Moment Estimators for the 3-Parameter Weibull Distribution

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Key Words **— Weibull distribution, Weibull moments, Moment estimator**

Reader Aids **—**
Purpose: Widen state of the art
Special math needed for explanations: Statistics
Special math needed to use results: Same
Results useful to: Reliability statisticians

Abstract **— Weibull moments are defined generally and then calculated for the 3-parameter Weibull distribution with non-negative location parameter. Sample estimates for these moments are given and used to estimate the parameters. The results of a simulation investigation of the properties of the parameter estimates are discussed briefly. A simple method of deciding whether the location parameter can be considered zero is described.**

INTRODUCTION

Newby [1] has derived the basic properties of the conventional moment estimators for the 3-parameter Weibull distribution; in particular he discussed the usefulness of moment estimation when the shape parameter does not exceed 2. This paper considers an estimation procedure based on Weibull moments, as defined by Weibull [2, p 226]. This procedure has two advantages over conventional method of moments:

- Sample Weibull moments are functions of differences of the observations, rather than powers;
- The parameter estimates are explicit functions of low-order sample moments.

The recommended use of such Weibull moment estimates, as for conventional method of moments estimation, is to provide quick, preliminary estimates of the parameters. These estimates can be used to assess —

- Whether the location parameter can be considered zero
- The approximate magnitude of the shape parameter, leading to a more efficient method of estimation such as maximum likelihood.

Notation

a — location parameter, $a \geqslant 0$
b — scale parameter, $b > 0$
c — shape parameter, $c > 0$
Sf$\{x; a, b, c\}$ — Sf of the 3-parameter Weibull distribution with parameters a, b, c. The Cdf$\{\cdot\}$ is similarly defined.
$\bar{\mu}_k$ — Weibull moment k, $k = 1, 2, \ldots$
$x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(n)}$ — ordered random sample of size n
$S_n(x)$ — sample Cdf
$\bar{m}_k$ — sample Weibull moment k, $k = 1, 2, \ldots$.
$*$ — implies Weibull moment estimate or estimator
$*(x)$ — Weibull moment estimation based on x-values.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

WEIBULL MOMENTS

Weibull [2, p 226] introduced the following class of moments of the continuous r.v. X:

$$\int_{x_a}^{x_b} [\mathrm{Sf}\{x\}]^k \, dx, \quad k = 1, 2, \ldots;$$

the limits x_a, x_b are chosen according as the distribution is complete, truncated or censored.

For the 3-parameter Weibull distribution with Sf$\{x; a, b, c\}$ = weifc$((x - a)/b; c)$ and $a \geqslant 0$, the Weibull moment k is defined as:

$$\bar{\mu}_k \equiv \int_0^\infty [\mathrm{weifc}((x - a)/b; c)]^k \, dx, \quad k = 1, 2, \ldots.$$

$$= a + b\Gamma(1 + 1/c)/k^{1/c}. \tag{1}$$

The location parameter a is restricted to be non-negative, as is desirable in reliability and life-testing applications.

The parameters can be expressed in terms of the lower order moments:

$$c = \frac{\log_e 2}{\log(\bar{\mu}_1 - \bar{\mu}_2) - \log(\bar{\mu}_2 - \bar{\mu}_4)}, \tag{2a}$$

$$a = \frac{\bar{\mu}_1\bar{\mu}_4 - \bar{\mu}_2^2}{\bar{\mu}_1 + \bar{\mu}_4 - 2\bar{\mu}_2}, \tag{2b}$$

$$b = \frac{\bar{\mu}_1 - a}{\Gamma(1 + 1/c)}. \tag{2c}$$

ESTIMATION OF $\bar{\mu}_k$ AND a, b, c

Given the ordered random sample

$$x_{(1)} \leqslant x_{(2)} \leqslant \ldots \leqslant x_{(n)}$$

from the 3-parameter Weibull distribution, the Cdf$\{x; a, b, c\}$ is estimated by $S_n(x)$:

$$S_n(x) = 0, \quad x < x_{(1)},$$

$$= r/n, \; x_{(r)} \leqslant x < x_{(r+1)}, \quad r = 1, \ldots, n - 1,$$

$$= 1, \quad x_{(n)} \leqslant x.$$

Then $\bar{\mu}_k$ is estimated by:

$$\bar{m}_k = \int_0^\infty \{1 - S_n(x)\}^k dx$$

$$= \sum_{r=0}^{n-1} \left(1 - \frac{r}{n}\right)^k (x_{(r+1)} - x_{(r)}), \quad x_{(0)} = 0. \qquad (3)$$

In particular, $\bar{m}_1 = \bar{x}$, the sample mean.

