

Modified Moment, Maximum Likelihood and Percentile Estimators for the Parameters of the Power Function Distribution

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Abstract

This paper is concerned with the modifications of maximum likelihood, moments and percentile estimators of the two parameter Power function distribution. Sampling behavior of the estimators is indicated by Monte Carlo simulation. For some combinations of parameter values, some of the modified estimators appear better than the traditional maximum likelihood, moments and percentile estimators with respect to bias, mean square error and total deviation.

Keywords: Parameter estimation, Percentile estimators, Maximum likelihood estimators, Moment estimators, Modified estimators, Monte Carlo study, Total deviation, Mean square error.

1. Introduction

The Power function distribution is a flexible life time distribution model that may offer a good fit to some sets of failure data. Theoretically Power function distribution is the inverse of Pareto distribution. An excellent account of this distribution and its properties is given in Kleiber and Kotz (1970). Meniconi and Barry (1995) discussed the application of Power function distribution. They proved that the Power function distribution is the best distribution to check the reliability of any electrical component. They used Exponential distribution, Lognormal distribution and Weibull distribution and showed from reliability and hazard function that Power function distribution is the best distribution.

The probability distribution of Power function distribution is

$$f(t) = \frac{\gamma t^{\gamma-1}}{\beta^\gamma}; \quad 0 < t < \beta \quad (1)$$

With shape parameter γ and scale parameter β , the interval $(0, \beta)$

Rider (1964) derived Distributions of the product and quotients of the Order Statistics from a Power function distribution. Moments of Order Statistics for a Power function Distribution were calculated by Malik (1967). Lwin (1972) discussed Bayesian estimation for the scale parameter of the Pareto Distribution using a Power function prior. Ahsanullah and Kabir (1975) discussed the Estimation of the location and scale parameters of a Power function distribution.

Cohen and Whitten (1982) used the moment and Modified Moment Estimators for the Weibull Distribution. Samia and Mohammad (1993) used five modifications of moments to estimate the parameters of the Pareto Distribution. Lalitha and Anand (1996) used Modified Maximum Likelihood to estimate the scale parameter of the Rayleigh Distribution. Rafiq (1996) discussed the parameters of the Gamma Distribution. Rafiq (1999) discussed the method of Fractional Moments to estimate the parameters of Weibull Distribution. Kang and Young (1997) estimated the parameters of a Pareto Distribution by Jackknife and Bootstrap Methods. Neil (2005) estimated the parameters of Weibull Distribution with the help of percentiles. He called it Common Percentile Method. Zaka and Akhter (2013) derived the different estimation methods for the parameters of Power function distribution. Zaka and Akhter (2013) discussed the different modifications of the parameter estimation methods and proved that the modified estimators appear better than the traditional maximum likelihood, moments and percentile estimators.

In this paper, we use the percentiles method, maximum likelihood and moment method to estimate the two parameters of the Power function distribution. The present paper introduces the modified estimators for parameters of the Power function distribution. The resulting estimators are easier to calculate than the maximum likelihood and, for certain combinations of parameter values; they improve the estimated values and mean square error. We examine these methods using two parameters Power function distribution to find the most accurate method (the method which has least M.S.E).

2. Methodology

2.1. Percentile Estimator (P.E)

Let $t_1, t_2, t_3, \dots, t_n$ be a random sample of size n drawn from probability density function of Power function distribution. The cumulative distribution function for a Power function distribution with shape and scale parameters β and γ , respectively

$$F(t_i) = \left(\frac{t_i}{\beta}\right)^\gamma \quad (2)$$

By solving we get

$$t_i = \beta(R_i)^{1/\gamma} \quad (3)$$

Where $R_i = F(t_i)$

Let P_{75} and P_{25} are used.

therefore (3) becomes

$$P_{75} = \beta(.75)^{1/\gamma} \quad (4)$$

$$P_{25} = \beta(.25)^{1/\gamma} \quad (5)$$

Dividing (4) and (5) we get

$$\left(\frac{P_{75}}{P_{25}}\right)^{\gamma} = \left(\frac{.75}{.25}\right)$$

$$\gamma \ln\left(\frac{P_{75}}{P_{25}}\right) = \ln\left(\frac{.75}{.25}\right)$$

$$\hat{\gamma} = \frac{\ln\left(\frac{.75}{.25}\right)}{\ln\left(\frac{P_{75}}{P_{25}}\right)}$$

$$\text{and } \hat{\beta} = \frac{P_{75}}{(.75)^{1/\hat{\gamma}}}$$

$$\hat{\gamma} = \frac{\ln\left(\frac{H}{L}\right)}{\ln\left(\frac{P_H}{P_L}\right)} \quad (6)$$

$$\text{and } \hat{\beta} = \frac{P_H}{(H)^{1/\hat{\gamma}}} \quad (7)$$

Where $H+L = 1$.

