Statistics
A Journal of Theoretical and Applied Statistics

ISSN: 0233-1888 (Print) 1029-4910 (Online) Journal homepage: www.tandfonline.com/journals/gsta20

Comparison of estimation methods for the Weibull distribution

Mahdi Teimouri, Seyed M. Hoseini & Saralees Nadarajah

To cite this article: Mahdi Teimouri, Seyed M. Hoseini & Saralees Nadarajah (2013) Comparison of estimation methods for the Weibull distribution, Statistics, 47:1, 93-109, DOI: 10.1080/02331888.2011.559657

To link to this article: https://doi.org/10.1080/02331888.2011.559657

Published online: 31 Mar 2011.

Statistics, 2013
Vol. 47, No. 1, 93–109, http://dx.doi.org/10.1080/02331888.2011.559657

Comparison of estimation methods for the Weibull distribution

Mahdi Teimouri[a], Seyed M. Hoseini[b] and Saralees Nadarajah[c]*

[a]*Department of Statistics, Gonbad Kavous University, Gonbad, Iran;* [b]*Department of Engineering, Islamic Azad University, Aliabad Katoul, Branch, Aliabad Katoul, Iran;* [c]*School of Mathematics, University of Manchester, Manchester M13 9PL, UK*

(*Received 07 August 2010; final version received 30 January 2011*)

Weibull distributions have received wide ranging applications in many areas including reliability, hydrology and communication systems. Many estimation methods have been proposed for Weibull distributions. But there has not been a comprehensive comparison of these estimation methods. Most studies have focused on comparing the maximum likelihood estimation (MLE) with one of the other approaches. In this paper, we first propose an L-moment estimator for the Weibull distribution. Then, a comprehensive comparison is made of the following methods: the method of maximum likelihood estimation (MLE), the method of logarithmic moments, the percentile method, the method of moments and the method of L-moments.

Keywords: L-moment; logarithmic moment; maximum likelihood estimation; moment; Weibull distribution; percentile method; *TL*-moment

1. Introduction

The probabilistic Weibull model was first introduced by Dr. Walodi Weibull to represent the distribution of the breaking strength of materials and later to describe the behavior of systems or events that have some degree of variability. It is a flexible distribution that can encompass characteristics of several other distributions. This property has given rise to widespread applications. For example, the most popular distribution for failure data analysis is the Weibull distribution. The process of modeling failure data is commonly known as Weibull analysis, even when the underlying failure distribution is not Weibull.

The Weibull distribution is widely applied in radar systems to model the dispersion of the received signal level caused by clutters. In wireless communications, the Weibull fading model exhibits good fits to experimental fading channel measurements, so it is used for fading channel modeling. Other application areas include tensile characteristics of ring and rotor yarns [1], examination of tyre rubber cure [2], estimation of wind power potential [3], temporal patterns of ventricular tachyarrhythmia recurrences [4], food products drying technology [5], distribution for asset returns [6], significant wave height simulation and prediction [7], modeling air drying of coroba slices [8], model for growth/decline in product sales [9], interoccurrence times of earthquakes [10], combustion behavior of pine sawdust [11], optimum adhesive thickness in structural adhesives joints [12] and diametric distribution model for thinning eucaliptus

*Corresponding author. Email: mbbsssn2@manchester.ac.uk

stands [13]. For further details on applications, we refer the readers to Nelson [14], Meeker and Escobar [15], Bartolucci *et al.* [16], Heo *et al.* [17], Salas *et al.* (2001), Murthy *et al.* [18] and Dodson [19].

Because of its importance, many estimation methods have been proposed for the Weibull distribution for both complete and censored data. We refer the readers to Wu [20], Dodson [19], Zhang *et al.* [21], Balakrishnan and Kateri [22] and Ahmed *et al.* [23] for the main estimation methods. The most common estimation method is the MLE, see Nelson [14], Johnson *et al.* [24], Murthy *et al.* [18], and Dodson [19]. MLE has attractive efficiency properties and is asymptotically unbiased.

The use of the proposed estimation methods depends on the area of application. The estimation methods may have a different appeal for different users. For example, a user may prefer to use the uniformly minimum variance estimator even if it does not have a closed form expression.

There has been some studies comparing the different estimation methods [21,25–28]. But to the best of our knowledge all of these studies have compared at most two of the methods, one of them being the MLE. For example, Zanakis and Kyparisis [29] provided an interesting review and comparison of the known methods for the MLE. Since then, three or more methods have been proposed. The bias of the MLE for the two-parameter Weibull distribution has been studied by Ross [30], Watkins [31] and Montanari *et al.* [32]. In [26], a comparison between the MLE and a bootstrap robust estimator was made. In [22], the MLE for censored data was derived (see also [33,34]). In [21], a comparison of methods based on the Weibull probability plot (WPP) was made.

Comprehensive comparisons of estimation methods for other less prominent distributions have been performed in the literature: see Kundu and Raqab [35] for generalized Rayleigh distributions and Alkasasbeh and Raqab [36] for generalized logistic distributions. However, such a study for the more widely used Weibull distribution has not been performed.

In this paper, we provide a comprehensive comparison of the main estimation methods for the Weibull distribution: the Method of MLE, the Method of Logarithm Moment (MLM), the Percentile Method (PM), the newly proposed L-moment method (LM), and the method of moments (MM). We use the following criteria for comparison: the bias and the mean squared error.

The structure of the paper is as follows. In Section 2, we propose the new L-moment estimators for the Weibull distribution. The known estimation methods are reviewed in Section 3. A numerical comparison of the new and the known methods is given in Section 4. Finally, some conclusions are noted in Section 5.

2. L-moment estimator for the Weibull distribution

The probability density function of the three-parameter Weibull distribution is given by [14,19,24]

$$f_X(x) = \frac{\alpha}{\beta}\left(\frac{x-\theta}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x-\theta}{\beta}\right)^{\alpha}\right) \tag{1}$$

for $x > \theta$, $\alpha > 0$, $\beta > 0$ and $\theta \in \mathbb{R}$. The parameters α, β and θ are known as the shape, scale and location parameters, respectively. By a shift transformation, Equation (1) can be reduced to the two-parameter version:

$$f_X(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} \exp\left(-\left(\frac{x}{\beta}\right)^{\alpha}\right) \tag{2}$$

for $x > 0$, $\alpha > 0$ and $\beta > 0$.

The new estimation methods proposed here are based on L-moments and *TL*-moments. These are defined in terms of order statistics and have their origin in Hosking [37] and Elamir and Seheult [38], respectively. Until now, estimation methods based on L-moments and *TL*-moments have been applied only for the generalized Rayleigh and the exponential distributions, see Kundu and Raqab [35] and Abdul-Moniem [39]. These estimation methods are efficient and yield robust estimators against outliers. *TL*-moment estimators are more robust than L-moment estimators. Unlike L-moment estimators, *TL*-moment estimators can be applied to distributions with infinite moments (for example, Levy-stable and Pearson-type VII distributions).

TL-moments are much more recent than the L-moments. So, they have not yet received widespread applications. Of the known applications, we mention: robust averaging of event-related potentials [40], characterization of distributions [41,42], dynamic quantile models [43], estimation of quantile mixtures [42], hedge funds modeling [44], assessing symmetry [45]. For other practical uses of *TL*-moments, we refer the readers to Hosking [46].

