Comparison of nonlinear height–diameter functions for major Alberta tree species

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Twenty nonlinear height–diameter functions were fitted and evaluated for major Alberta species based on a data set consisting of 13 489 felled trees for 16 different species. All functions were fitted using weighted nonlinear least squares regression ($w_i = 1/DBH_i$) because of the problem of unequal error variance. The examination and comparison of the weighted mean squared errors, the asymptotic t-statistics for the parameters, and the plots of studentized residuals against the predicted height show that many concave and sigmoidal functions can be used to describe the height–diameter relationships. The sigmoidal functions such as the Weibull-type function, the modified logistic function, the Chapman–Richards function, and the Schnute function generally gave the most satisfactory results.

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Vingt modèles non linéaires exprimant la hauteur en fonction du diamètre ont été ajustés et évalués pour les principales espèces de l'Alberta. La base de données utilisée consistait en 13 489 arbres abattus répartis en 16 différentes espèces. Tous les modèles de régression non linéaire ont été ajustés par la méthode des moindres carrés pondérés ($w_i = 1/DHP_i$) dû à la présence d'hétéroscédasticité. L'examen et la comparaison des erreurs résiduelles pondérées, la statistique t des coefficients et le graphique des résidus standardisés en fonction de la hauteur estimée démontrent que plusieurs modèles concaves et sigmoïdes peuvent décrire les relations hauteur–diamètre. Les fonctions sigmoïdes telles que celles de Weibull, de Chapman–Richards et de Schnute présentent généralement les meilleurs résultats.

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Introduction

Predicting total tree height based on observed diameter at breast height outside bark is routinely required in practical management and silvicultural research work (Meyer 1940). The estimation of tree volume, as well as the description of stands and their development over time, relies heavily on accurate height–diameter functions (Curtis 1967). Many growth and yield models also require height and diameter as two basic input variables, with all or part of the tree heights predicted from measured diameters (Burkhart *et al.* 1972; Curtis *et al.* 1981; Wykoff *et al.* 1982). In the cases where the actual measurements of height growth are not available, height–diameter functions can also be used to indirectly predict height growth (Larsen and Hann 1987).

Curtis (1967) summarized a large number of available height–diameter functions and used Furnival's index of fit to compare the performance of 13 linear functions fitted to second-growth Douglas-fir (*Pseudotsuga menziesii* (Mirb.) Franco) data. Since then, many new height–diameter functions have been developed. With the relative ease of fitting nonlinear functions and the nonlinear nature of the height–diameter relationships, nonlinear height–diameter functions have now been widely used in height predictions (Schreuder *et al.* 1979; Curtis *et al.* 1981; Wykoff *et al.* 1982; Wang and Hann 1988; Farr *et al.* 1989; Arabatzis and Burkhart 1992).

For 16 Alberta tree species in 9 groups, this study compared 11 published nonlinear height–diameter functions as well as 9 nonlinear functions that apparently have not been

applied to height–diameter relationships. The primary objectives of the study are to evaluate the relative performance of a variety of potential height–diameter functions on a large, regional data set covering numerous species, and to identify the most appropriate height–diameter functions for major Alberta tree species.

The data

Alberta Forest Service provided felled-tree data for this analysis. Collected over the last 2 decades, the 13 489 trees were randomly selected throughout the inventory areas of the province to provide representative information for a variety of densities, heights, species compositions, stand structures, ages, and site conditions. The data set was initially used for developing individual tree volume equations and includes many different variables for individual trees and qualitative characteristics of their surrounding environment. A detailed description of how the data are collected and recorded can be found in *Alberta Phase 3 Forest Inventory: Tree Sectioning Manual* (Alberta Forest Service 1988). Two variables available from the records, diameter at breast height (DBH) outside bark and total tree height (H) for each tree, were selected to be used in this analysis.

The 13 489 trees included 16 different species. To facilitate the analysis, species are classified into different species groups according to their similarity, management objectives, and number of observations (Table 1). Summary statistics including the mean, minimum, maximum, and standard deviation for total tree height and DBH by species group are shown in Table 2. The variation in number of sample trees by

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TABLE 1. Species and species groups

Species group	Species	Scientific name
1	White spruce	*Picea glauca* (Moench) Voss
2a	Lodgepole pine	*Pinus contorta* var. *latifolia* Engelm.
	Whitebark pine	*Pinus albicaulis* Engelm.
	Limber pine	*Pinus flexilis* James
2b	Jack pine	*Pinus banksiana* Lamb.
3	Aspen	*Populus tremuloides* Michx.
4a	White birch	*Betula papyrifera* Marsh.
4b	Balsam poplar	*Populus balsamifera* L.
5	Black spruce	*Picea mariana* (Mill.) B.S.P.
	Engelmann spruce	*Picea engelmannii* Parry
6a	Balsam fir	*Abies balsamea* (L.) Mill.
6b	Douglas-fir	*Pseudotsuga menziesii* (Mirb.) Franco
	Alpine fir	*Abies lasiocarpa* (Hook.) Nutt.
	Alpine larch	*Larix lyallii* Parl.
	Tamarack	*Larix laricina* (Du Roi) K. Koch
	Western larch	*Larix occidentalis* Nutt.

