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Parameter Estimation Through Weighted Least-Squares Rank Regression with Specific Reference to the Weibull and Gumbel Distributions

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Probability plots are often used to estimate the parameters of distributions. Using large sample properties of the empirical distribution function and order statistics, weights to stabilize the variance in order to perform weighted least squares regression are derived. Weighted least squares regression is then applied to the estimation of the parameters of the Weibull, and the Gumbel distribution. The weights are independent of the parameters of the distributions considered. Monte Carlo simulation shows that the weighted least-squares estimators outperform the usual least-squares estimators totally, especially in small samples.

Keywords Estimation; Gumbel distribution; Probability plot; Rank regression; Weibull distribution; Weighted least-squares regression.

Mathematics Subject Classification 62G30; 62F10; 62G20.

1. Introduction

The Weibull distribution is widely used in life testing and reliability theory especially in engineering and the analysis of medical lifetime data. In this article, a weighted least squares regression method is derived to estimate the parameters of the two-parameter Weibull distribution for complete samples. The weights are proportional to the inverse of the large sample variances of a function of the order statistics. We use Monte Carlo simulation to assess the performance of the weighted least squares procedure. The following aspects are investigated.

- The quality of the variance approximation, from which the weights are derived, particularly for small samples.
- To compare the mean squared error of estimation of the proposed weighted regression procedure with that of the usual simple least squares procedure.

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- The performance of different empirical estimators of the cumulative distribution function (cdf) evaluated at the order statistics in the calculation of the weights.
- To assess the relative efficiency of the weighted regression procedure and compare it with maximum likelihood estimation (MLE) and the generalized least squares (GLS) median rank regression results found in the study of Engeman and Keefe (1982).

This article suggests that the proposed large sample approximation of the variances is good even for small samples and relative efficiencies with respect to the Cramér-Rao bound indicates performance close to that of the GLS model and MLE. The GLS model outperforms MLE with respect to efficiency in complete small samples (Engeman and Keefe, 1982). The linear weighted regression totally outperforms the simple least squares method.

The cumulative distribution of the two-parameter Weibull distribution is

$$F(x;\alpha,\beta) = 1 - \exp(-(x/\alpha)^{\beta}), \quad x, \alpha, \beta \geq 0, \tag{1}$$

α the scale parameter and β the shape parameter. The probability density function (pdf) of the Weibull distribution is given by

$$f(x) = (\beta/\alpha)x^{\beta-1}\exp(-(x/\alpha)^{\beta}), \quad x \geq 0.$$

Two of the most important methods of estimation are MLE and linear least squares estimation based on order statistics. Least squares linear models provide closed form solutions of the estimators. The MLE estimators have to be computed by making use of nonlinear methods and the sampling properties can only be determined for large samples when asymptotic theory can be applied.

Cohen (1965) and White (1963) considered the MLE method and derived the Cramér-Rao bounds of $\hat{\beta}$ and $\hat{\alpha}$ are, respectively, $0.608\beta^2/n$ and $1.109(\alpha/\beta)^2/n$. Balakrishnan and Kateri (2008) derived an easy to apply graphical estimation procedure of the shape parameter based on the MLE equations. They give a closed form expression of the shape parameter for censored data and large samples, and the existence and uniqueness of the MLE estimator is shown as a byproduct. A comparative study of nonparametric estimation methods in Weibull regression was conducted by Kim (2011).

Beginning with the article of Lloyd (1952) estimators which are linear functions of the order statistics of a sample have attracted much attention. These linear estimation methods generally exploit the fact that the expected value of an order statistic $X_{(r)}$ from a distribution depending on a location parameter μ and a scale parameter σ, can be written in the linear model form, namely $E(X_{(r)}) = \mu + \sigma\alpha_r$, $r = 1, \ldots, n$. The α_r, $r = 1, \ldots, n$ are standardized variables which are distributed parameter free. Approximations of the expected values and of the covariance matrix of the α_r can be calculated. Blom (1962) gave a linear estimator which is nearly best and unbiased. Generalized Weibull linear models are considered by Prudente and Cordeiro (2010).

Genschel and Meeker (2010) compared maximum likelihood (ML) estimation against median rank regression with the order statistic the dependent variable. They found that MLE methods may be better to apply in the case of censored data. A comparison of various estimation procedures, including a method based on L

moments, was conducted by Teimouri et al. (2011), and they found that L moments estimation performs well and compares favourably with MLE estimation.

Overviews of estimation procedures based on order statistics are given in the books of Balakrishnan and Cohen (1991) and David and Nagaraja (2003). The Weibull distribution is reviewed in the books by Murthy et al. (2004) and Dodson (2006). Asymptotic properties of order statistics are given by DasGupta (2008). Other aspects which attract attention in current research are estimation procedures on complete and censored samples for the parameters of the two and three-parameter Weibull distribution, variations of the distribution and Weibull regression.