Hosking's [37] idea works on the basis of computing the rth moment of the ith order statistic. Equating the sample *TL*-moment (L-moment) to the population counterpart gives the *TL*-moment estimate (L-moment estimate). To begin, the *TL*-moment μ_r^t is given by [38,39]

$$\mu_r^t = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} E(X_{r+t-k:r+2t}) \tag{3}$$

for $t, r = 1, 2, 3, \ldots$, where C_i^n denotes the binomial coefficient $n!/(i!(n-i)!)$ and $X_{i:n}$ denotes the ith order statistic in a sample of size n. Note that L moments correspond to taking $t = 0$ in Equation (3). The sample *TL*-moments are defined as:

$$m_r^t = \frac{1}{r}\sum_{i=t+1}^{n-t} \frac{\sum_{k=0}^{r-1}(-1)^k C_k^{r-1} C_{t+k}^{n-i} C_{r+t-k-1}^{i-1}}{C_{r+2t}^n} X_{i:n}.$$

The expectation of the ith order statistic can be derived using Lemma 2.1. Its proof is based on an easy extension of the results in [24]), so omitted.

LEMMA 2.1 *The pth moment of the ith order statistic from Equation* (1) *is given by*

$$E(X_{i:n}^p) = iC_i^n \sum_{m=0}^{p}\sum_{j=0}^{i-1}(-1)^j \frac{C_j^{i-1} C_m^p \theta^m \beta^{p-m}}{(n-i+j+1)^{(p-m)/\alpha+1}} \Gamma\left(\frac{p-m}{\alpha}+1\right) \tag{4}$$

for $\alpha > 0$, $\beta > 0$, $\theta \in \mathbb{R}$ and $p = 1, 2, \ldots$, where $\Gamma(\cdot)$ denotes the gamma function. If $\theta = 0$, then Equation (4) *should be interpreted as*

$$E(X_{i:n}^p) = \beta^p \Gamma\left(\frac{p}{\alpha}+1\right) iC_i^n \sum_{j=0}^{i-1}(-1)^j \frac{C_j^{i-1}}{(n-i+j+1)^{p/\alpha+1}}$$

for $\alpha > 0$, $\beta > 0$ and $p = 1, 2, \ldots$.

THEOREM 2.1 *The rth TL-moment and L-moment of* (1) *are given by*

$$\mu_r^t = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1}(r+t-k)C_{r+t-k}^{r+2t} \times \sum_{m=0}^{1}\sum_{j=0}^{r+t-k-1} C_j^{r+t-k-1}\frac{(-1)^j\theta^m\beta^{1-m}}{(t+k+j+1)^{(1-m)/\alpha+1}}\Gamma\left(\frac{1-m}{\alpha}+1\right) \tag{5}$$

and

$$\mu_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k C_k^{r-1}(r-k)C_{r-k}^{r} \times \sum_{m=0}^{1}\sum_{j=0}^{r-k-1} C_j^{r-k-1}\frac{(-1)^j\theta^m\beta^{1-m}}{(k+j+1)^{(1-m)/\alpha+1}}\Gamma\left(\frac{1-m}{\alpha}+1\right), \tag{6}$$

respectively, for $\alpha > 0$, $\beta > 0$, $\theta \in \mathbb{R}$, $r = 1, 2, \ldots$ *and* $t = 0, 1, \ldots$. *If* $\theta = 0$, *then* (5) *and* (6) *should be interpreted as*

$$\mu_r^t = \frac{\beta}{r}\Gamma\left(\frac{1}{\alpha}+1\right)\sum_{k=0}^{r-1}(-1)^k C_k^{r-1}(r+t-k)C_{r+t-k}^{r+2t}\sum_{j=0}^{r+t-k-1} C_j^{r+t-k-1}\frac{(-1)^j}{(t+k+j+1)^{1/\alpha+1}}$$

and

$$\mu_r = \frac{\beta}{r}\Gamma\left(\frac{1}{\alpha}+1\right)\sum_{k=0}^{r-1}(-1)^k C_k^{r-1}(r-k)C_{r-k}^{r}\sum_{j=0}^{r-k-1} C_j^{r-k-1}\frac{(-1)^j}{(k+j+1)^{1/\alpha+1}},$$

respectively, for $\alpha > 0$, $\beta > 0$, $r = 1, 2, \ldots$ *and* $t = 0, 1, \ldots$.

Proof The *TL*-moment is obtained by substituting Equation (4) into Equation (3) for $p = 1$, $i = r + t - k$ and $n = r + 2t$. The *L*-moment is obtained by a similar substitution. ■

Now consider estimating the parameters θ and β in terms of the *TL* moments (assuming α is known). From Theorem 2.1, we have

$$\mu_1^1 = \frac{3\beta\Gamma(1/\alpha+1)}{2^{1/\alpha}} - \frac{2\beta\Gamma(1/\alpha+1)}{3^{1/\alpha}} + \theta \tag{7}$$

and

$$\mu_2^1 = 6\beta\Gamma\left(\frac{1}{\alpha+1}\right)\sum_{j=0}^{2} C_j^2\frac{(-1)^j}{(j+2)^{1/\alpha+1}} - 6\beta\Gamma\left(\frac{1}{\alpha+1}\right)\sum_{j=0}^{1}\frac{(-1)^j}{(j+3)^{1/\alpha+1}}. \tag{8}$$

The first two sample *TL*-moments are

$$m_1^1 = \frac{6}{n(n-1)(n-2)}\sum_{i=2}^{n-1}(i-1)(n-i)X_{i:n} \tag{9}$$

and

$$m_2^1 = \frac{12}{n(n-1)(n-2)(n-3)}\left[\sum_{i=3}^{n-1} C_1^{n-i}C_2^{i-1}X_{i:n} - \sum_{i=2}^{n-2} C_2^{n-i}C_1^{i-1}X_{i:n}\right]. \tag{10}$$

Now equating Equations (7) and (9), we obtain

$$\hat{\theta}_{\mathrm{TLM}} = m_1^1 - \frac{3\beta\Gamma(1/\alpha+1)}{2^{1/\alpha}} + \frac{2\beta\Gamma(1/\alpha+1)}{3^{1/\alpha}}.$$

Similarly, equating (8) and (10), we obtain

$$\hat{\beta}_{\mathrm{TLM}} = \frac{m_2^1}{6\Gamma(1/\alpha+1)} \left[\sum_{j=0}^{2} \frac{C_j^2(-1)^j}{(j+2)^{1/\alpha+1}} - \sum_{j=0}^{1} \frac{(-1)^j}{(j+3)^{1/\alpha+1}} \right]^{-1}.$$

Now consider estimating the parameters α and β in terms of the L moments (assuming $\theta = 0$). From Theorem 2.1, we have

$$\mu_1^0 = \beta\Gamma\left(\frac{1}{\alpha+1}\right)$$

and

$$\mu_2^0 = \beta\Gamma\left(\frac{1}{\alpha+1}\right)\left[1 - \frac{1}{2^{1/\alpha}}\right].$$

The first two sample L-moments are

$$m_1^0 = \frac{1}{n}\sum_{i=1}^{n} X_{i:n} = \bar{X}$$

and

$$m_2^0 = \frac{2}{n(n-1)}\sum_{i=1}^{n} (i-1)X_{i:n} - \bar{X}.$$

Now, equating μ_1^0 and μ_2^0 with m_1^0 and m_2^0, respectively, we obtain the estimators

$$\hat{\alpha}_{\mathrm{LM}} = -\frac{\ln(2)}{\ln(1 - m_2^0/m_1^0)}$$

and

$$\hat{\beta}_{\mathrm{LM}} = \frac{m_1^0}{\Gamma(1/\alpha+1)}.$$

3. Known estimators for the Weibull distribution

Here, we review some of the main estimation methods known for the two-parameter and three-parameter Weibull distributions given by Equations (1) and (2), respectively. Throughout, we suppose $x_1, x_2, \ldots, x_n$ is a random sample from Equation (1) or Equation (2).

3.1. *Maximum likelihood estimation (MLE)*

The MLE for the three-parameter Weibull distribution suffers from regularity problems since the distribution has a parameter-dependent lower bound. Also the MLE does not give closed form expressions and requires iterative computations. It is asymptotically normal and efficient for large sample sizes. However, for small sample sizes, the MLE is biased.