2.2. First Modified Percentile Estimator (M.P.E.1)

In this modification of the percentile estimators, equation (5) is replaced by the coefficient of variation of Power function distribution.

$$c.v = \frac{1}{\sqrt{\gamma(\gamma+2)}} \quad (8)$$

$$\text{and } P_{75} = \beta(.75)^{1/\gamma} \quad (4)$$

$$\text{From (8)} \quad \frac{S}{\bar{t}} = \frac{1}{\sqrt{\gamma(\gamma+2)}}$$

Taking square in both sides we get

$$\left(\frac{S}{\bar{t}}\right)^2 = \frac{1}{\gamma(\gamma+2)}$$

$$\gamma(\gamma+2) = \left(\frac{\bar{t}}{S}\right)^2$$

After Simplification we get

$$\hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (9)$$

$$\text{From (4)} \quad \hat{\beta} = \frac{P_{75}}{(.75)^{1/\hat{\gamma}}}$$

$$\hat{\beta} = \frac{P_{75}}{(.75)^{1/(-1+\sqrt{1+\frac{\tilde{t}^2}{S^2}})}}$$

$$\hat{\beta} = \frac{P_H}{(H)^{1/(-1+\sqrt{1+\frac{\tilde{t}^2}{S^2}})}}$$
(10)

2.3. Second Modified Percentile Estimator (M.P.E.2)

In this modification of the percentile estimators, equation (5) is replaced by the median of Power function distribution.

$$\tilde{t} = \frac{\beta}{2^{1/\gamma}}$$

i.e. $\hat{\beta} = \tilde{t} 2^{1/\gamma}$ (11)

From (4) $\hat{\beta} = \frac{P_{75}}{(.75)^{1/\gamma}}$

therefore $\tilde{t} 2^{1/\gamma} = \frac{P_{75}}{(.75)^{1/\gamma}}$

$$(2 * .75)^{1/\gamma} = \frac{P_{75}}{\tilde{t}}$$

Taking ln on both sides $\frac{1}{\gamma} \ln(2 * .75) = \ln\left(\frac{P_{75}}{\tilde{t}}\right)$

$$\hat{\gamma} = \frac{\ln(2 * .75)}{\ln\left(\frac{P_{75}}{\tilde{t}}\right)}$$

$$\hat{\gamma} = \frac{\ln(2 * H)}{\ln\left(\frac{P_H}{\tilde{t}}\right)}$$
(12)

2.4. Third Modified Percentile Estimator (M.P.E.3)

Using $E(F(t_1)) = F(t_1)$ of Power function distribution and (4), by neglecting(5).

$$P_{75} = \beta (.75)^{1/\gamma}$$
(4)

$$\hat{\beta} = \frac{P_{75}}{(.75)^{1/\gamma}}$$
(13)

And $E(F(t_1)) = F(t_1)$ (14)

$$\frac{1}{n+1} = \left(\frac{t_{(1)}}{\beta}\right)^\gamma$$

$$\frac{t_{(1)}}{\beta} = \left(\frac{1}{n+1}\right)^{1/\gamma}$$

$$\hat{\beta} = t_{(1)}(n+1)^{1/\gamma}$$
(15)

Equating (13) and (15)

$$\frac{P_{75}}{(.75)^{1/\gamma}} = t_{(1)}(n+1)^{1/\gamma}$$

$$\frac{P_{75}}{t_{(1)}} = (.75 * (n+1))^{1/\gamma}$$

taking ln on both sides $\ln\left(\frac{P_{75}}{t_{(1)}}\right) = \ln(.75 * (n+1))^{1/\gamma}$

$$\ln\left(\frac{P_{75}}{t_{(1)}}\right) = \frac{1}{\gamma} [\ln(.75 * (n+1))]$$

$$\hat{\gamma} = \frac{\ln(.75*(n+1))}{\ln\left(\frac{P_{75}}{t_{(1)}}\right)}$$

$$\hat{\gamma} = \frac{\ln(H*(n+1))}{\ln\left(\frac{P_H}{t_{(1)}}\right)} \quad (16)$$

2.5. Maximum Likelihood Method (M.L.E)

Let $t_1, t_2, t_3, \dots, t_n$ be a random sample of size n drawn from probability density function of Power function distribution. The likelihood function of this random sample is the joint density of the n random variables and is a function of the unknown parameters. Thus

$$f(t) = \frac{\gamma t^{\gamma-1}}{\beta^\gamma} \quad 0 < t < \beta$$

Likelihood function is

$$L(\gamma, \beta) = \prod_{i=1}^n f(t_i, \gamma, \beta)$$

$$L(\gamma, \beta) = \frac{\gamma^n \prod_{i=1}^n t_i^{\gamma-1}}{\beta^{n\gamma}}$$

taking natural log on both sides we get

$$\ln L = n \ln \gamma + \sum_{i=1}^n \ln t_i^{\gamma-1} - n\gamma \ln(\beta)$$

The maximum likelihood estimator (MLE) of the parameter is the value of the parameter that maximizes L and MLE for 2 parameter of Power function distribution can be obtained by solving the equations resulting from setting the two partial derivatives of $L(\gamma, \beta)$ to zero;

$$\frac{\partial \ln L}{\partial \beta} = 0 + 0 - \frac{n\gamma}{\beta} = 0 \quad (17)$$

β does not exist. but the likelihood function can be maximize by taking

$$\hat{\beta} = t_n \quad (18)$$

Where t_n is the largest value in the sample data.