2. Weighted Least Squares Regression for the Weibull and Gumbel Distributions

Let $x_1, \dots, x_n$ denote a sample of size n with distribution function $F(x) = F(x; \alpha, \beta)$, α and β unknown parameters, and the support of the distribution is not a function of a parameter. The corresponding order statistics of $x_1, \dots, x_n$ are denoted by $x_{(1)} \leq \cdots \leq x_{(n)}$. Let $\widehat{F}_r = \widehat{F}(x_{(r)}; \alpha, \beta)$ be some consistent nonparametric estimate of $F(x_{(r)}) = F(x_{(r)}; \alpha, \beta)$.

Consider a model $\Lambda(\widehat{F}_r) = \Lambda(F(x_{(r)})) + u_r$, Λ a function of the form $\Lambda(x_{(r)}) = \Lambda(F(x_{(r)}))$ with a continuous derivatives in the points $\widehat{F}_r, r = 1, \dots, n,$ and $E(\Lambda(\widehat{F}_r) - \Lambda(F(x_{(r)}))) = 0$. The variance of the residual u_r can be approximated and is

$$\begin{aligned} Var(u_r) &= Var(\Lambda(\widehat{F}_r) - \Lambda(F(x_{(r)}))) \\ &= Var(\Lambda(F(x_{(r)}))) \\ &\approx \frac{\widehat{F}_r(1-\widehat{F}_r)}{n}\left(\frac{d\Lambda(F(x_{(r)}))}{dF(x_{(r)})}\right)^2_{F(x_{(r)})=\widehat{F}_r}, \end{aligned} \tag{2}$$

using the large sample approximation of the variance of function of a random variable and the large sample approximate variance of an order statistic. The weight w_r for a squared residual is the inverse of the approximate variance $w_r = 1/Var(u_r)$. The derivation of this result is given in Appendix A.

The estimation procedure can be carried out by minimizing the weighted sum of squares with respect to the parameters α and β, thus

$$\min_{\alpha,\beta} \sum_{r=1}^{n} w_r(\Lambda(\widehat{F}_r) - \Lambda(F(x_{(r)}; \alpha, \beta)))^2.$$

For a sample of size n from the Weibull distribution with corresponding order statistics $x_{(1)} \leq \cdots \leq x_{(n)}$, Eq. (1) can be rewritten as

$$\log(-\log(1 - F(x_{(r)}; \alpha, \beta))) = \beta \log(x_{(r)}) - \beta \log(\alpha). \tag{3}$$

For the sample this leads to the regression model

$$\log(-\log(1 - \widehat{F}_r)) = \beta \log(x_{(r)}) - \beta \log(\alpha) + u_r,$$

where r is the order number and u_r a random error with mean zero and a finite variance. Various estimates of the cdf, $\widehat{F}_r = \widehat{F}(x_{(r)}; \alpha, \beta)$, was suggested for example $\widehat{F}_r = E(F(x_{(r)}, \alpha, \beta)$ or $\widehat{F}_r = m_r = r/(n+1)$, the median estimate (Hossain and Zimmer, 2003) and Bernard's median rank estimator $\widehat{F}_r = m_r^b = (r-0.3)/(n+0.4)$ (Bernard and Bosi-Levenbach, 1953). The Bernard median rank estimator was found to perform good on complete data sets (Zhang et al., 2007) and it will be used in this work.

Equation (3) is used to estimate the parameters β and α where the logarithm of the order statistics are the independent variables. The order statistics $x_{(1)} \leq \cdots \leq x_{(n)}$do not have constant variance, nor do the log transformed order statistics, so that the regression model is heteroscedastic. Using the result (2), it follows that the approximate large sample variance is

$$var(u_r) = var(\log(-\log(1 - F(x_{(r)}; \alpha, \beta))) \approx \frac{\widehat{F}_r(1-\widehat{F}_r)}{n(\log(1-\widehat{F}_r))^2(1-\widehat{F}_r)^2}. \tag{4}$$

The relationships (3) and (4) are used to perform weighted regression with $\mathrm{w_r} = 1/var(log(-log(1-\widehat{F}_r)))$. The weighted least-squares regression equation is solved by letting $\mathbf{y}' = [\log(-\log(1-\widehat{F}_1)), \ldots, \log(-\log(1-\widehat{F}_n))]$,

$$X = \begin{pmatrix} 1 & \log(x_{(1)}) \\ \vdots & \vdots \\ 1 & \log(x_{(n)}) \end{pmatrix}, \quad W = diag(w_1, \ldots, w_n), \quad \hat{\boldsymbol{\theta}} = (X'WX)^{-1}X'W\mathbf{y},$$