Many authors have tried to refine the MLE for the two-parameter and three-parameter Weibull distributions. Cohen and Whitten [47] considered a modified MLE involving complicated numerical computations. Gourdin *et al.* [48] examined numerical problems associated with the MLE. Dodson [19] derived the MLE for the shape parameter graphically.

Here, we assume that the shift parameter $\theta = 0$. Then, the solutions of the MLE are [24]:

$$\hat{\beta}_{\mathrm{MLE}} = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^{\alpha}\right)^{\alpha}$$

and

$$\frac{1}{\hat{\alpha}_{\mathrm{MLE}}} = \frac{\sum_{i=1}^{n} x_i^{\alpha}\ln(x_i)}{\sum_{i=1}^{n} x_i^{\alpha}} - \frac{1}{n}\sum_{i=1}^{n}\ln(x_i).$$

It can be seen that $\hat{\beta}_{\mathrm{MLE}}$ depends on α and also that $\hat{\alpha}_{\mathrm{MLE}}$ must be computed numerically. Several approaches based on graphical- and iterative-based methods have been proposed to overcome these problems.

3.2. *Method of logarithmic moment (MLM)*

The log-moment estimates of the parameters of Equation (2) are given by [14,19,24]

$$\hat{\alpha}_{\mathrm{MLM}} = \sqrt{\frac{\pi^2}{6S^2}} \tag{11}$$

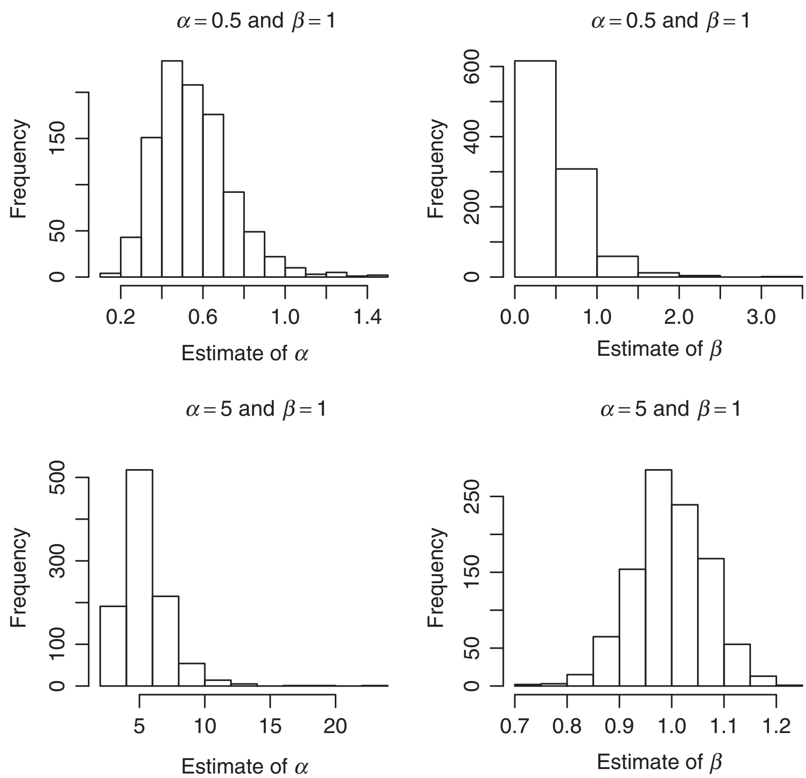

Figure 1. Histograms of the LM estimators for α and β for $(\alpha, \beta) = (0.5, 1), (5, 1)$ and $n = 10$.

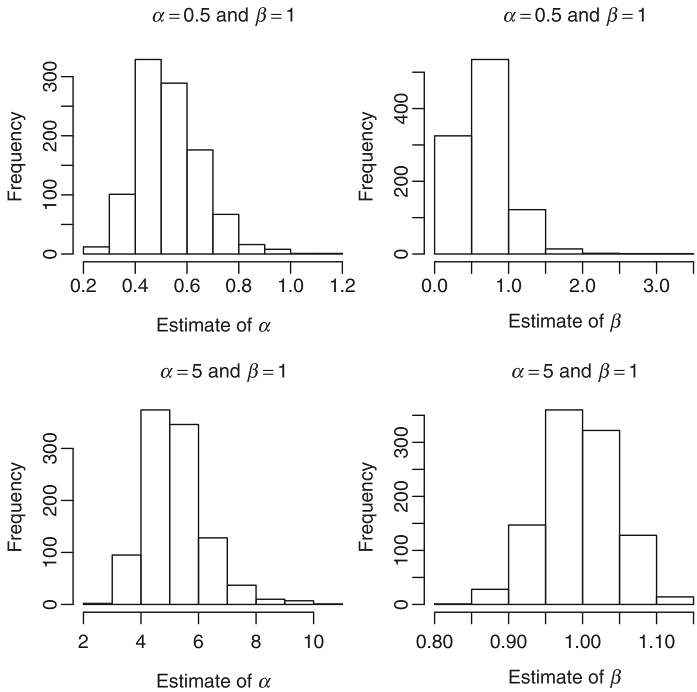

Figure 2. Histograms of the LM estimators for α and β for $(\alpha, \beta) = (0.5, 1), (5, 1)$ and $n = 20$.

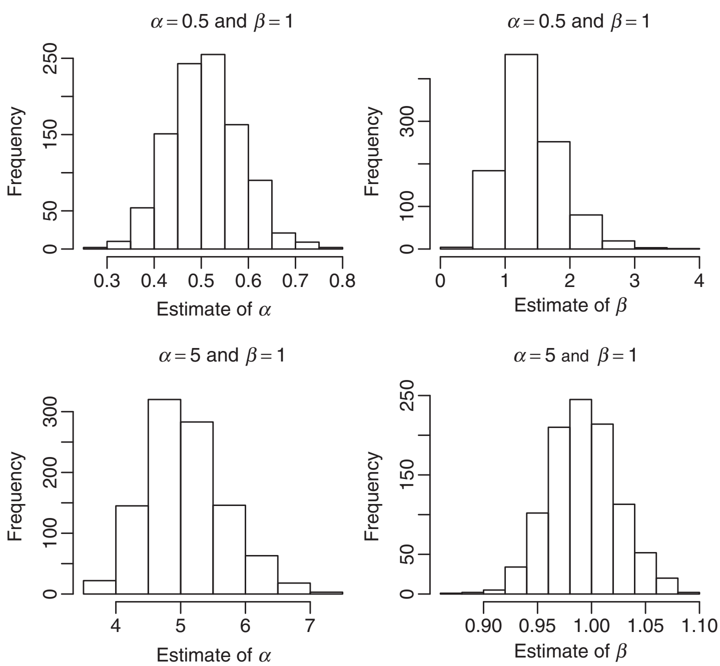

Figure 3. Histograms of the LM estimators for α and β for $(\alpha, \beta) = (0.5, 1), (5, 1)$ and $n = 50$.

and

$$\hat{\beta}_{\text{MLM}} = \exp\left(\frac{M_1 - \Psi(1)}{\hat{\alpha}_{\text{LM}}}\right), \tag{12}$$

where S^2 and M_1 are the sample variance and the mean of log-transformed data, respectively. Also $\Psi(1) = -0.5772156$. It can be shown that (11) and (12) are both asymptotically unbiased and consistent [24].