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \ln(\beta) = 0 \quad (19)$$

$$\hat{\gamma} = \left(\frac{n}{n \ln(\beta) - \sum_{i=1}^n \ln t_i} \right) \quad (20)$$

2.6. First Modified Maximum Likelihood Estimator (M.M.L.E.1)

By neglecting (17) and replacing

$$E(F(t_1)) = F(t_1)$$

$$\frac{1}{n+1} = \left(\frac{t_{(1)}}{\beta} \right)^\gamma$$

$$\frac{t_{(1)}}{\beta} = \left(\frac{1}{n+1} \right)^{1/\gamma}$$

$$\hat{\beta} = t_{(1)}(n+1)^{1/\gamma} \quad (21)$$

Put in (19) we get

$$\frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \ln(t_{(1)}(n+1)^{1/\gamma}) = 0$$

$$\frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \left\{ \ln t_{(1)} + \frac{1}{\gamma} \ln(n+1) \right\} = 0$$

$$\frac{n}{\gamma} - \frac{n}{\gamma} \ln(n+1) = n \ln t_{(1)} - \sum_{i=1}^n \ln t_i$$

$$\frac{n}{\gamma} \{1 - \ln(n+1)\} = n \ln t_{(1)} - \sum_{i=1}^n \ln t_i$$

therefore
$$\hat{\gamma} = \frac{n\{1 - \ln(n+1)\}}{n \ln t_{(1)} - \sum_{i=1}^n \ln t_i}$$

$$\hat{\gamma} = \frac{\{1 - \ln(n+1)\}}{\ln t_{(1)} - \frac{\sum_{i=1}^n \ln t_i}{n}} \quad (22)$$

2.7. Second Modified Maximum Likelihood Estimator (M.M.L.E. 2)

From equation (17) and (19)

$$\frac{n\gamma}{\beta} = 0 \quad (23)$$

$$\frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \ln(\beta) = 0 \quad (24)$$

Replacing (23) by median of Power function distribution

$$\begin{aligned}\tilde{t} &= \frac{\beta}{2^{1/\gamma}} \\ \text{i.e. } \hat{\beta} &= \tilde{t} 2^{1/\gamma}\end{aligned}\quad (25)$$

$$\text{Put in (24)} \quad \frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \ln(\tilde{t} 2^{1/\gamma}) = 0$$

$$\begin{aligned}\frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \left\{ \ln(\tilde{t}) + \frac{1}{\gamma} \ln(2) \right\} &= 0 \\ \frac{n}{\gamma} - \frac{n}{\gamma} \ln(2) &= n \ln(\tilde{t}) - \sum_{i=1}^n \ln t_i \\ \frac{n}{\gamma} \{1 - \ln(2)\} &= n \ln(\tilde{t}) - \sum_{i=1}^n \ln t_i \\ \hat{\gamma} &= \frac{n\{1 - \ln(2)\}}{n \ln(\tilde{t}) - \sum_{i=1}^n \ln t_i} \\ \hat{\gamma} &= \left[\frac{\{1 - \ln(2)\}}{\ln(\tilde{t}) - \frac{\sum_{i=1}^n \ln t_i}{n}} \right]\end{aligned}\quad (26)$$

2.8. Third Modified Maximum Likelihood Estimator (M.M.L.E.3)

Using c.v of Power function distribution and neglecting $\frac{\partial \ln L}{\partial \beta}$

$$\frac{\partial \ln L}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \ln(\beta) = 0 \quad (19)$$

$$\text{c.v} = \frac{1}{\sqrt{\gamma(\gamma+2)}} \quad (27)$$

$$\frac{S}{\bar{t}} = \frac{1}{\sqrt{\gamma(\gamma+2)}}$$

$$\text{by solving we get} \quad \hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (28)$$

Where S is standard deviation and \bar{t} is mean.

$$\begin{aligned}\text{and } \frac{\partial \ln L}{\partial \gamma} &= \frac{n}{\gamma} + \sum_{i=1}^n \ln t_i - n \ln(\beta) = 0 \\ n \ln(\beta) &= \sum_{i=1}^n \ln t_i + \frac{n}{\gamma} \\ \ln(\beta) &= \frac{\sum_{i=1}^n \ln t_i}{n} + \frac{1}{\gamma} \\ \hat{\beta} &= e^{\left\{ \frac{\sum_{i=1}^n \ln t_i}{n} + \frac{1}{\gamma} \right\}}\end{aligned}\quad (29)$$

2.9. Moment Estimators (M.E)

The method of moments is another technique commonly used in the field of estimation of parameters. If the numbers $t_1, t_2, t_3, \dots, t_n$ represent a set of data, then an unbiased estimator for the k^{th} origin moment is

$$m'_k = \frac{\sum_{i=1}^n t_i^k}{n}$$

Where m'_k stands for the k^{th} sample moment.