$\hat{\alpha} = \exp(-\hat{\theta}_1/\hat{\beta})$, $\hat{\beta} = \hat{\theta}_2$. For $\widehat{F}_r = r/(n+1)$ it follows that

$$w_r = \frac{n(n-r+1)}{r}\left(\log\left(\frac{n-r+1}{n+1}\right)\right)^2, \quad r = 1, \ldots, n.$$

If it is know that $\beta = 1$, or that the data is from an exponential distribution which is a special case of the Weibull, the equations simplify and the cdf is $F(x; \alpha) = 1 - \exp(-x/\alpha)$. The regression equation is

$$-\log(1 - F(x_{(r)}; \alpha)) = x_{(r)}/\alpha,$$

and the weights are $w_r = 1/var(-\log(1-\widehat{F}_r)) \propto (n-r+1)/r$, $r = 1, \ldots, n$.

The Type I smallest extreme value distribution has the density

$$f(x) = \beta\exp((x-\mu)/\beta - \exp(-(x-\mu)/\beta)), \quad -\infty \leq x \leq \infty,$$

and is related to the Weibull distribution since it is the distribution of the log of a Weibull distributed random variable. The related Gumbel or largest extreme value distribution function is $F(x) = \exp(-\exp(-(x-\mu)/\beta)))$ and the pdf is

$$f(x) = (1/\beta)\exp(-(x-\mu)/\beta - \exp(-(x-\mu)/\beta)), \quad -\infty \leq x \leq \infty.$$

The relationship $-\log(-\log(F(x_{(r)}; \mu, \beta)) = x_{(r)}/\beta - \mu/\beta$ is used to perform rank regression. The approximate variance of $u_r = -\log(-\log(F(x_{(r)}; \mu, \beta)) - x_{(r)}/\beta +$

μ/β from which the weights are calculated is $var(u_r) \approx (1-\widehat{F}_r)/(n(\log(\widehat{F}_r))^2\widehat{F}_r)$, $r = 1, \ldots, n$.

3. Simulation Study

3.1. *Variance Approximation*

For the Weibull distribution, the residuals in the regression Eq. (2) are given by

$$u_r = \log(-\log(1-\widehat{F}_r)) - (\beta\log(x_{(r)}) - \beta\log(\alpha)), \quad r = 1, \ldots, n.$$

The variance approximation (4) for the variances of the residuals was compared to the sample variances of u_r, $r = 1, \ldots, n$, estimated from 5,000 simulated samples of size n for α and β given. Each variance, $var(u_r)$, $r = 1, \ldots, n$ was estimated from 5,000 observed residuals for every r. The simulated and approximate variances are plotted for $r = 1, \ldots, n$ in Figs. 1–3.

Approximate variances using Bernard's median rank estimator, namely $\widehat{F}_r = m_r^b = (r-0.3)/(n+0.4)$ instead of $\widehat{F}_r = m_r = r/(n+1)$, were also calculated. The approximation of the variance of the residuals is good even for a relatively small sample size of $n = 15$ (Fig. 1) and the variances calculated using Bernard's weights gave the best approximation.

In Fig. 2 with a larger sample size of $n = 100$, it can be seen that the approximation is very close.

For the Gumbel distribution with parameters μ, β, let $u_r = -\log(-\log(m_r)) - x/\beta + \mu/\beta$. The true and approximated variances calculated from 5,000 samples of size 15 are shown in Fig. 3.

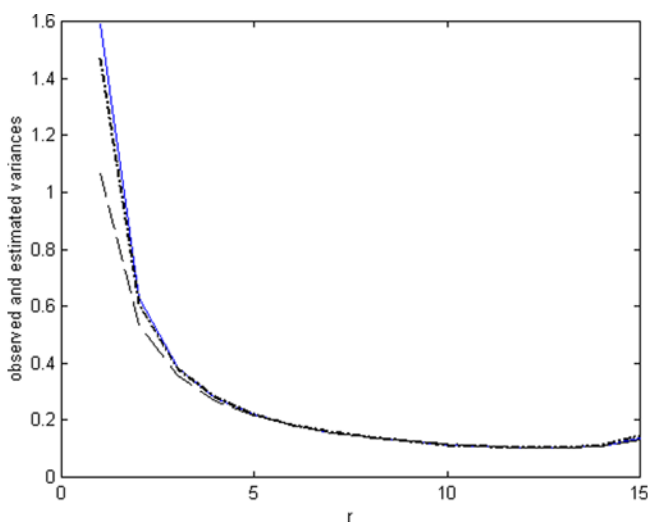

Figure 1. Variances of 5,000 residuals, u_r, $r = 1, \ldots, n$ in a sample of size $n = 15$, from a Weibull distribution with $\alpha = 1$, $\beta = 0.5$. The solid line denotes the observed variance, the dashed line the estimated variances using the Bernard method, and the dashdot line the usual estimated variances. (color figure available online.)