3.3. *Percentile method*

The quantile function corresponding to Equation (2) is [14,19,24]

$$x_p = \beta[-\ln(1-p)]^{1/\alpha}. \tag{13}$$

Setting $p = 1 - \exp(-1) \cong 0.632$ (63.2th percentile) into Equation (13), we obtain an estimate for β. Substituting this estimate for β, we see that the percentile-based estimators for α and β become

$$\hat{\beta}_{\text{PM}} = x_{1-\exp(-1)}$$

and

$$\hat{\alpha}_{\text{PM}} = \left(\frac{\ln[-\ln(1-p)]}{\ln(x_p) - \ln(x_{0.632})}\right),$$

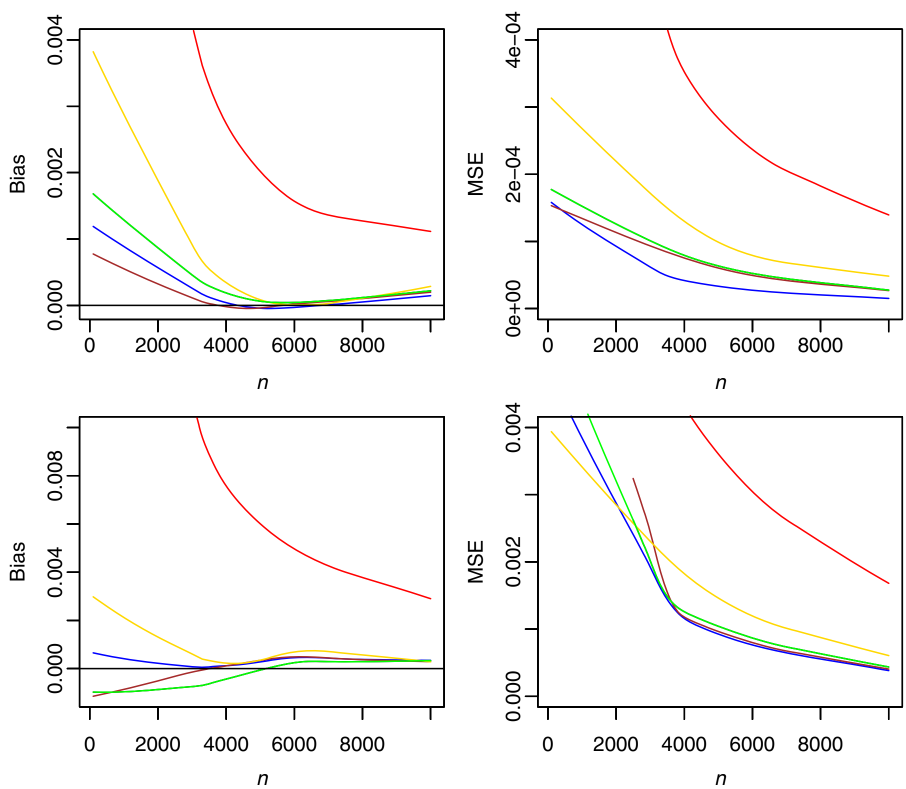

Figure 4. Biases of $\hat{\alpha}$ (top left), mean squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 0.5$ and $\beta = 1$.

respectively, where $0 < x_p < x_{0.632}$. There are some suggestions for the optimum value of p. Wang and Keats [49] showed by simulation that 0.15 would be the correct choice for p. Seki and Yokoyama [26] suggested $p = 0.31$. See also Hassanein [50]. Statistical tools show that percentile-based estimators are, in general, asymptotically normal and unbiased [14].

3.4. *Method of moments (MM)*

Making inferences about statistical distributions using their moments is recognized as a traditional technique. The MMs is based on equating the population moments to their sample counterparts and solving the resulting equations to obtain estimates of the unknown parameters. Unfortunately, the MMs for the two-parameter and three-parameter Weibull distributions suffers from numerical computations [51]. Besides, the moment-based estimators are not efficient.

The kth non-central moment for the two-parameter Weibull distribution is [14,19,24]:

$$M_r = \beta^r \Gamma\left(\frac{r}{\alpha + 1}\right).$$

Equating the mean and variance (M_1 and $M_2 - M_1^2$) with the sample counter parts ($\bar{X}$ and S^2), we see that

$$\hat{\beta}_{\mathrm{MLM}} = \frac{\bar{X}}{\Gamma(1/\hat{\alpha} + 1)}$$

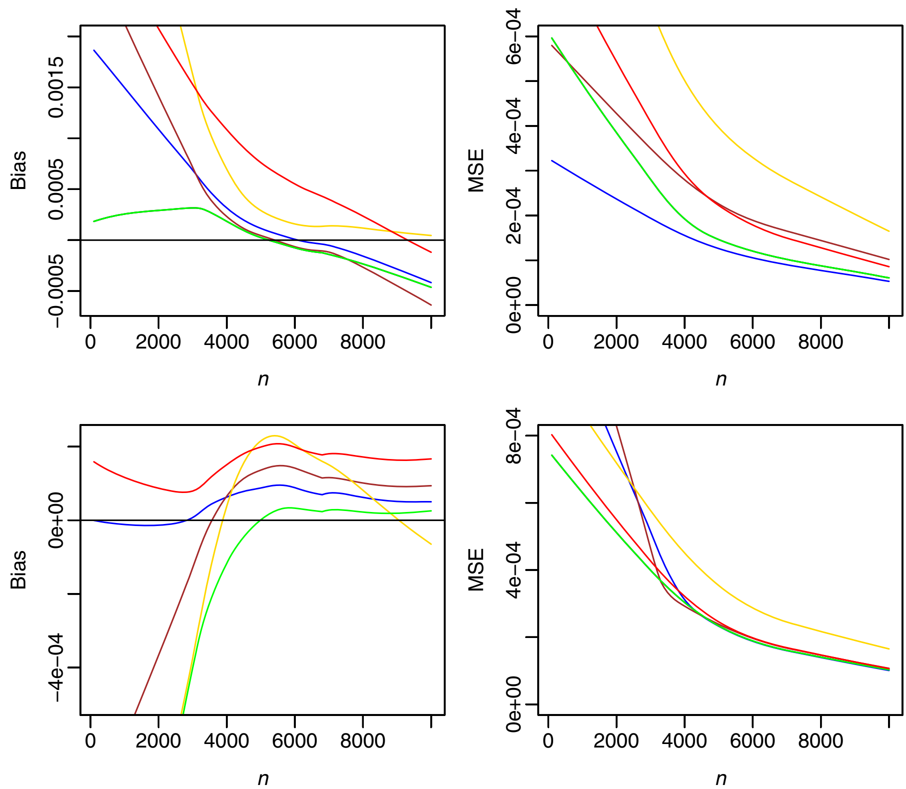

Figure 5. Biases of $\hat{\alpha}$ (top left), mean squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 1$ and $\beta = 1$.

and that $\hat{\alpha}$ is root of the equation

$$\frac{\Gamma(1+2/\alpha)}{\Gamma^2(1+1/\alpha)}+\frac{S^2}{\bar{X}}-1=0.$$

4. Numerical study

This section is in two parts. Firstly, we study the small sample behavior of the LM estimator. Secondly, we compare the performances of all of the estimation methods in Sections 2 and 3.