The first moment of Power function distribution is

$$\begin{aligned} E(t) &= \int_{-\infty}^{\infty} t f(t) dt \\ E(t) &= \int_0^{\beta} t \left(\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}} \right) dt \\ E(t) &= \frac{\gamma}{\beta^{\gamma}} \int_0^{\beta} t^{\gamma} dt \\ E(t) &= \frac{\gamma}{\beta^{\gamma}} \left(\frac{\beta^{\gamma+1}}{\gamma+1} \right) \\ E(t) &= \frac{\beta\gamma}{\gamma+1} \end{aligned}$$

And $E(t^2) = \int_0^{\beta} t^2 \left(\frac{\gamma t^{\gamma-1}}{\beta^{\gamma}} \right) dt$

$$\begin{aligned} E(t^2) &= \frac{\gamma}{\beta^{\gamma}} \int_0^{\beta} t^2 (t^{\gamma-1}) dt \\ E(t^2) &= \frac{\gamma}{\beta^{\gamma}} \int_0^{\beta} t^{\gamma+1} dt \\ E(t^2) &= \frac{\beta^2\gamma}{\gamma+2} \end{aligned}$$

Therefore by equating sample and population moments we get

$$\begin{aligned} m'_1 &= \mu'_1 = E(t) \\ m'_1 &= \frac{\beta\gamma}{\gamma+1} \end{aligned} \tag{30}$$

And $m'_2 = \mu'_2 = E(t^2)$

$$m'_2 = \frac{\beta^2\gamma}{\gamma+2} \tag{31}$$

From (30) $\bar{t} = \frac{\beta\gamma}{\gamma+1}$

$$\bar{t}(\gamma + 1) = \beta\gamma$$

$$\hat{\beta} = \frac{\bar{t}(\gamma+1)}{\gamma} \quad (32)$$

Put in (31), we get

$$m_2' = \frac{\bar{t}^2(\gamma + 1)^2\gamma}{\gamma^2(\gamma + 2)}$$

$$m_2' = \frac{\bar{t}^2(\gamma + 1)^2}{\gamma(\gamma + 2)}$$

we know that $m_2 = m_2' - (m_1')^2$

$$m_2' = S^2 + \bar{t}^2 \quad ;$$

Where $m_2 = S^2$ and $m_1' = \bar{t}$

therefore $S^2 + \bar{t}^2 = \frac{\bar{t}^2(\gamma + 1)^2}{\gamma(\gamma + 2)}$

$$(S^2 + \bar{t}^2)\gamma(\gamma + 2) = \bar{t}^2(\gamma + 1)^2$$

$$\gamma^2 S^2 + \gamma^2 \bar{t}^2 + 2\gamma S^2 + 2\gamma \bar{t}^2 = \gamma^2 \bar{t}^2 + \bar{t}^2 + 2\gamma \bar{t}^2$$

$$\bar{t}^2 = \gamma^2 S^2 + 2\gamma S^2$$

$$\gamma^2 + 2\gamma = \frac{\bar{t}^2}{S^2}$$

$$\gamma^2 + 2\gamma - \frac{\bar{t}^2}{S^2} = 0$$

$$\hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (33)$$

2.10. First Modified Moment Estimator (M.M.E.1)

In this modification of the moment estimators, the second moment of two parameters Power function distribution is replaced by the coefficient of variation of Power function distribution.

$$c.v = \frac{1}{\sqrt{\gamma(\gamma+2)}} \quad (34)$$

and $m_1' = \frac{\beta\gamma}{\gamma+1}$ (30)

$$\frac{S}{\bar{t}} = \frac{1}{\sqrt{\gamma(\gamma + 2)}}$$

Taking square in both sides we get

$$\left(\frac{S}{\bar{t}}\right)^2 = \frac{1}{\gamma(\gamma+2)}$$

$$\gamma(\gamma+2) = \left(\frac{\bar{t}}{S}\right)^2$$

After Simplification we get

$$\hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (35)$$

2.11. Second Modified Moment Estimator (M.M.E.2)

In this modification of the moment estimators, the second moment of two parameters Power function distribution is replaced by the variance of Power function distribution.

i.e. $S^2 = \frac{\beta^2 \gamma}{(\gamma+2)(\gamma+1)^2}$ (36)

and $m'_1 = \frac{\beta \gamma}{\gamma+1}$ (30)

From (36) $\hat{\beta} = \frac{S^2(\gamma+1)(\gamma+2)}{\bar{t}}$

put in (30) $m'_1 = \frac{S^2(\gamma+1)(\gamma+2)}{\bar{t}} * \frac{\gamma}{\gamma+1}$

$$\frac{m'^2_1}{S^2} = \gamma^2 + 2\gamma$$

$$\gamma^2 + 2\gamma - \frac{m'^2_1}{S^2} = 0$$

Where $m'_1 = \bar{t}$

Therefore $\gamma^2 + 2\gamma - \frac{\bar{t}^2}{S^2} = 0$

$$\hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (37)$$

and $\hat{\beta} = \frac{S^2(\hat{\gamma}+1)(\hat{\gamma}+2)}{\bar{t}}$ (38)