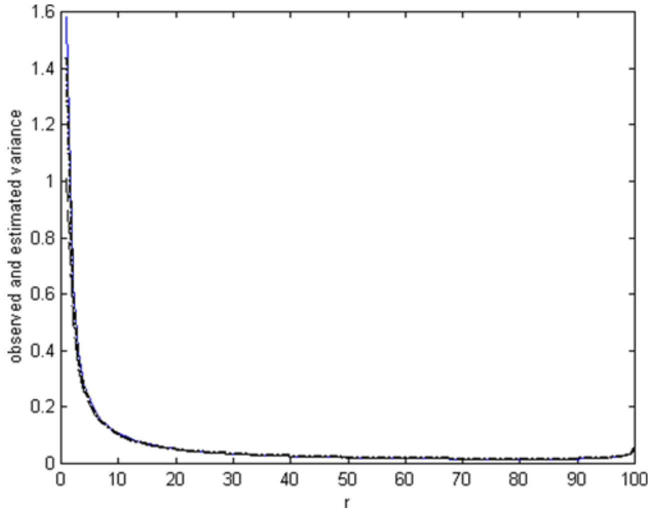

Figure 2. Variances of 5,000 residuals, $u_r, r = 1, \ldots, n$ in a sample of size $n = 100$, from a Weibull distribution with $\alpha = 1, \beta = 0.5$. The solid line denotes the observed variance, the dashed line the estimated variances using the Bernard method and the dashdot line the usual estimated variances. (color figure available online.)

The variance approximation is reasonable even for this small sample size, and again using Bernard's median rank method to calculate the weights, resulted in a better approximation of the variances of the regression residuals.

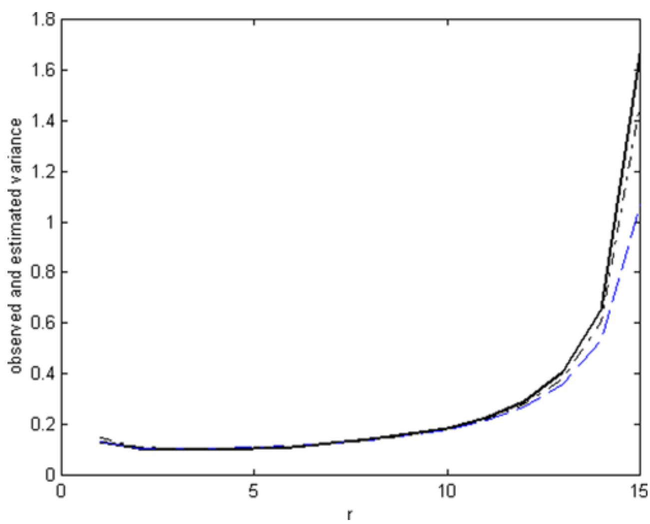

Figure 3. Variances of 5,000 residuals, $u_r, r = 1, \ldots, n$ in a sample of size $n = 15$, from a Gumbel distribution with $\mu = 0.5, \beta = 2.0$. The solid line denotes the observed variance, the dashdot line the estimated variances using the Bernard method and the dashed line the usual estimated variances. (color figure available online.)

Table 1
Means and MSE's of estimated parameters of the Weibull distribution with $\alpha = 1.0$, $\beta = 0.25$ (5,000 simulated samples)

$\beta = 0.25$ $\alpha = 1.0$	MSE LS (β) (Bernard)	MSE LS (α) (Bernard)	MSE weighted LS (β)	MSE weighted LS (α)	MSE weighted LS (β) (Bernard)	MSE weighted LS (α) (Bernard)
$n = 10$	0.2427	2.5508	0.2162	2.3572	0.2368	2.1115
	(0.0067)	(20.7387)	(0.0056)	(14.4691)	(0.0055)	(11.5233)
$n = 15$	0.2395	1.9313	0.2211	1.7848	0.2368	1.6551
	(0.0042)	(7.2900)	(0.0036)	(5.8577)	(0.0034)	(5.0147)
$n = 30$	0.2398	1.4564	0.2313	1.3617	0.2406	1.3092
	(0.0021)	(1.8359)	(0.0016)	(1.4464)	(0.0015)	(1.3239)
$n = 100$	0.2445	1.1525	0.2442	1.1053	0.2475	1.0911
	(0.0007)	(0.3004)	(0.0005)	(0.2537)	(0.0004)	(0.2455)
$n = 250$	0.2468	1.0602	0.2475	1.0357	0.2488	1.0301
	(0.0003)	(0.0914)	(0.0002)	(0.0819)	(0.0002)	(0.0808)

