoid" diameter distribution of an uneven-aged old-growth stand. However, it may not be sufficient when a diameter frequency distribution is multimodal or highly irregular in shape.

**Key words** Diameter frequency distribution · Weibull function · Finite mixture model · Model fitting and comparison

## Introduction

The Weibull model has been popular for quantifying the diameter frequency distributions of forest stands among various probability density functions (e.g., Bailey and Dell 1973; Little 1983; Maltamo et al. 1995; Nanang 1998). A single Weibull function is sufficient to characterize (1) the regular and unimodal diameter distributions of even-aged stands, and (2) the balanced and reverse J-shaped diameter distributions of uneven-aged stands. However, the use of unimodal statistical distributions can lead to an oversimplified description for irregular stand structures (e.g., Murphy and Farrar 1981; Maltamo et al. 2000). Minowa and Hirata (1993) modified an exponential distribution by applying a quadratic function transformation. A power transformation was also used to modify the Weibull function. They found that the modified exponential and Weibull functions were flexible to describe the bimodal and rotated sigmoid diameter distributions of uneven-aged forest stands (Minowa and Hirata 1993). In recent years, a finite mixture of Weibull functions has been utilized to model the multimodal structure of mixed-species stands, highly skewed and irregular shapes of uneven-aged stands, and rotated sigmoid diameter distribution of old-growth stands (Zhang et al. 2001; Liu et al. 2002; Zasada and Cieszewski 2005). The purpose of this study was to compare the model fitting of a traditional two-parameter Weibull model, a modified two-parameter Weibull model, and a finite mixture of two-parameter Weibull models to describe irregular diameter distributions using four artificial example plots. The three models are briefly described as follows:

The probability density function (pdf) of a two-parameter Weibull function $f(x)$ is given by

$$f(x) = \left(\frac{\gamma}{\beta}\right)\left(\frac{x}{\beta}\right)^{\gamma-1} \exp\left[-\left(\frac{x}{\beta}\right)^{\gamma}\right] \qquad 0 \le x < \infty, \beta > 0, \gamma > 0 \tag{1}$$

where $x$ is a random variable (i.e., tree diameter), and $\beta$ and $\gamma$ are the scale and shape parameters, respectively (Bailey and Dell 1973). Because the artificially generated example plots in this study all have zero as the minimum diameter, it is reasonable to assume the location parameter of the Weibull is zero.

Minowa and Hirata (1993) derived the modified Weibull as follows: the exponential distribution can be written in the form of a differential equation

L. Zhang (✉)
Faculty of Forest and Natural Resources Management, State University of New York College of Environmental Science and Forestry, 1 Forestry Drive, Syracuse, NY 13210, USA
Tel. +1-31-5470-6558; Fax +1-31-5470-6535
e-mail: lizhang@esf.edu

C. Liu
Forintek Canada Corporation, 319 rue Franquet, Sainte-Foy, QC, Canada

$$\varphi(x) = \frac{1}{f(x)} \frac{df(x)}{dx} = -a \tag{2}$$

where $x$ is a random variable, $f(x)$ is the pdf, $\varphi(x)$ is the relative derivative of $f(x)$, and $a$ is a positive constant. Equation 2 can be modified by introducing a quadratic function such that

$$\varphi(x) = \frac{1}{f(x)} \frac{df(x)}{dx} = ax^2 + bx + c \tag{3}$$

where $a$, $b$, and $c$ are constants. Then, $f(x)$ can be solved as follows:

$$f(x) = \exp\left(\frac{ax^3}{3} + \frac{bx^2}{2} + cx + d\right) \tag{4}$$

where $d$ is an integral constant. Thus, Eq. 4 is the modified exponential function. The modified Weibull function can be derived by applying the power transformation, $z = \theta x^\delta$, to Eq. 4 as follows:

$$f(z) = \frac{1}{\theta\delta}\left(\frac{\theta}{z}\right)^{(\delta-1)/\delta} \exp\left[\frac{a}{3}\left(\frac{z}{\theta}\right)^{3/\delta} + \frac{b}{2}\left(\frac{z}{\theta}\right)^{2/\delta} + c\left(\frac{z}{\theta}\right)^{1/\delta} + d\right]. \tag{5}$$

A frequency distribution made up of two or more component distributions is defined as a "mixture" distribution. Suppose a mixture distribution consists of $k$ components, the distribution of the $i$th individual component is described by a specific pdf, $f_i(x)$. Then the general pdf, $f(x)$, for the mixture distribution can be expressed as

$$f(x) = \sum_{i=1}^{k} \rho_i f_i(x) = \rho_1 f_1(x) + \ldots + \rho_k f_k(x) \tag{6}$$

where $\rho_i$ is the relative abundance of the $i$th component as a proportion of the total population, and must satisfy the constraints $0 \le \rho_i \le 1$ and $\sum_{i=1}^{k} \rho_i = 1$. In this study we chose the Weibull function (Eq. 1) as the common component pdf, $f_i(x)$, with different means and variances (Zhang et al. 2001; Liu et al. 2002).