2.12. Third Modified Moment Estimator (M.M.E.3)

In this modification of the moment estimators, the first moment of two parameters Power function distribution is replaced by the variance of Power function distribution.

i.e. $S^2 = \frac{\beta^2 \gamma}{(\gamma+2)(\gamma+1)^2}$ (36)

and $m'_2 = \frac{\beta^2 \gamma}{\gamma+2}$ (31)

$$\text{from (36)} \quad \beta^2 = \frac{(\gamma + 2)(\gamma + 1)^2 S^2}{\gamma}$$

$$\hat{\beta} = S(\hat{\gamma} + 1) \sqrt{(1 + \frac{2}{\hat{\gamma}})}$$

$$\text{put in (31)} \quad m'_2 = \frac{(\gamma + 2)(\gamma + 1)^2 S^2}{\gamma} * \frac{\gamma}{(\gamma + 2)}$$

$$m'_2 = (\gamma + 1)^2 S^2$$

$$S^2 + \bar{t}^2 = (\gamma + 1)^2 S^2$$

$$\frac{S^2 + \bar{t}^2}{S^2} = (\gamma + 1)^2$$

$$1 + \frac{\bar{t}^2}{S^2} = (\gamma + 1)^2$$

$$\gamma^2 + 1 + 2\gamma = 1 + \frac{\bar{t}^2}{S^2}$$

$$\gamma^2 + 2\gamma - \frac{\bar{t}^2}{S^2} = 0$$

$$\hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (39)$$

$$\text{and } \hat{\beta} = S(\hat{\gamma} + 1) \sqrt{(1 + \frac{2}{\hat{\gamma}})} \quad (40)$$

2.13. Fourth Modified Moment Estimator (M.M.E.4)

Using co-efficient of variation of Power function distribution and m'_2 , by neglecting m'_1 .

$$m'_2 = \frac{\beta^2 \gamma}{\gamma + 2} \quad (31)$$

$$c.v = \frac{1}{\sqrt{\gamma(\gamma + 2)}} \quad (34)$$

$$\text{From (31)} \quad \beta^2 = m'_2 * \frac{(\gamma + 2)}{\gamma}$$

$$\text{From (34)} \quad \gamma(\gamma + 2) = \frac{\bar{t}^2}{S^2}$$

$$\text{therefore } \hat{\gamma} = -1 + \sqrt{1 + \frac{\bar{t}^2}{S^2}} \quad (41)$$

$$\text{and } \hat{\beta} = \sqrt{\frac{(\hat{\gamma} + 2)m'_2}{\hat{\gamma}}} \quad (42)$$

Three approximations for $\hat{F}(t_i)$ based on its being uniformly distributed on interval $[0, 1]$ show in Table.

Method		$F(t_i)$
(a)	MEDIAN RANK	$\frac{i - .3}{n + .4}$
(b)	MEAN RANK	$\frac{i}{n + 1}$
(c)	SYMMETRICAL CDF	$\frac{i - .5}{n}$
Methods for estimating $F(t_i)$		

3. Performance Indices (Goodness of Fit Analysis)

Some methods of goodness of fit analysis are employed here. Mean square error MSE and total deviation TD are two measurements that give an indication of the accuracy of parameter estimation. AL-Fawzan (2000) referred to the use of the procedure of MSE and TD.

3.1. Mean Square Errors (MSE)

The MSE can be calculated as below

$$\text{Standard bias, Bias} = E(\hat{\beta}) - \beta \quad \text{and} \quad \text{M.S.E}(\hat{\beta}) = E[(\hat{\beta} - \beta)^2]$$

$$\text{Standard bias, Bias} = E(\hat{\gamma}) - \gamma \quad \text{and} \quad \text{M.S.E}(\hat{\gamma}) = E[(\hat{\gamma} - \gamma)^2]$$

3.2. Total Derivation (TD)

The total derivation TD, calculated for each method is as follows

$$TD = \left| \frac{\hat{\gamma} - \gamma}{\gamma} \right| + \left| \frac{\hat{\beta} - \beta}{\beta} \right|$$

Where γ and β are the known parameters, and $\hat{\gamma}$ and $\hat{\beta}$ are the estimated parameters by any method. These techniques are used to measure the variability of parameter estimates for each simulation. These are used to determine the overall “best” parameter estimation method.

4. Application

A simulation study is used in order to compare the performance of the proposed estimation methods. We carry out this comparison by taking the samples of sizes as $n = 20, 60$ and 100 with pairs of $(\beta, \gamma) = \{(1, 2), (3, 2), (4, 3)\}$. We generate random samples of different sizes by observing that if R is uniform $(0, 1)$, then $t_i = \beta R_i^{1/\gamma}$ is the random number generator of Power function distribution with (β, γ) parameters. All results are based on 10,000 replications. Such generated data have been used to obtain estimates of the unknown parameters. The results obtained from parameters estimation of

the 2-parameters Power function distribution using different sample sizes and different values of parameters with mean square error MSE and total deviation TD.