3.2. *Performance of the Weighted Least Squares Estimators*

It will be shown that the weighted least squares estimation, especially when using the Bernard weights, outperforms the usual least squares estimation. In Tables 1–3 the performance (MSEs) of the weighted least squares and the usual least squares method for estimating the parameters of the Weibull distribution are compared. For the samples sizes investigated, the MSE of the weighted methods outperforms the

Table 2
Means and MSE's of estimated parameters of the Weibull distribution with $\alpha = 1.0$, $\beta = 1.0$ (exponential distribution). Estimated using weighted least squares and the usual regression method based on 5,000 simulated samples

$\beta = 1.0$ $\alpha = 1.0$	MSE LS (β) (Bernard)	MSE LS (α) (Bernard)	MSE Weighted LS (β)	MSE weighted LS (α)	MSE weighted LS (β) (Bernard)	MSE weighted LS (α) (Bernard)
$n = 10$	0.9633	1.0779	0.8575	1.0655	0.9390	1.0348
	(0.1017)	(0.1372)	(0.0870)	(0.1279)	(0.0835)	(0.1194)
$n = 15$	0.9565	1.0577	0.8832	1.0419	0.9457	1.0214
	(0.0648)	(0.0869)	(0.0569)	(0.0795)	(0.0527)	(0.0761)
$n = 30$	0.9617	1.0400	0.9294	1.0248	0.9669	1.0144
	(0.0334)	(0.0433)	(0.0253)	(0.0392)	(0.0233)	(0.0383)
$n = 100$	0.9797	1.0162	0.9764	1.0073	0.9893	1.0048
	(0.0106)	(0.0127)	(0.0071)	(0.0119)	(0.0069)	(0.0118)
$n = 250$	0.9885	1.0080	0.9907	1.0027	0.9960	1.0014
	(0.0044)	(0.0048)	(0.0028)	(0.0046)	(0.0028)	(0.0046)

Table 3
Means and MSE's of estimated parameters of the Weibull distribution with $\alpha = 1.0$, $\beta = 1.5$. Estimated using weighted least squares and the usual regression method based on 5,000 simulated samples

$\beta = 1.5$ $\alpha = 1.0$	MSE LS (β) (Bernard)	MSE LS (α) (Bernard)	MSE weighted LS (β)	MSE weighted LS (α)	MSE weighted LS (β) (Bernard)	MSE weighted LS (α) (Bernard)
$n = 10$	1.4473	1.0397	1.2908	1.0315	1.4142	1.0115
	(0.2244)	(0.0547)	(0.1915)	(0.0514)	(0.1841)	(0.0493)
$n = 15$	1.4431	1.0302	1.3286	1.0204	1.4221	1.0069
	(0.1508)	(0.0358)	(0.1253)	(0.0334)	(0.1160)	(0.0326)
$n = 30$	1.4455	1.0178	1.3941	1.0080	1.4500	1.0011
	(0.0744)	(0.0180)	(0.0561)	(0.0165)	(0.0516)	(0.0163)
$n = 100$	1.4658	1.0094	1.4620	1.0034	1.4813	1.0012
	(0.0248)	(0.0054)	(0.0167)	(0.0051)	(0.0163)	(0.0051)
$n = 250$	1.4790	1.0050	1.4839	1.0012	1.4919	1.0003
	(0.0101)	(0.0021)	(0.0065)	(0.0020)	(0.0064)	(0.0020)

usual least squares method with respect to MSE, and the use of the Bernard weights decreased the bias.

In Table 4 and 5, the relative efficiencies with respect to the Cramér-Rao bounds (White, 1963) of $\hat{\beta}$ and $\hat{\alpha}$ are, respectively, $0.608\beta^2/n$ and $1.109(\alpha/\beta)^2/n$, are given for a few values of the parameters. Results for the usual linear regression and weighted regression are compared. The weights calculated using $\widehat{F}_r = r/(n+1)$ and the Bernard method $\widehat{F}_r = (r - 0.3)/(n + 0.4)$ were calculated. Relative efficiency is calculated as the ratio of the Cramér-Rao bound to the MSE of the estimated parameter.