4.1. *Small sample behavior of the LM*

Here, we study the small sample behavior of the LM estimator for the two-parameter Weibull distribution. We simulate 1000 samples of size $n = 10, 20, 50$ from Equation (2) for $(\alpha, \beta) = (0.5, 1), (5, 1)$. We estimate the LM estimators for the 1000 samples. The histograms of the LM estimators for α and β are shown in Figures 1–3 for $n = 10, 20, 50$. The following observations can be made:

1. the distribution of $\hat{\alpha}$ appears skewed to the right for $n = 10, 20, 50$ (see the top left and bottom left plots in Figures 1–3);
2. the distribution of $\hat{\beta}$ for $\alpha = 0.5$ appears skewed to the right for $n = 10, 20, 50$ (see the top right plots in Figures 1–3);

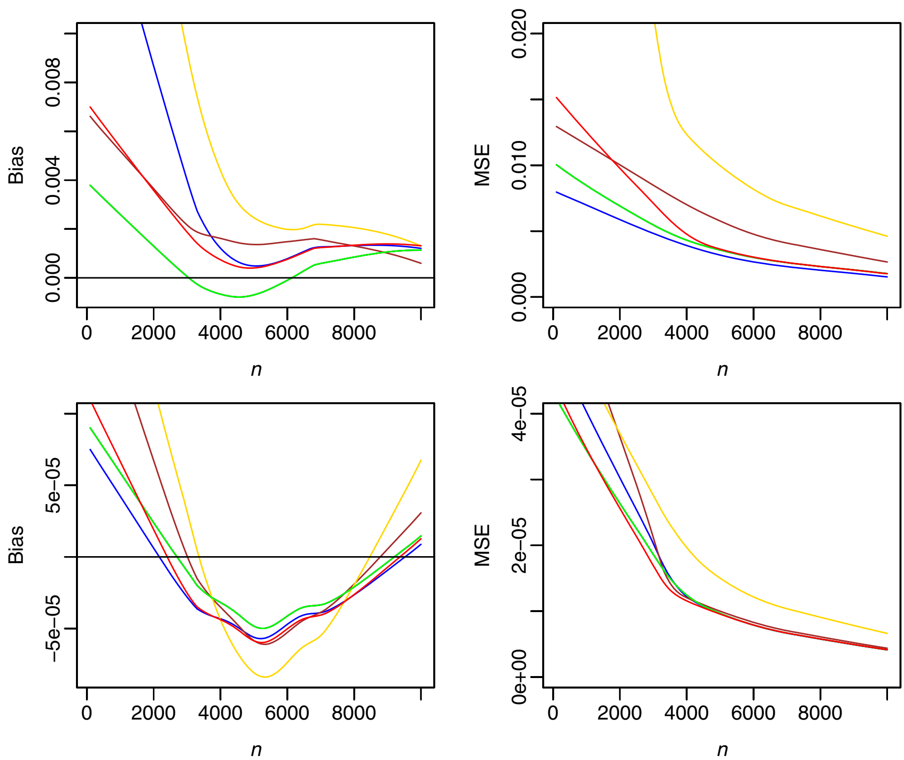

Figure 6. Biases of $\hat{\alpha}$ (top left), mean squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 5$ and $\beta = 1$.

3. the distribution of $\hat{\beta}$ for $\alpha = 5$ appears skewed to the left for $n = 10, 20, 50$ (see the bottom right plots in Figures 1–3);
4. the distributions of $\hat{\alpha}$ and $\hat{\beta}$ appear less skewed for $n = 50$ (see Figure 3).

The observations for the number of replications greater than 1000 were similar.

4.2. *Comparison of all estimators*

Here, we compare the performances of the MLE, the MLM, the PM, the LM, and the MM for the two-parameter Weibull distribution. The comparison is based on: the bias and the mean-squared error (MSE) criteria defined by

$$\text{Bias } (\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha),$$

$$\text{Bias } (\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta),$$

$$\text{MSE } (\hat{\alpha}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_i - \alpha)^2$$

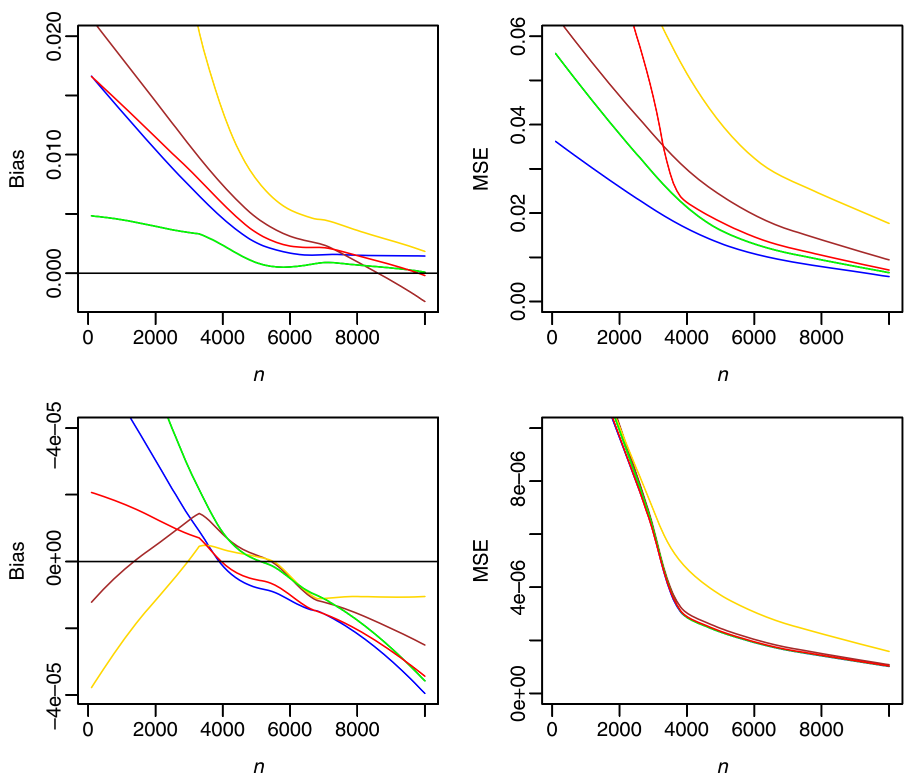

Figure 7. Biases of $\hat{\alpha}$ (top left), mean squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 10$ and $\beta = 1$.

and

$$\text{MSE }(\hat{\beta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\beta}_i - \beta)^2,$$

where N is the number of replications, $\hat{\alpha}_i$ the estimator of α in the ith replication and $\hat{\beta}_i$ the estimator of β in the ith replication. Larger values of the bias and the mean-squared error correspond to less efficient estimators.

The bias and the mean-squared error were computed for sample sizes of $10, 20, \ldots, 10{,}000$, number of replications $N = 1000, 2000, \ldots, 10{,}000$ and the parameter values $(\alpha, \beta) = (0.5, 1)$, $(1, 1)$, $(5, 1)$, $(10, 1)$, $(2, 1)$, $(2, 2)$, $(2, 5)$, $(2, 10)$. The plots of the computed biases and the mean squared errors for all five estimators are shown in Figures 4–11 for $N = 1000$. The results for $N = 1000, 2000, \ldots, 10{,}000$ were stable and similar, so the plots for $N > 1000$ are not reported here. The following color scheme is used in Figures 4–11: blue for the MLE, brown for the MLM, gold for the PM, green for the LM and red for the MM.

The following observations can be made on how the biases and mean-squared errors of the estimators vary with respect to α and β. For sufficiently large n (say $n \geq 6000$):

1. the bias $(\hat{\alpha})$ generally increases with increasing α (see the top left plots in Figures 4–7);
2. the mean-squared error $(\hat{\alpha})$ generally increases with increasing α (see the top right plots in Figures 4–7);
3. the bias $(\hat{\beta})$ generally decreases with increasing α (see the bottom left plots in Figures 4–7);
4. the mean-squared error $(\hat{\beta})$ generally decreases with increasing α (see the bottom right plots in Figures 4–7);