Table 1: (Comparison between maximum likelihood, moments and modified estimators for the parameters β and γ of Power function distribution under the sample size 20)

Methods	True Values		Estimated Values		M.S.E		T.D
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	
M.L.E	1	2	0.978645	2.200732	0.0008578	0.313847	0.121721
	4	3	3.933556	3.264606	0.008168	0.518843	0.104813
	3	2	2.917646	2.126088	0.012277	0.209215	0.090495
M.M.L.E.1	1	2	1.137105	1.897594	0.088734	0.521512	0.188307
	4	3	4.378462	2.750874	0.611629	1.053479	0.177657
	3	2	3.461701	1.817926	1.342053	0.483105	0.244937
M.M.L.E.2	1	2	0.971083	3.324441	0.054715	23.89807	0.691138
	4	3	3.942013	3.17264	0.355711	536.4506	0.072043
	3	2	2.926568	3.262925	0.591672	352.0646	0.65594
M.M.L.E.3	1	2	1.014889	2.070333	0.002964	0.284255	0.050055
	4	3	4.010238	3.136546	0.028394	0.583318	0.048075
	3	2	3.027586	2.003834	0.029478	0.206902	0.011112
M.E	1	2	1.00223	2.070333	0.002549	0.284255	0.037396
	4	3	3.989649	3.136546	0.027886	0.583318	0.048103
	3	2	2.991901	2.003834	0.032327	0.206902	0.004617
M.M.E.1	1	2	1.00223	2.070333	0.002549	0.284255	0.037396
	4	3	3.989649	3.136546	0.027886	0.583318	0.048103
	3	2	2.991901	2.003834	0.032327	0.206902	0.004617
M.M.E.2	1	2	1.00223	2.070333	0.002549	0.284255	0.037396
	4	3	3.989649	3.136546	0.027886	0.583318	0.048103
	3	2	2.991901	2.003834	0.032327	0.206902	0.004617
M.M.E.3	1	2	1.00223	2.070333	0.002549	0.284255	0.037396
	4	3	3.989649	3.136546	0.027886	0.583318	0.048103
	3	2	2.991901	2.003834	0.032327	0.206902	0.004617
M.M.E.4	1	2	1.00223	2.070333	0.002549	0.284255	0.037396
	4	3	3.989649	3.136546	0.027886	0.583318	0.048103
	3	2	2.991901	2.003834	0.032327	0.206902	0.004617

Table 2: (Comparison between maximum likelihood, moments and modified estimators for the parameters β and γ of Power function distribution under the sample size 60)

Methods	True Values		Estimated Values		M.S.E		T.D
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	
M.L.E	1	2	0.990868	2.034965	0.00017	0.050894	0.026615
	4	3	3.977701	3.064252	0.00093	0.131827	0.026992
	3	2	2.976511	2.046065	0.000955	0.070244	0.030862
M.M.L.E.1	1	2	1.107412	1.883062	0.07121	0.363556	0.165881
	4	3	4.258624	2.77774	0.306968	0.688143	0.138743
	3	2	3.345987	1.834348	0.509059	0.33646	0.198155
M.M.L.E.2	1	2	0.958338	2.646938	0.020348	4.393358	0.365131
	4	3	3.969671	3.560702	0.162279	2.6524	0.194483
	3	2	3.042986	2.172272	0.189356	0.63329	0.100465
M.M.L.E.3	1	2	0.999941	2.009726	0.000847	0.065919	0.004922
	4	3	3.997524	3.037836	0.009031	0.178646	0.013231
	3	2	3.021874	1.992693	0.008609	0.074065	0.010945
M.E	1	2	0.994954	2.009726	0.000902	0.065919	0.009909
	4	3	3.991493	3.037836	0.009727	0.178646	0.014739
	3	2	3.013006	1.992693	0.00945	0.074065	0.007989
M.M.E.1	1	2	0.994954	2.009726	0.000902	0.065919	0.009909
	4	3	3.991493	3.037836	0.009727	0.178646	0.014739
	3	2	3.013006	1.992693	0.00945	0.074065	0.007989
M.M.E.2	1	2	0.994954	2.009726	0.000902	0.065919	0.009909
	4	3	3.991493	3.037836	0.009727	0.178646	0.014739
	3	2	3.013006	1.992693	0.00945	0.074065	0.007989
M.M.E.3	1	2	0.994954	2.009726	0.000902	0.065919	0.009909
	4	3	3.991493	3.037836	0.009727	0.178646	0.014739
	3	2	3.013006	1.992693	0.00945	0.074065	0.007989
M.M.E.4	1	2	0.994954	2.009726	0.000902	0.065919	0.009909
	4	3	3.991493	3.037836	0.009727	0.178646	0.014739
	3	2	3.013006	1.992693	0.00945	0.074065	0.007989

Table 3: (Comparison between maximum likelihood, moments and modified estimators for the parameters β and γ of Power function distribution under the sample size 100)