Table 4
Relative efficiencies of estimation methods of β for various $\beta's$, $\alpha = 1$, sample size 25, based on 5,000 simulations

	$n = 25$			$n = 100$		
$\alpha = 1$	Efficiency (β) (Bernard)	Efficiency weighted LS (β)	Efficiency weighted LS (β) (Bernard)	Efficiency (β) (Bernard)	Efficiency weighted LS (β)	Efficiency weighted LS (β) (Bernard)
$\beta = 0.25$	0.6039	0.7700	0.8304	0.5545	0.8467	0.9768
$\beta = 0.5$	0.6975	0.8079	0.8573	0.5568	0.8296	0.8543
$\beta = 0.75$	0.6157	0.7754	0.8559	0.5531	0.8329	0.8580
$\beta = 1.0$	0.5948	0.7477	0.8120	0.5560	0.8250	0.8508
$\beta = 1.5$	0.5941	0.7554	0.8151	0.5442	0.8328	0.8567
$\beta = 2$	0.6016	0.7570	0.8141	0.5501	0.8261	0.8518

Table 5
Relative efficiencies of estimation methods of α for various $\beta's$, $\alpha = 1$, sample size 25, based on 5,000 simulations

	$n = 25$			$n = 100$		
$\alpha = 1$	Efficiency (α) (Bernard)	Efficiency weighted LS (α)	Efficiency weighted LS (α) (Bernard)	Efficiency (α) (Bernard)	Efficiency weighted LS (α)	Efficiency weighted LS (α) (Bernard)
$\beta = 0.25$	0.2818	0.3619	0.4017	0.5856	0.6977	0.8768
$\beta = 0.5$	0.6975	0.8079	0.8573	0.7947	0.8755	0.8843
$\beta = 0.75$	0.7906	0.8822	0.9167	0.8825	0.9487	0.9572
$\beta = 1.0$	0.8539	0.9388	0.9624	0.9140	0.9709	0.9768
$\beta = 1.5$	0.8922	0.9574	0.9748	0.8836	0.9394	0.9422
$\beta = 2$	0.9062	0.9745	0.9835	0.9403	0.9924	0.9941

Engeman and Keefe (1982) compared the GLS estimator based on the methods of Blom (1962), using the full covariance matrix and the order statistics from the sample as the dependent variable, with other methods of estimation. They found the GLS estimator best with respect to relative efficiency when estimating the shape parameter β, and close to the maximum likelihood estimator when estimating the scale parameter α. The sample size is $n = 25$. In Table 6, their results are presented together with the relative efficiencies of the weighted regression method, with weights calculated using the Bernard estimate. It can be seen that the relative very simple weighted procedure compares very well to the more complicated models, and better than MLE with respect to the shape parameter.

Results for the Gumbel distribution are given in Table 7 and 8.

As is the case with the Weibull distribution, weighted regression outperforms the simple least squares regression, and Bernard's median ranks estimate is preferred when calculating the weights.

4. Discussion

Our simulation results show that the weighted least squares method using the Bernard weights, which are easy to calculate, outperforms simple least squares regression, especially for small sample sizes.

Table 6
Relative efficiencies weighted least squares with GLS, ML

	Relative efficiencies β			Relative efficiencies α		
$\alpha = 1$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$	$\beta = 0.5$	$\beta = 1$	$\beta = 2$
GLS	0.9560	0.8803	0.7402	0.9083	0.9542	1.2300
MLE	0.6341	0.5942	0.6702	0.9662	0.9709	1.0071
WLS (Bernard)	0.8573	0.8120	0.8141	0.8573	0.9624	0.9835

Table 7
MSE (and mean) of estimates of parameters of the Gumbel distribution with $\mu = 0.5$, $\beta = 2.0$ (5,000 simulated samples)

$\beta = 2.0$ $\mu = 0.5$	MSE LS (β) (Bernard)	MSE LS (μ) (Bernard)	MSE weighted LS (β)	MSE weighted LS (μ)	MSE weighted LS (β) (Bernard)	MSE weighted LS (μ) (Bernard)
$n = 10$	2.2738	0.4477	2.4997	0.4671	2.2807	0.5264
	(0.6158)	(0.4628)	(0.7324)	(4514)	(0.4747)	(0.4574)
$n = 15$	2.2401	0.4399	2.3791	0.4700	2.2208	0.5105
	(0.4249)	(0.3016)	(0.4154)	(0.2919)	(0.2838)	(0.2946)
$n = 30$	2.1541	0.4632	2.2037	0.4925	2.1188	0.5133
	(0.1980)	(0.1569)	(0.1494)	(0.1514)	(0.1149)	(0.5127)
$n = 100$	2.0678	0.4793	2.0626	0.4977	2.0357	0.5043
	(0.0531)	(0.0466)	(0.0332)	(0.0448)	(0.0302)	(0.0449)
$n = 250$	2.0363	0.4870	2.0261	0.4983	2.0153	0.5010
	(0.0206)	(0.0188)	(0.0123)	(0.0180)	(0.0118)	(0.0180)

In the cases considered in the simulation study, the weighted procedure compares rather favorably with the GLS and maximum likelihood methods, particularly in view of the relative simplicity of the weighted least squares method. With GLS slightly higher efficiencies can be achieved, especially when interest is in the estimation of the location parameter.