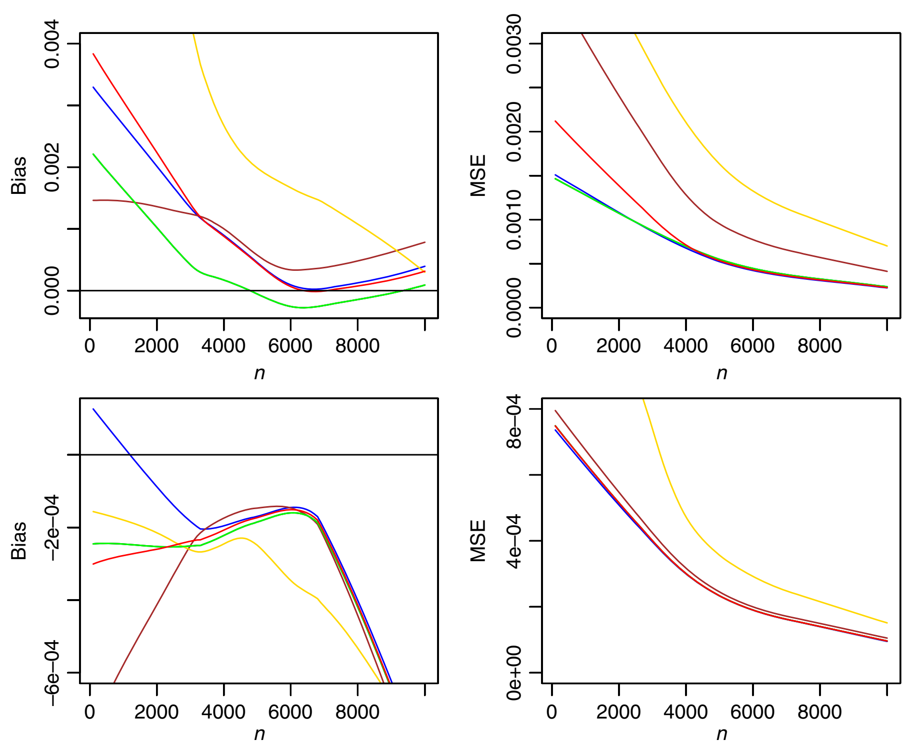

Figure 8. Biases of $\hat{\alpha}$ (top left), mean squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 2$ and $\beta = 1$.

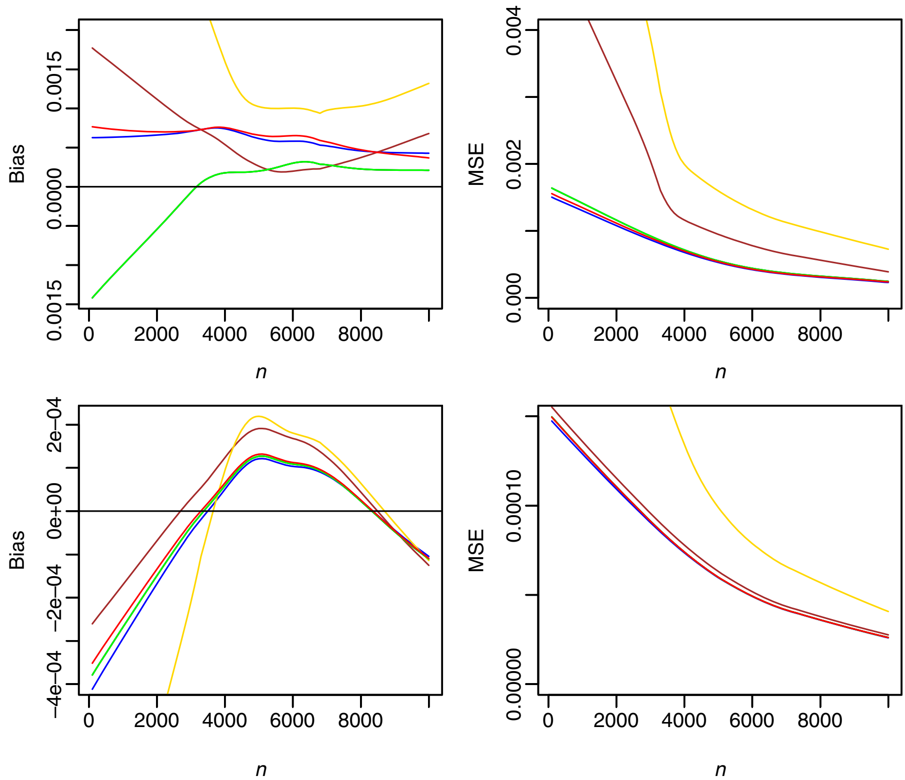

Figure 9. Biases of $\hat{\alpha}$ (top left), mean squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 2$ and $\beta = 2$.

5. excluding Figure 9 and considering the bottom left plots in Figures 8, 10; 11, the bias ($\hat{\beta}$) generally increases with increasing β;
6. the mean-squared error ($\hat{\beta}$) generally increases with increasing β, see the bottom right plots in Figures 8–11.

The following observations can be made on how the biases and mean-squared errors of the estimators vary with respect to n:

1. the mean-squared error of $\hat{\alpha}$ decreases (see the top right plots in Figures 4–11);
2. the mean-squared error of $\hat{\beta}$ also decreases (see the bottom right plots in Figures 4–11);
3. the bias of $\hat{\alpha}$ does not show a clear pattern of change with respect to n (see the top left plots in Figures 4–11);
4. the bias of $\hat{\beta}$ also does not show a clear pattern of change with respect to n (see the bottom left plots in Figures 4–11).

The following observations can be made on how bias ($\hat{\alpha}$) and mean-squared error ($\hat{\alpha}$) compare with bias ($\hat{\beta}$) and mean-squared error ($\hat{\beta}$). For sufficiently large n (say $n \geq 6000$):

1. the mean square error of $\hat{\beta}$ is generally greater than that of $\hat{\alpha}$ for $(\alpha, \beta) = (0.5, 1)$, $(2, 5)$, $(2, 10)$ (see the top right and bottom right plots in Figures 4, 10, 11);
2. the mean square error of $\hat{\beta}$ is generally smaller than that of $\hat{\alpha}$ for $(\alpha, \beta) = (5, 1)$, $(10, 1)$, $(2, 1)$, $(2, 2)$ (see the top right and bottom right plots in Figures 6–9);

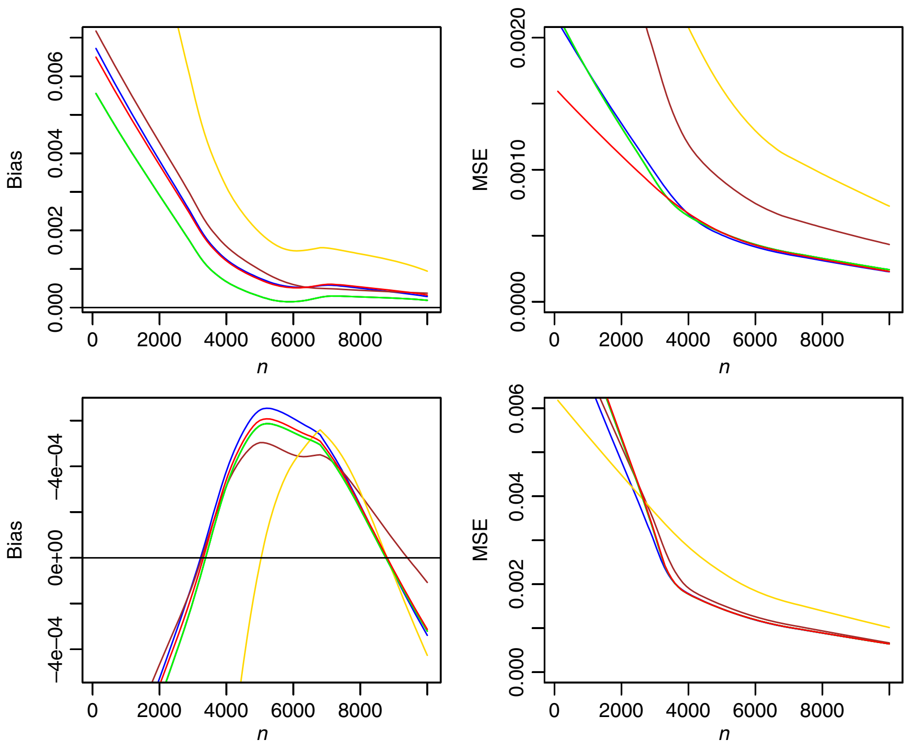

Figure 10. Biases of $\hat{\alpha}$ (top left), mean-squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 2$ and $\beta = 5$.