Methods	True Values		Estimated Values		M.S.E		T.D
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	
M.L.E	1	2	0.995338	2.048911	3.71E-05	0.040732	0.029117
	4	3	3.986661	3.047854	0.000323	0.109404	0.019286
	3	2	2.98528	2.044074	0.00041	0.039428	0.026944
M.M.L.E.1	1	2	1.081902	1.908326	0.035466	0.290455	0.127739
	4	3	4.265531	2.780829	0.334422	0.706234	0.13944
	3	2	3.269395	1.858852	0.348456	0.228624	0.160372
M.M.L.E.2	1	2	0.984178	2.293329	0.013389	0.652912	0.162486
	4	3	3.985102	3.252392	0.094201	0.904084	0.087855
	3	2	3.04372	2.098991	0.11714	0.361612	0.064069
M.M.L.E.3	1	2	1.000854	2.030636	0.000452	0.045108	0.016173
	4	3	3.998787	3.033221	0.00458	0.147128	0.011377
	3	2	3.01057	2.012123	0.00437	0.037805	0.009585
M.E	1	2	0.998244	2.030636	0.000516	0.045108	0.017075
	4	3	3.996833	3.033221	0.005528	0.147128	0.011866
	3	2	3.006474	2.012123	0.004818	0.037805	0.008219
M.M.E.1	1	2	0.998244	2.030636	0.000516	0.045108	0.017075
	4	3	3.996833	3.033221	0.005528	0.147128	0.011866
	3	2	3.006474	2.012123	0.004818	0.037805	0.008219
M.M.E.2	1	2	0.998244	2.030636	0.000516	0.045108	0.017075
	4	3	3.996833	3.033221	0.005528	0.147128	0.011866
	3	2	3.006474	2.012123	0.004818	0.037805	0.008219
M.M.E.3	1	2	0.998244	2.030636	0.000516	0.045108	0.017075
	4	3	3.996833	3.033221	0.005528	0.147128	0.011866
	3	2	3.006474	2.012123	0.004818	0.037805	0.008219
M.M.E.4	1	2	0.998244	2.030636	0.000516	0.045108	0.017075
	4	3	3.996833	3.033221	0.005528	0.147128	0.011866
	3	2	3.006474	2.012123	0.004818	0.037805	0.008219

Table 4: (Comparison between percentile estimators and modified percentile estimators for the parameters β and γ of Power function distribution under the sample size 20)

Methods	True Values		Estimated Values		M.S.E		T.D
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	
P.E _(5,95)	1	2	0.963314	1.914608	0.000842	0.336271	0.049109
	4	3	3.93834	2.68987	0.006584	0.79654	0.099876
	3	2	3.05117	2.51692	0.009876	0.33698	0.096325
M.P.E.1 _(5, 95)	1	2	1.006413	1.94920	0.0005443	0.34987	0.052698
	4	3	4.009277	2.790714	0.004107	0.64841	0.072081
	3	2	3.003777	1.828275	0.005991	0.288481	0.087121
M.P.E.2 _(5, 95)	1	2	0.962884	1.98285	0.009865	0.31587	0.069878
	4	3	3.90627	4.70869	0.008954	0.87598	1.00258
	3	2	3.05623	2.32768	1.02356	0.44569	0.99987
M.P.E.3 _(5, 95)	1	2	0.955534	2.81742	0.02568	0.69874	0.08974
	4	3	3.99683	1.51702	0.36547	0.89547	1.12547
	3	2	3.05095	2.52579	0.12458	1.02365	0.09999
P.E _(25,75)	1	2	1.02118	1.94920	0.0067588	0.399743	0.037396
	4	3	4.033476	2.68987	0.05474	0.746732	0.012429
	3	2	3.06322	2.51692	0.056018	0.502218	0.01792
M.P.E.1 _(25,75)	1	2	1.014671	2.028446	0.002549	0.284255	0.028894
	4	3	4.14214	2.987819	0.027886	0.583318	0.048103
	3	2	3.014928	2.025889	0.032327	0.206902	0.004617
M.P.E.2 _(25,75)	1	2	1.06012	1.55491	0.002549	0.284255	0.037396
	4	3	3.99252	4.10038	0.027886	0.583318	0.048103
	3	2	3.11731	2.18263	0.032327	0.206902	0.004617
M.P.E.3 _(25,75)	1	2	0.977914	2.75832	0.002549	0.284255	0.037396
	4	3	4.55513	1.42423	0.027886	0.583318	0.048103
	3	2	3.06314	2.51746	0.032327	0.206902	0.004617

Table 5: (Comparison between percentile estimators and modified percentile estimators for the parameters β and γ of Power function distribution under the sample size 60)