In conclusion, the weighted least squares procedure for estimating the parameters of the two-parameter Weibull or Gumbel distributions is easy to apply,

Table 8
MSE (and mean) of estimates of parameters of the Gumbel distribution with $\mu = 0.5$, $\beta = 4.0$ (5,000 simulated samples)

$\beta = 4.0$ $\mu = 0.5$	MSE LS (β) (Bernard)	MSE LS (μ) (Bernard)	MSE weighted LS (β)	MSE weighted LS (μ)	MSE weighted LS (β) (Bernard)	MSE weighted LS (μ) (Bernard)
$n = 10$	4.5499	0.4633	5.0103	0.4983	4.5733	0.6170
	(2.4401)	(1.8837)	(2.9664)	(1.8366)	(1.9283)	(1.8776)
$n = 15$	4.4738	0.3955	4.7635	0.4488	4.4493	0.5295
	(1.6849)	(1.2971)	(1.6926)	(1.2501)	(1.1627)	(1.2629)
$n = 30$	4.2933	0.4344	4.3991	0.4899	4.2304	0.5315
	(0.7700)	(0.6335)	(0.5961)	(0.6056)	(0.4615)	(0.6108)
$n = 100$	4.1404	0.4565	4.1280	0.4932	4.0743	0.5064
	(0.2136)	(0.1947)	(0.1353)	(0.1860)	(0.1229)	(0.1866)
$n = 250$	4.0724	0.4780	4.0533	0.4996	4.0317	0.5050
	(0.0820)	(0.0782)	(0.0496)	(0.0758)	(0.0476)	(0.0760)

outperforms the usual simple least regression method, and performs nearly as well as its best, but computationally complex, competitors like GLS and MLE.

Appendix A

Let $x_1, \ldots, x_n$ denote a sample of size n with distribution function $F(x) = F(x; \boldsymbol{\theta})$, $\boldsymbol{\theta}$ a vector of unknown parameters, and the support of the distribution is not a function of a parameter. The corresponding order statistics of $x_1, \ldots, x_n$ are denoted by $x_{(1)} \leq \cdots \leq x_{(n)}$. Let $\widehat{F}_r = \widehat{F}(x_{(r)}; \boldsymbol{\theta})$ be some consistent nonparametric estimate of $F(x_{(r)}) = F(x_{(r)}; \boldsymbol{\theta})$.

A regression model of the form $E(\Lambda(x_{(r)}) = \Lambda(x_{(r)}) + u_r$, will be considered. The function Λ need not be a linear function of the order statistics and will almost always be of the form $\Lambda(F(x_{(r)}))$, a function of the distribution function of the order statistic. Order statistics are not independently distributed, but by using the least squares estimation as proposed here, independence is assumed.

The weighted least squares expression used to estimate the parameters is $\min_{\boldsymbol{\theta}} \sum_{r=1}^{n} w_r [E(\Lambda(x_{(r)})) - \Lambda(x_{(r)}; \boldsymbol{\theta})]^2$. The weight for the rth squared residual

$$u_r^2 = [E(\Lambda(x_{(r)})) - \Lambda(x_{(r)})]^2 \quad \text{is } w_r = 1/var(\Lambda(x_{(r)})), \quad r = 1, \ldots, n,$$

where $var(\Lambda(x_{(r)})) = var(u_r)$. Let X_r be such that $F^{-1}(X_r) = r/n = m_r$ (Kendall et al., 1987 p. 452), then asymptotically $var(x_{(r)}) = \frac{m_r(1-m_r)}{n(f(X_r))^2}$. The approximate variance of $\Lambda(x_{(r)})$ is

$$\begin{aligned} var(\Lambda(x_{(r)})) &\approx var(x_{(r)}) \left(\frac{d\Lambda(x_{(r)})}{dx_{(r)}} \right)^2_{x_{(r)}=X_r} \\ &= \frac{m_r(1-m_r)}{n(f(X_r))^2} \left(\frac{d\Lambda(x_{(r)})}{dx_{(r)}} \right)^2_{x_{(r)}=X_r}. \end{aligned}$$

Furthermore, if $\Lambda(x_{(r)})$ is of the form as $\Lambda(x_{(r)}) = \Lambda(F(x_{(r)}))$ which is the usual form in probability plot estimation, it can be seen that

$$\begin{aligned} var(\Lambda(F(x_{(r)}))) &\approx \frac{m_r(1-m_r)}{n(f(X_r))^2} \left(\frac{d\Lambda(F(x_{(r)}))}{dF(x_{(r)})} \right)^2_{x_{(r)}=X_r} \left(\frac{dF(x_{(r)})}{dx_{(r)}} \right)^2_{x_{(r)}=X_r} \\ &= \frac{m_r(1-m_r)}{n} \left(\frac{d\Lambda(F(x_{(r)}))}{dF(x_{(r)})} \right)^2_{F(x_{(r)})=E(F(x_{(r)}))} \end{aligned} \tag{5}$$