The Weibull moment estimates a^*, b^*, c^* are obtained from (2) by substituting $\bar{m}_1$, $\bar{m}_2$, $\bar{m}_4$ for $\bar{\mu}_1$, $\bar{\mu}_2$, $\bar{\mu}_4$ respectively and appending * to a, b, c. From (2a) and (2c) it can be shown that the estimates c^* and b^* are non-positive and hence inadmissible when $\bar{m}_2 \geqslant \frac{1}{2}(\bar{m}_1 + \bar{m}_4)$: an alternative method of estimation must then be used. In addition it is possible that the estimate a^* is inadmissible by being negative or by exceeding $x_{(1)}$; in the former case, estimate a by zero; in the latter case one possibility is to use the alternative estimate:

$$a^{**} = x_{(1)} - b^*\Gamma(1 + 1/c^*)/n^{1/c^*}.$$

PROPERTIES OF a^*, b^*, c^*

The properties of the Weibull moment estimators a^*, b^* and c^* have been evaluated by means of a simulation experiment, described in the appendix. The main conclusions drawn from the simulation results are summarised below; more specific information on bias and variation is given in tables 1 and 2. For all (c, n) combinations studied, a^* appears to be negatively biased, b^* and c^* are positively biased. All three estimators show considerable variation: for a^* and c^* this increases dramatically as c increases; the pattern of variation of b^* tends to be more erratic but the variation is a concave-upwards function of c with minimum lying between $c = 1$ and $c = 2$. The correlation coefficients Corr$\{a^*, b^*\}$ and Corr$\{a^*, c^*\}$ are negative, Corr$\{b^*, c^*\}$ is positive and the magnitudes of these coefficients are large, particularly as c increases. These results are similar to those obtained in a similarly designed simulation study of the properties of the conventional moment estimators (as defined by Newby [1]), described in [3].

TABLE 1
Bias of Weibull Moment Estimators

The following results are derived from the simulations described in the appendix: $\bar{a}^*(y)$, $\bar{b}^*(y)$, $\bar{c}^*(y)$ are the means of the 1000 estimates of a, b, c respectively for samples of size n from the standard Weibull distribution with shape parameter c.

Bias of a^* The function tabulated is $n\ \bar{a}^*(y)$ which estimates n Bias$\{a^*(y)\}$. The corresponding results for the Weibull distribution with parameters (a, b, c) are obtained by multiplying the tabulated values by b and adding a.

n \ c	0.5	1.0	1.5	2.0	2.5	3.5
25	−1.02	−0.78	−1.04	−0.95	−2.85	−13.75
50	−1.04	−0.75	−0.74	−0.87	−2.11	− 8.15
100	−1.10	−0.76	−1.21	−1.02	−1.36	− 4.02

Bias of b^* The function tabulated is $n\{\bar{b}^*(y) - 1\}$ which estimates $n \cdot$ Bias$\{b^*(y)\}$. The corresponding results for the Weibull distribution with parameters (a, b, c) are obtained by multiplying the tabulated values by b.

n \ c	0.5	1.0	1.5	2.0	2.5	3.5
25	5.25	1.0	1.25	0.75	2.75	13.5
50	6.0	1.5	1.0	0.5	2.0	8.0
100	5.0	1.0	1.0	1.0	1.0	4.0

Bias of c^* The function tabulated is $n\{\bar{c}^*(y) - c\}$ which estimates $n \cdot$ Bias$\{c^*(y)\}$; the tabulated values also hold for the general Weibull distribution.

n \ c	0.5	1.0	1.5	2.0	2.5	3.5
25	2.38	3.25	5.50	5.75	14.00	65.75
50	2.40	3.50	4.00	5.00	10.50	40.00
100	2.20	3.00	6.00	5.00	9.00	24.00

TABLE 2
Standard Deviations of Weibull Moment Estimators

The following results are derived from the simulations described in the appendix: SD$\{a^*(y)\}$, SD$\{b^*(y)\}$, SD$\{c^*(y)\}$ are the standard deviations of the 1000 estimates of a, b, c respectively for samples of size n from the standard Weibull distribution with shape parameter c.

The functions tabulated are $\sqrt{n} \cdot$ SD$\{a^*(y)\}$ etc. which estimates $\sqrt{n} \cdot x$ standard deviation of the estimator $a^*(y)$, etc. respectively.

Standard deviation of $a^*(y)$ The corresponding results for the Weibull distribution with parameters (a, b, c) are obtained by multiplying the tabulated values by b.

n \ c	0.5	1.0	1.5	2.0	2.5	3.5
25	0.42	0.60	1.12	1.50	3.50	38.60
50	0.35	0.54	0.83	1.17	1.69	7.78
100	0.31	0.52	0.82	1.12	1.42	3.38

Standard deviation of $b^*(y)$ The corresponding results for the general Weibull distribution are obtained as above.

n \ c	0.5	1.0	1.5	2.0	2.5	3.5
25	2.86	1.41	1.48	1.74	3.62	38.65
50	2.82	1.34	1.25	1.42	1.87	7.85
100	2.55	1.45	1.28	1.40	1.63	3.47

Standard deviation of $c^*(y)$ The results also hold for the general Weibull distribution.

n \ c	0.5	1.0	1.5	2.0	2.5	3.5
25	0.73	1.52	3.37	4.88	13.85	165.50
50	0.71	1.36	2.34	3.70	6.02	34.51
100	0.65	1.27	2.17	3.39	5.19	16.60

LOCATION PARAMETER

Estimation is simpler for the 2-parameter Weibull distribution (location parameter $a = 0$, b and c unknown) than for the 3-parameter distribution, eg, in the former case maximum likelihood estimation is regular for $c > 0$ whereas in the latter it is regular only for $c > 2$.