Methods	True Values		Estimated Values		M.S.E		T.D
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	
P.E _(5,95)	1	2	1.00362	2.020935	0.07121	0.134476	0.011973
	4	3	4.01174	3.36567	0.306968	0.688143	0.138743
	3	2	3.03379	1.67547	0.509059	0.33646	0.198155
M.P.E.1 _(5, 95)	1	2	1.001506	2.11504	0.000234	0.363556	0.165881
	4	3	4.002723	2.945079	0.0015	0.293768	0.018988
	3	2	3.006631	1.932092	0.001793	0.133337	0.036164
M.P.E.2 _(5, 95)	1	2	1.01130	1.60909	0.020348	4.393358	0.365131
	4	3	4.04036	2.29499	0.162279	2.6524	0.194483
	3	2	3.02366	1.88107	0.189356	0.63329	0.100465
M.P.E.3 _(5, 95)	1	2	1.00519	1.98730	0.000847	0.065919	0.004922
	4	3	4.01585	3.15379	0.009031	0.178646	0.013231
	3	2	3.04060	1.56118	0.008609	0.339364	0.010945
P.E _(25,75)	1	2	0.987804	1.949088	0.002205	0.163036	0.029359
	4	3	3.93727	3.36567	0.016628	0.339364	0.014739
	3	2	3.18543	1.67547	0.022423	0.163036	0.007989
M.P.E.1 _(25, 75)	1	2	1.003903	2.11504	0.000902	0.065919	0.009909
	4	3	3.997267	3.031052	0.009727	0.178646	0.011034
	3	2	3.019546	2.019861	0.00945	0.074065	0.016446
M.P.E.2 _(25, 75)	1	2	1.04515	1.49485	0.000902	0.065919	0.009909
	4	3	4.13844	2.12615	0.009727	0.178646	0.014739
	3	2	3.20107	1.62901	0.00945	0.074065	0.007989
M.P.E.3 _(25, 75)	1	2	0.995826	1.99633	0.000902	0.065919	0.009909
	4	3	3.95577	3.19068	0.009727	0.178646	0.014739
	3	2	3.24010	1.52439	0.00945	0.074065	0.007989

Table 6: (Comparison between percentile estimators and modified percentile estimators for the parameters β and γ of Power function distribution under the sample size 100)

Methods	True Values		Estimated Values		M.S.E		T.D
	β	γ	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\beta}$	$\hat{\gamma}$	
P.E _(5,95)	1	2	0.99228	2.40349	0.035466	0.290455	0.127739
	4	3	3.95831	3.34118	0.334422	0.706234	0.13944
	3	2	3.02558	1.90685	0.348456	0.228624	0.160372
M.P.E.1 _(5, 95)	1	2	1.000372	2.000956	0.00016	0.097682	0.00085
	4	3	4.000468	2.940868	0.000956	0.196453	0.019828
	3	2	3.00697	1.97611	0.001332	0.083339	0.014269
M.P.E.2 _(5, 95)	1	2	0.993887	2.22899	0.013389	0.652912	0.162486
	4	3	3.94751	4.06448	0.094201	0.904084	0.087855
	3	2	3.02276	1.97528	0.11714	0.361612	0.064069
M.P.E.3 _(5, 95)	1	2	0.988208	2.96813	0.000452	0.045108	0.016173
	4	3	3.99627	2.06027	0.00458	0.147128	0.011377
	3	2	3.03948	1.62927	0.00437	0.037805	0.009585
P.E _(25,75)	1	2	0.988892	2.40349	0.000516	0.045108	0.017075
	4	3	4.01869	3.34118	0.005528	0.147128	0.011866
	3	2	3.05026	1.90685	0.004818	0.037805	0.008219
M.P.E.1 _(25,75)	1	2	1.000876	2.014617	0.001124	0.077664	0.008184
	4	3	3.991898	3.069985	0.009282	0.272487	0.025354
	3	2	3.000934	2.030725	0.010081	0.067682	0.015674
M.P.E.2 _(25,75)	1	2	1.00128	2.17713	0.000516	0.045108	0.017075
	4	3	3.96475	3.96322	0.005528	0.147128	0.011866
	3	2	3.04261	1.93913	0.004818	0.037805	0.008219
M.P.E.3 _(25,75)	1	2	0.965212	3.01376	0.000516	0.045108	0.017075
	4	3	4.25638	2.00387	0.005528	0.147128	0.011866
	3	2	3.13577	1.61152	0.004818	0.037805	0.008219

5. Conclusions

All the results are listed in tables 1 to 6. A Monte Carlo simulation study shows that if we compare the M.L.E, M.E and their modifications, we find that for small sample size their modified estimators produce better results. But as the sample size increases than only M.L.E provides best results. M.E does not show any reliable results.

If we compare the Percentile estimators (P.E) and its modifications then we see that for small sample size the Modified percentile estimators (M.P.E) and in some cases Percentile estimators (P.E) produce the best results for β and γ . The estimated values of β and γ are close to their true values. But as the sample size increases the only M.P.E.1 (First Modified Percentile Estimator) provides the best estimators (the method which has least M.S.E).

Among all the Percentile estimators and their modifications, $P_{(5, 25)}$ produces best results, which shows that larger data produce better results. The best estimator among all the Percentile estimators and Modified percentile estimators is M.P.E.1 $_{(5, 95)}$.

Finally, we recommend using M.P.E.1 and M.L.E modifications methods for small sample size and only M.L.E when the sample size is large.

It is very interesting that the result of M.E and its modifications are equal but their formulae are different.

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