so that the term $(f(X_r))^2$ cancels in the approximation. In such a case the weights are a function of m_r and not of the parameters of the distribution under consideration, and it is possible to apply this method if a function Λ can be constructed which gives a relationship between the parameters and the cdf $F(x_{(r)})$. By using similar methods the approximate covariance between $\Lambda(F(x_{(r)}))$ and $\Lambda(F(x_{(s)}))$ can be derived and is

$$cov(\Lambda(F(x_{(r)})), \Lambda(F(x_{(s)})) \approx \frac{m_r(1-m_s)}{n}\left(\frac{d\Lambda(F(x_{(r)}))}{dF(x_{(r)})}\frac{d\Lambda(F(x_{(s)}))}{dF(x_{(s)})}\right)_{F(x_{(r)})=m_r, F(x_{(s)})=m_s}$$

Instead of using m_r other general estimates $\widehat{F}_r = \widehat{F}(x_{(r)}; \boldsymbol{\theta})$ which are a consistent nonparametric estimates of $F(x_{(r)}; \boldsymbol{\theta})$, such as Bernard's median rank estimator (Bernard and Bosi-Levenbach, 1953), $\widehat{F}_r = m_r^b = (r-0.3)/(n+0.4)$, can be used to calculate the approximate variance.

The statistics $F(x_{(1)}), \ldots, F(x_{(n)})$ are beta distributed with $F(x_{(r)}) \sim Beta(r, n-r+1)$. $E(F(x_{(r)})) = m_r = r/(n+1)$, $var(F(x_{(r)})) = \frac{r(n-r+1)}{(n+2)(n+1)^2} = \frac{m_r(1-m_r)}{n+2}$. This approach can also be used to derive a least squares model with $m_r = r/(n+1)$.

A general asymptotic result can be derived by making use of the multivariate delta method and the asymptotic normality of order statistics, which also yields the result that the errors are asymptotically normally distributed. If the errors are normally distributed, the least squares estimators are equal to those found when using a regression model likelihood with the error terms $u_r = (\Lambda(F(m_r))) - \Lambda(F(x_{(r)}))$ normally distributed.

Asymptotically $\sqrt{n}(\mathbf{X} - E(\mathbf{X})) \sim N(\mathbf{0}, \Sigma)$ under certain conditions (DasGupta, 2008), where $\mathbf{X} = (x_{(1)}, \ldots, x_{(n)})'$, $E(\mathbf{X}) = (X_1, \ldots, X_n)'$ and $\sigma_{rs} = \frac{m_r(1-m_s)}{f(X_r)f(X_s)}$. Assume the matrix of partial derivatives $\Phi = [\frac{\partial^2\Lambda(\mathbf{X})}{\partial x_{(r)}\partial x_{(s)}}{}_{|x_{(r)}=X_r, x_{(s)}=X_s}]_{r,s}$, $r, s = 1, \ldots, n$ exist and are continuous in the points $x_{(r)} = X_r$, $x_{(s)} = X_s$.

It follows by making use of the asymptotic multivariate distribution of the vector of order statistics and the multivariate delta method that $\sqrt{n}[\Lambda(\mathbf{X}) - E(\Lambda(\mathbf{X}))] \sim N(\mathbf{0}, \Phi\Sigma\Phi')$. The covariance matrix can be used in a generalized least squares (GLS) model. The log-likelihood L is of the form $L \propto -\frac{n}{2}(\Lambda(\mathbf{X}) - E(\Lambda(\mathbf{X})))'(\Phi\Sigma\Phi')^{-1}(\Lambda(\mathbf{X}) - E(\Lambda(\mathbf{X})))$.

If Λ is a vector valued function of $\mathbf{X}$ which is of the form, $\Lambda(\mathbf{F}) = (\Lambda(F(x_{(1)}), \ldots, \Lambda(F(x_{(n)}))'$ and the order statistics are assumed to be independent, the resulting approximate covariance matrix using the delta method would be diagonal, with diagonal elements the same variances derived above, independent of the parameters of the distribution.

An approximation for the bias can be found by using the second-order term of the Taylor expansion of $\Lambda(x_{(r)})$. Let $h_r = x_{(r)} - X_r$, $E(h_r) = 0$, the Taylor expansion of $\Lambda(x_{(r)})$ up to the second-order term is

$$\Lambda(x_{(r)}) \approx \Lambda(X_r) + h_r\Lambda'(X_r) + \frac{1}{2}h^2\Lambda''(X_r) + o_p(1),$$

and it follows that the estimators are asymptotically efficient. The bias term is

$$E[\Lambda(X_r) - \Lambda(x_{(r)})] \approx -\frac{1}{2}\Lambda''(X_r))(var(\Lambda(x_{(r)})).$$

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