3. the bias of $\hat{\beta}$ is generally smaller than that of $\hat{\alpha}$ for $(\alpha, \beta) = (1, 1), (5, 1), (10, 1), (2, 1)$ (see the top left and bottom left plots in Figures 5–8).

We now discuss the relative performances of the five estimators with respect to the mean-squared error criterion. The MLE estimator gives the best overall performance for all sufficiently large n. However, the LM estimator performs equally well for many cases considered. In fact, the mean-squared errors of $\hat{\alpha}$ for the MLE and the LM estimators are approximately equal for all sufficiently large n for $(\alpha, \beta) = (2, 1), (2, 2), (2, 5)$, see the top right plots in Figures 8–10. Further, the mean squared errors of $\hat{\beta}$ for the MLE and the LM estimators are approximately equal for all sufficiently large n for $(\alpha, \beta) = (1, 1), (5, 1), (10, 1), (2, 1), (2, 2), (2, 5), (2, 10)$, see the bottom right plots in Figures 5–11. In most of the other cases, the LM estimator gives the second best performance.

For $(\alpha, \beta) = (0.5, 1)$, the worst performance is by the MM estimator, see the top right and bottom right plots in Figure 4. For all other cases, the PM estimator gives the worst performance for all sufficiently large n. The PM estimator gives the second worst performance for $(\alpha, \beta) = (0.5, 1)$ for all sufficiently large n, see the top right and bottom right plots in Figure 4. For all other cases, the MLM estimator gives the second worst performance with respect to $\hat{\alpha}$ for all sufficiently large n.

We finally discuss the relative performances of the five estimators with respect to the bias criterion. The LM estimator gives the best overall performance for the biases of $\hat{\alpha}$ and $\hat{\beta}$. The MM and the PM estimators give the worst performance for the biases of $\hat{\alpha}$ and $\hat{\beta}$. For $(\alpha, \beta) = (0.5, 1)$, $(1, 1)$, the MM estimator gives the worst performance, see the top left and bottom left plots in

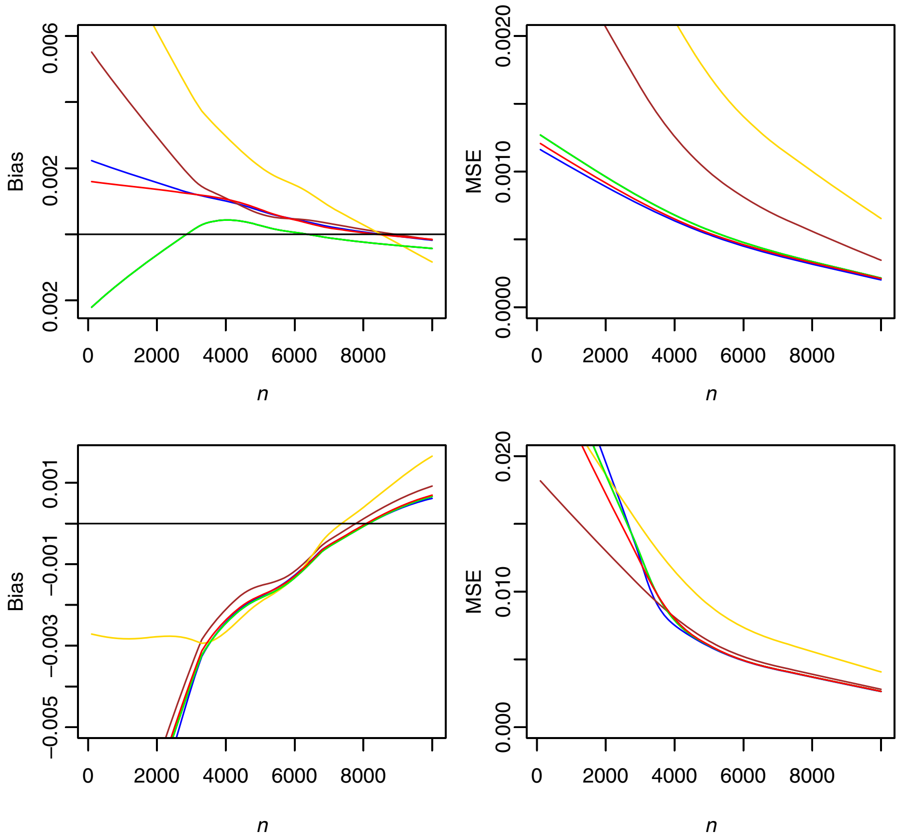

Figure 11. Biases of $\hat{\alpha}$ (top left), mean-squared errors of $\hat{\alpha}$ (top right), biases of $\hat{\beta}$ (bottom left), and mean-squared errors of $\hat{\beta}$ (bottom right) for $\alpha = 2$ and $\beta = 10$.

Figures 4–5. For $(\alpha, \beta) = (5, 1), (10, 1), (2, 1), (2, 2), (2, 5), (2, 10)$, the PM estimator gives the worst performance, see the top left and bottom left plots in Figures 6–11.

5. Conclusions

We have proposed a new estimation method for the Weibull distribution based on *TL* moments and *L* moments. We have performed simulations to compare this method with four other competitors including the MLE. The results show that the new method:

1. gives the best overall performance with respect to the biases of the estimators of the shape and scale parameters;
2. performs as well as the MLE or else second only to the MLE with respect to the mean-squared errors of the estimators of the shape and scale parameters.

It should also be noted that the MLE does not give closed form estimators. The new method does have closed form estimators.

Of the five methods considered, the best performances are by the LM and the MLE estimators. The worst performances are by the MM, the PM and the MLM estimators.

Acknowledgements

The authors would like to thank the Editor and the two referees for their careful reading and for their comments which greatly improved the paper.