## Example plots and modeling methods

### Example plots

Because we did not have appropriate field data to fit and compare the three models, we generated four example plots to mimic the irregular frequency distributions of tree diameters for mixed-species, uneven-aged, and old-growth forest stands. These artificial data were inspired by the graphics published in the literature. We attempted to follow the patterns suggested in these research reports. Plot 1 represents a mixed-species plot with two species components. Plot 2 shows a mixed-species plot with three species components. The combination of the three species produced a distinct modal in the middle of the distribution. Plot 3 mimicks the "rotated-sigmoid" form of an uneven-aged old-growth plot suggested by Goff and West (1975) and Leak (1996). Plot 4 represents an irregular uneven-aged plot due to disturbances such as harvests, fires, competition control, seed crops, weather, or insect and diseases attacks (Baker et al. 1996).

### Model fitting

In this study, Statistical Analysis System (SAS) (SAS Institute 2002) was used to estimate the parameters of the traditional Weibull (Eq. 1) and the modified Weibull (Eq. 5). MIX software (Macdonald and Pitcher 1979; Haughton 1997) was used to estimate the parameters of the mixture Weibull (Eq. 6).

### Model comparison

We used root mean square error and the $\chi^2$ test to compare model fitting to the four example plots. The root mean square error (RMSE) for the diameter sums was computed as follows:

$$\text{RMSE} = \sqrt{\frac{\sum_{j=1}^{m}\left(N_j - \hat{N}_j\right)^2}{m}}$$

where $N_j$ and $\hat{N}_j$ are the observed number and predicted number of trees for the $j$th diameter-class in a plot, respectively, and $m$ is the number of diameter classes. The likelihood-ratio $\chi^2$ test was chosen for testing goodness of fit such that

$$\chi^2 = -2\sum_{j=1}^{m} N_j \cdot \log\left(\frac{\hat{N}_j}{N_j}\right).$$

The $\chi^2$ has $(m - k - 1)$ degrees of freedom, where $k$ is the number of estimated parameters.

## Results and discussion

The parameter estimates of the three models are given in Table 1. Note that plots 1 and 3 are assumed to consist of two individual components, while plots 2 and 4 are composed of three components. The predicted frequencies by diameter classes were obtained from each model for each plot. The predictions from each model were compared with the observed frequencies. The RMSE, $\chi^2$, and $P$ value for the $\chi^2$ test were computed for each model and each example plot (Table 2). The observed frequency distribution (histograms) and the three prediction curves are illustrated for each plot in Fig. 1.

**Table 1.** Parameter estimates of the three Weibull models for the four example plots

| Plot | Weibull | | Modified Weibull | | | | | | Mixture Weibull | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | β | γ | a | b | c | d | θ | δ | $\beta_1$ | $\gamma_1$ | $\beta_2$ | $\gamma_2$ | $\beta_3$ | $\gamma_3$ |
| Plot 1 | 11.0619 | 1.6514 | –0.00001 | 0.00195 | –0.0784 | 3.4183 | 1.6471 | 0.5008 | 7.1378 | 1.6066 | 17.0039 | 0.0588 | – | – |
| Plot 2 | 11.4416 | 1.9326 | –0.00009 | –0.00551 | 0.0870 | 3.3880 | 1.0267 | 0.8809 | 4.6296 | 1.5603 | 10.3842 | 4.4297 | 17.0358 | 6.2098 |
| Plot 3 | 4.7103 | 0.8297 | 0.00024 | 0.0192 | –0.3539 | 4.7255 | 0.9869 | 0.6635 | 2.7058 | 1.1472 | 13.5685 | 4.3701 | – | – |
| Plot 4 | 6.2854 | 1.0438 | –0.00040 | 0.0212 | –0.2814 | 5.3606 | 0.9953 | 0.6974 | 3.3091 | 1.1057 | 9.9305 | 8.8039 | 14.9031 | 3.5967 |