A simple procedure for assessing whether $a = 0$ is reasonable is to estimate the parameters in the 3-parameter model and also in the 2-parameter model and compare the corresponding estimates of b and c. If they are reasonably close then it can be assumed that $a = 0$; otherwise the 3-parameter model is used.

The method of Weibull moment estimation is particularly well suited for the above procedure. The Weibull moments for the 2-parameter Weibull distribution are given by (1) with $a = 0$. Hence the Weibull moment estimates of c and b are respectively

$$c^{**} = (\log_e 2)/(\log \overline{m}_1 - \log \overline{m}_2),$$

$$b^{**} = \overline{m}_1/\Gamma(1 + 1/c^{**}).$$

These estimates can be computed simultaneously with the 3-parameter estimates.

EXAMPLES

The method of Weibull moment estimation is applied to two data sets, each of size 40, given in Harter & Moore [4, table 1]. The conventional method-of-moments estimates are found by the method of Newby [1].

Example 1 (Ex.W2)

Weibull population	$a = 10$	$b = 100$	$c = 2$
Maximum likelihood	2.48	101.8	2.20
Method of moments	−8.67	114.2	2.51
Weibull moments (3)	−3.62	108.7	2.37
Weibull moments (2)	0	104.7	2.26

The estimates of a using the method of moments and Weibull moments (3 parameters) are inadmissible. Since $b^* = 108.7$, $b^{**} = 104.7$, and $c^* = 2.37$, $c^{**} = 2.26$ are approximately the same, and $|a^*|/b^* = 0.03$ is small, it is not unreasonable to consider the location parameter to be zero. Although the true value of a is 10, this is equivalent to 0.1 scale units which is close enough to zero.

Example 2 (Ex.W3)

Weibull population	$a = 20$	$b = 100$	$c = 3$
Maximum likelihood	30.80	83.5	2.33
Method of moments	25.65	89.2	2.52
Weibull moments (3)	27.81	86.8	2.45
Weibull moments (2)	0	116.5	3.47

In this case the two Weibull moment estimates for b and c are substantially different and a^* is positive with the ratio $a^*/b^* = 0.32$ not small. Hence the 3-parameter model with positive location parameter is preferred.

SUMMARY & CONCLUSIONS

Expressions for Weibull moment k and its estimate are derived for the Weibull distribution with non-negative location parameter. Using low order moment estimates the parameters of the 2- and 3-parameter Weibull distributions can be estimated simultaneously, providing a simple procedure for assessing whether the location parameter a is zero and the approximate magnitude of the shape parameter c. The bias and standard deviation of the estimators a^*, b^*, c^* have been investigated in a simulation experiment.

The Weibull distribution is a very widely used family of probability distributions in the analysis of reliability data. Also, in many studies a major interest is to discover whether a "threshold" value exists. The procedure described is a quick, simple method of estimating the parameters of the Weibull distribution and also indicates the likelihood of a "threshold" effect.

APPENDIX

As is well known, properties of the Weibull moment estimators of the parameters of the Weibull distribution with Cdf$\{x; a, b, c\}$ can be inferred from simulations based on the standard Weibull distribution, viz, with Cdf$\{y; 0, 1, c\}$ — as described below.

Let $y_{(1)} \leqslant y_{(2)} \leqslant \dots \leqslant y_{(n)}$ be an ordered pseudo-random sample of size n generated from the standard Weibull distribution giving the Weibull moment estimates $a^*(y)$, $b^*(y)$, $c^*(y)$. For chosen location and scale parameter values a and b, let $x_{(i)} = a + b\, y_{(i)}$. Then $x_{(1)} \leqslant x_{(2)} \leqslant \dots \leqslant x_{(n)}$ can be considered as an ordered pseudo-random sample of size n generated from the Weibull distribution Cdf$\{x; a, b, c\}$. Denote the Weibull moment estimates from this sample by $a^*(x)$, $b^*(x)$, $c^*(x)$.

From (3) the sample Weibull moments for the two samples are related by

$$\overline{m}_k(x) = a + b\, \overline{m}_k(y), \qquad k = 1, 2, \dots$$

hence from (2) it can be shown that

$$c^*(x) = c^*(y)$$

$$a^*(x) = a + b \cdot a^*(y)$$

$$b^*(x) = b \cdot b^*(y).$$

1000 Samples were generated for each (c, n) combination where —

$$c = 0.5,\ 1,\ 1.5,\ 2,\ 2.5,\ 3.5$$

$$n = 5,\ 10,\ 25,\ 50,\ 100.$$

For each combination, the summary statistics such as mean, standard deviation, percentile points, and correlations were computed for the Weibull moment estimates of the parameters. Samples providing non-positive estimates c^* and b^* were omitted: for $n = 25, 50, 100$ there were only 11 such samples, all associated with $c = 3.5$; the frequency of such inadmissible estimates increases as n decreases and c increases. Inadmissible estimates of a were included in the summary statistics rather than introduce subjectively chosen alternative estimators.

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