References

[1] K. Raghunathan, V. Subramaniam, and V.R. Srinivasamoorthy, *Studies on the tensile characteristics of ring and rotor yarns using modified Weibull distribution*, Indian J. Fibre Textile Res. 27 (2002), pp. 358–361.
[2] Y.J. Chiang, C.D. Shih, C.C. Lin, and Y.Y. Tseng, *Examination of tyre rubber cure by Weibull distribution functions*, Int. J. Mater. Product Technol. 20 (2004), pp. 210–219.
[3] A. Genc, M. Erisoglu, A. Pekgor, G. Oturanc, A. Hepbasli, and K. Ulgen, *Estimation of wind power potential using Weibull distribution*, Energy Sources 27 (2005), pp. 809–822.
[4] M.A. Wood, B. Gunderson, A. Xia, X.H. Zhou, V. Padmanabhan, and K.A. Ellenbogen, *Temporal patterns of ventricular tachyarrhythmia recurrences follow a Weibull distribution*, J. Cardiovasc. Electrophysiol. 16 (2005), pp. 181–185.
[5] P. Garcia-Pascual, N. Sanjuan, R. Melis, and A. Mulet, *Morchella esculenta (morel) rehydration process modeling*, J. Food Eng. 72 (2006), pp. 346–353.
[6] S. Nadarajah and S. Kotz, *The modified Weibull distribution for asset returns*, Quant. Finance 6 (2006), pp. 449–449.
[7] G. Muraleedharan, A.D. Rao, P.G. Kurup, N.U. Nair, and M. Sinha, *Modified Weibull distribution for maximum and significant wave height simulation and prediction*, Coastal Eng. 54 (2007), pp. 630–638.
[8] O. Corzo, N. Bracho, A. Pereira, and A. Vasquez, *Weibull distribution for modeling air drying of coroba slices*, LWT—Food Sci. Technol. 41 (2008), pp. 2023–2028.
[9] A. Diaconu, *Weibull distribution as a model for growth/decline in product sales*, Metalurgia Int. 14 (2009), pp. 184–185.
[10] T. Hasumi, T. Akimoto, and Y. Aizawa, *The Weibull–log Weibull distribution for interoccurrence times of earthquakes*, Physica A—Statist. Mech. Appl. 388 (2009), pp. 491–498.
[11] K.C. Lo, K.T. Wu, C.S. Chyang, and W.T. Ting, *A new study on combustion behavior of pine sawdust characterized by the Weibull distribution*, Chinese J. Chem. Eng. 17 (2009), pp. 860–868.
[12] J.M. Arenas, J.J. Narbon, and C. Alia, *Optimum adhesive thickness in structural adhesives joints using statistical techniques based on Weibull distribution*, Int. J. Adhesion Adhesives 30 (2010), pp. 160–165.
[13] D.H.B. Binoti, H.G. Leite, G.S. Nogueira, M.L.M. da Silva, S.L.R. Garcia, and J.P. da Cruz, *Three-parameter Weibull distribution in a diametric distribution model for thinning eucaliptus stands*, Revista Arvore 34 (2010), pp. 147–156.
[14] W. Nelson, *Applied Life Data Analysis*, Wiley, New York, 1982.
[15] W.Q. Meeker and L.A. Escobar, *Statistical Methods for Reliability Data*, Wiley, New York, 1998.
[16] A.A. Bartolucci, K.P. Singh, A.D. Bartolucci, and S. Bae, *Applying medical survival data to estimate the three-parameter Weibull distribution by the method of probability weighted moments*, Math. Comput. Simul. 48 (1999), pp. 385–392.
[17] J.H. Heo, J.D. Salas, and K.D. Kim, *Estimation of confidence intervals of quantiles for the Weibull distribution*, Stochast. Environ. Res. Risk Assessment 15 (2001), pp. 284–309.
[18] D.N.P. Murthy, M. Xie, and R. Jiang, *Weibull Models*, Wiley, New York, 2004.
[19] B. Dodson, *The Weibull Analysis Handbook*, 2nd ed., ASQ Quality Press, Milwaukee, 2006.
[20] S.-J. Wu, *Estimation of the parameters of Weibull distribution with progressively censored data*, J. Japan Statist. Soc. 32 (2002), pp. 155–163.
[21] L. Zhang, M. Xie, and L.C. Tang, *A study of two estimation approaches for parameters of Weibull distribution based on WPP*, Reliab. Eng. System Safety 92 (2007), pp. 360–368.
[22] N. Balakrishnan and M. Kateri, *On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data*, Statist. Probab. Lett. 78 (2008), pp. 2971–2975.
[23] A.O.M. Ahmed, H.S. Al-Kutubi, and N.A. Ibrahim, *Comparison of the Bayesian and maximum likelihood estimation for Weibull Distribution*, J. Math. Statist. 6 (2010), pp. 100–104.
[24] N.L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, 2nd ed., Vol. I, Wiley, New York, 1994.
[25] R.L. Smith and J.C. Naylor, *A comparison of maximum likelihood estimators and Bayesian estimators for the three-parameter Weibull distribution*, Appl. Statist. 36 (1987), pp. 359–369.
[26] T. Seki and S. Yokoyama, *Robust parameter-estimation using the bootstrap method for the 2-parameter Weibull distribution*, IEEE Trans. Reliab. 45 (1996), pp. 34–51.
[27] A.M. Hussain and W.J. Zimmer, *Comparison of estimation methods for Weibull parameters: Complete and censored samples*, J. Statist. Comput. Simul. 73 (2003), pp. 145–153.
[28] Y.M. Kantar and B. Penoolu, *A comparative study for the location and scale parameters of the Weibull distribution with given shape parameter*, Comput. Geosci. 34 (2008), pp. 1900–1909.
[29] S.H. Zanakis and J. Kyparisis, *A review of maximum likelihood estimation methods for the three parameter Weibull distribution*, J. Statist. Comput. Simul. 25 (1986), pp. 53–73.
[30] R. Ross, *Bias and standard deviation due to Weibull parameter estimation for small data sets*, IEEE Trans. Dielectrics Electrical Insulation 3 (1996), pp. 28–42.

[31] A.J. Watkins, *On maximum likelihood estimation for the two parameter Weibull distribution*, Microelectron. Reliab. 36 (1996), pp. 595–603.
[32] G.C. Montanari, G. Mazzanti, M. Cacciari, and J.C. Fothergill, *In search of convenient techniques for reducing bias in the estimation of Weibull parameters for uncensored tests*, IEEE Trans. Dielectrics Electrical Insulation 4 (1997), pp. 306–313.
[33] J. Wyckoff and M. Engelhardt, *Cramer-Rao lower bounds for estimators based on censored data*, Commun. Statist. Theory Methods 9 (1980), pp. 1385–1399.
[34] R. Viveros and N. Balakrishnan, *Interval estimation of parameters of life from progressively censored data*, Technometrics 36 (1994), pp. 84–91.
[35] D. Kundu and M.Z. Raqab, *Generalized Rayleigh distribution: Different methods of estimations*, Comput. Statist. Data Anal. 49 (2005), pp. 187–200.
[36] M.R. Alkasasbeh and M.Z. Raqab, *Estimation of the generalized logistic distribution parameters: Comparative study*, Statist. Methodol. 6 (2009), pp. 262–279.
[37] J. Hosking, *L-moments: Analysis and estimation of distributions using linear combinations of order statistics*, J. R. Statist. Soc. B 52 (1990), pp. 105–124.
[38] E.A.H. Elamir and A.H. Seheult, *Trimmed L-moments*, Comput. Statist. Data Anal. 43 (2003), pp. 299–314.
[39] I.B. Abdul-Moniem, *L-moments and TL-moments estimation for the exponential distribution*, Far East J. Theoret. Statist. 23 (2007), pp. 51–61.
[40] Z. Leonowicz, J. Karvanen, and S.L. Shishkin, *Trimmed estimators for robust averaging of event-related potentials*, J. Neurosci. Methods 142 (2005), pp. 17–26.
[41] J.R.M. Hosking, *On the characterization of distributions by their L-moments*, J. Statist. Plann. Inference 136 (2006), pp. 193–198.
[42] J. Karvanen and A. Nuutinen, *Characterizing the generalized lambda distribution by L-moments*, Comput. Statist. Data Anal. 52 (2008), pp. 1971–1983.
[43] C. Gourieroux and J. Jasiak, *Dynamic quantile models*, J. Econometrics, 147 (2008), pp. 198–205.
[44] S. Darolles, C. Gourieroux, and J. Jasiak, *L-performance with an application to hedge funds*, J. Empirical Finance 16 (2009), pp. 671–685.
[45] G.E. Thomas, *Assessing symmetry using quantiles and L-moments*, Commun. Statist. Simul. Comput. 38 (2009), pp. 335–354.
[46] J.R.M. Hosking, *Some theory and practical uses of trimmed L-moments*, J. Statist. Plann. Inference, 137 (2007), pp. 3024–3039.
[47] A.C. Cohen and B. Whitten, *Modified maximum likelihood and modified moment estimators for the three-parameter Weibull distribution*, Commun. Statist. Theory Methods 11 (1982), pp. 2631–2656.
[48] E. Gourdin, P. Hansen, and B. Jaumard, *Finding maximum likelihood estimators for the 3-parameter Weibull distribution*, J. Global Optim. 5 (1994), pp. 373–397.
[49] F.K. Wang and J.B. Keats, *Improved percentile estimation for the 2-parameter Weibull distribution*, Microelectron. Reliab. 35 (1995), pp. 883–892.
[50] K.M. Hassanein, *Percentile estimators for the parameters of the Weibull distribution*, Biometrika 58 (1971), pp. 673–676.
[51] G.W. Cran, *Moment estimators for the 3-parameter Weibull distribution*, IEEE Trans. Reliab. 37 (1988), pp. 360–363.