**Table 2.** The root mean square error (RMSE), and $\chi^2$ test of the three Weibull models for the four example plots

| Plot | Weibull | | | Modified Weibull | | | Mixture Weibull | | |
|---|---|---|---|---|---|---|---|---|---|
| | RMSE | $\chi^2$ | *P* value | RMSE | $\chi^2$ | *P* value | RMSE | $\chi^2$ | *P* value |
| Plot 1 | 274.07 | 174.32 | <0.0001 | 41.92 | 65.90 | <0.0001 | 34.14 | 19.50 | 0.2436 |
| Plot 2 | 292.50 | 151.57 | <0.0001 | 78.77 | 27.91 | 0.0219 | 14.69 | 5.61 | 0.9592 |
| Plot 3 | 117.81 | 132.84 | <0.0001 | 10.88 | 20.86 | 0.1411 | 8.61 | 9.28 | 0.7517 |
| Plot 4 | 444.80 | 160.28 | <0.0001 | 188.26 | 92.55 | <0.0001 | 24.18 | 14.43 | 0.3441 |

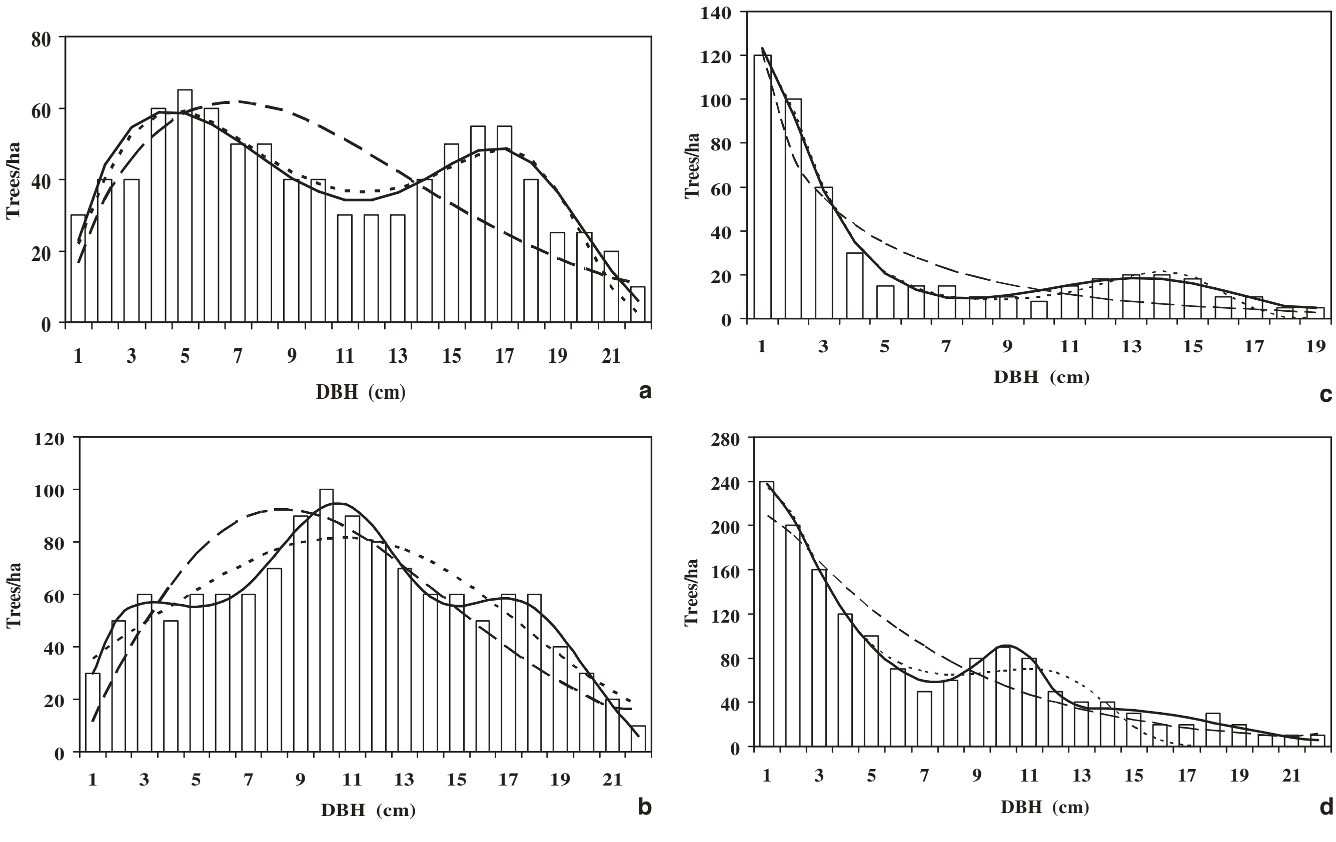


**Fig. 1a–d.** Model comparison for the four example plots. The histogram represents the observed diameter distribution with traditional Weibull (*dashed line*), modified Weibull (*dotted line*), and mixture Weibull (*solid line*) for **a** plot 1, **b** plot 2, **c** plot 3, and **d** plot 4

For plot 1, the mixture Weibull model is the only one that adequately fits the two peaks and the valley between the two distinct modes (Fig. 1a). The traditional Weibull model was definitely not flexible enough to fit the distribution at all. This single Weibull function missed the second peak as well as the valley between the two peaks. The modified Weibull model greatly improved fitting of the distribution (Fig. 1a), but the $\chi^2$ test indicated that the predicted frequency was significantly different from the observed one (Table 2).

Plot 2 represents a plot with three individual components. Again, the mixture Weibull model adequately fits the plot, while other two models fail to characterize the distribution (Table 2). Figure 1b shows that traditional Weibull underpredicts small and large trees and overpredicts middle-sized trees. For this plot, the modified Weibull performed better than the traditional Weibull model, but not as well as the mixture Weibull model. It still produced a smoothing unimodal curve (Fig. 1b).

Plot 3 had a reverse J-shape distribution up to the 10-cm diameter class followed by a hump for the "sigmoid" portion of the distribution. For this plot both modified and mixture Weibull models produced satisfactory fitting results. Both models fit the entire distribution well according to the $\chi^2$ tests (Table 2) and yield similar predictions across tree diameters (Fig. 1c). On the other hand, the traditional Weibull model did not fit the plot well, and definitely missed the "sigmoid" portion of the distribution (Fig. 1c).

For the highly irregular distribution of plot 4, the mixture Weibull model was again the only one that adequately fitted the plot according to the $\chi^2$ tests (Table 2). Both the traditional and modified Weibull models missed the irregular fluctuation of the diameter distribution due to disturbances (Table 2, Fig. 1d).

## Conclusions

It is evident that the mixture Weibull model was more flexible in fitting various irregular diameter distributions of uneven-aged forest stands, while the traditional Weibull model failed in every case to adequately describe these irregular frequency distributions. The modified Weibull model was a good choice for fitting the "rotated-sigmoid" diameter distribution of an uneven-aged old-growth stand, as indicated by Minowa and Hirata (1993). However, it may not be sufficient when a frequency distribution is multimodal or highly irregular in shape.

## Literature cited

Bailey RL, Dell TR (1973) Quantifying diameter distributions with the Weibull function. Forest Sci 19:97–104

Baker JB, Cain MD, Guldin JM, Murphy PA, Shelton MG (1996) Uneven-aged silviculture for loblolly and shortleaf pine forest cover types. USDA Forest Service General Technical Report SO-118

Goff FG, West D (1975) Canopy–understory interaction effects on forest population structure. Forest Sci 21:98–108

Haughton D (1997) Packages for estimating finite mixtures: a review. Am Stat 51:194–205

Leak WB (1996) Long-term structure change in uneven-aged northern hardwoods. Forest Sci 42:160–165

Little SN (1983) Weibull diameter distributions for mixed stands of western conifers. Can J Forest Res 13:85–88

Liu C, Zhang L, Davis CJ, Solomon DS, Gove JH (2002) A finite mixture model for characterizing the diameter distribution of mixed-species forest stands. Forest Sci 48:653–661

Macdonald PDM, Pitcher TJ (1979) Age-groups from size-frequency data: a versatile and efficient method of analyzing distribution mixtures. J Fish Res Board Can 36:987–1001

Maltamo M, Puumalainen J, Paivinen R (1995) Comparison of beta and Weibull functions for modeling basal area diameter distribution in stands of *Pinus sylvestris* and *Picea abies*. Scand J Forest Res 10:284–295

Maltamo M, Kangas A, Uuttera J, Torniainen T, Saramaki J (2000) Comparison of percentile based prediction methods and the Weibull distribution in describing the diameter distribution of heterogeneous Scots pine stands. Forest Ecol Manag 133:263–274

Minowa M, Hirata Y (1993) A modified exponential distribution for describing the stand structure of even-aged forests. J Jpn For Soc 75:449–451

Murphy PA, Farrar RM (1981) A test of the exponential distribution for stand structure definition in uneven-aged loblolly-shortleaf pine stands. USDA Forest Service Research Paper SO-164, p 4

Nanang DM (1998) Suitability of the normal, log-normal and Weibull distributions for fitting diameter distributions of neem plantations in Northern Ghana. Forest Ecol Manag 103:1–7

SAS Institute (2002) SAS/STAT users' guide, version 9.0. SAS Institute, Cary, NC

Zasada M, Cieszewski CJ (2005) A finite mixture distribution approach for characterizing tree diameter distributions by natural social class in pure even-aged Scots pine stands in Poland. Forest Ecol Manag 204:145–158

Zhang L, Gove JH, Liu C, Leak WB (2001) A finite mixture distribution for modeling the diameter distribution of rotated-sigmoid, uneven-aged stands. Can J Forest Res 31:1654–1659