TABLE 2. Tree summary statistics based on species group

Species group	No. of sample trees	DBH (cm)				Total tree height (m)			
		Mean	Min.	Max.	SD	Mean	Min.	Max.	SD
1	3101	26.41	1.20	89.00	12.19	20.09	1.70	38.40	6.98
2a	3199	22.10	1.10	66.60	8.59	18.11	1.72	37.60	5.18
2b	659	18.01	1.60	45.00	9.81	14.74	2.58	28.20	6.38
3	3647	21.36	1.10	64.40	10.12	18.77	2.23	31.94	5.46
4a	102	12.11	1.60	32.00	5.87	11.88	3.18	21.50	4.13
4b	510	22.75	1.10	52.90	9.79	17.76	2.90	31.95	4.88
5	1628	14.10	1.10	55.30	6.08	12.20	1.76	30.63	4.26
6a	508	21.15	1.30	53.00	9.19	16.11	1.78	31.40	5.50
6b	135	20.60	3.30	48.70	9.72	13.26	3.35	22.33	4.98

NOTE: See Table 1 for species groups. DBH, diameter at breast height.

species group is an indication of relative importance. Lack of consistent quantitative variables for all data prevented using stand characteristics as additional independent variables.

Functions selected for comparison

The selection of the height–diameter functions was based on the examination of the height–diameter relationship as revealed by plotting total tree height against DBH for various species groups. Two typical examples for white spruce (*Picea glauca* (Moench) Voss) and aspen (*Populus tremuloides* Michx.) are shown in Figs. 1 and 2. It is clear that the height–diameter relationship for white spruce (Fig. 1) has a typical sigmoidal shape, with an inflection point occurring in the lower portion of the data points. On the other hand, the shape of the height–diameter relationship for aspen (Fig. 2) may be regarded as either concave or sigmoidal, with no apparent inflection point. The sigmoidal–concave shape reflects the strong correlation between DBH and age. Both the typical concave functions and the sigmoidal functions were selected for evaluations. Additional nonlinear functions that are common in biological studies were also selected by considering the plots of height versus DBH compared with the typical graphs of the various functions. Table 3 provides a complete list of the selected functions. Notice that some of the functions (such as 1 and 6) often appear in transformed forms, and the

dependent variable may take the form of $H - 1.3$ (Curtis 1967). The quadratic height–diameter functions, first presented by Trorey (1932) and advocated by Ker and Smith (1955) and previously used in the Pacific Northwest (Staebler 1954) and British Columbia (Watts 1983), were not considered because extrapolation of the functions often leads to unrealistic height predictions.

Methods

A fundamental nonlinear least squares assumption is that the error terms in all 20 height–diameter functions are independent and identically distributed, with zero mean and constant variance. However, in many forestry situations there is a common pattern of increasing variation as values of the dependent variable increase. This is clearly evident from the scatterplots of height versus DBH in Figs. 1 and 2, where the values of the error are more likely to be small for small DBH and large for large DBH. When the problem of unequal error variances occurs, weighted nonlinear least squares (WLS) is applied, with the weights selected to be inversely proportional to the variance of the error terms.

The WLS estimates of the parameters use an iterative process with a starting value chosen and continually improved until the weighted error sum of squares is minimized. It should be noted that the use of the WLS changes the estimates of the

TABLE 3. Nonlinear height–diameter functions selected for comparison

Function No. and form†	References
[1] $H = 1.3 + aD^b$	Stoffels and van Soest 1953; Stage 1975; Schreuder et al. 1979
[2] $H = 1.3 + e^{a+b/(D+1)}$	Wykoff et al. 1982
[3] $H = 1.3 + aD/(b + D)$	Bates and Watts 1980; Ratkowsky 1990
[4] $H = 1.3 + a\left(1 - e^{-bD}\right)$	Meyer 1940; Farr et al. 1989; Moffat et al. 1991
[5] $H = 1.3 + D^2/(a + bD)^2$	Loetsch et al. 1973
[6] $H = 1.3 + a\,e^{b/D}$	Burkhart and Strub 1974; Burk and Burkhart 1984; Buford 1986
[7] $H = 1.3 + 10^a D^b$	Larson 1986
[8] $H = 1.3 + aD/(D + 1) + bD$	Watts 1983
[9] $H = 1.3 + a(D/(1 + D))^b$	Curtis 1967; Prodan 1968
[10] $H = 1.3 + e^{a+bD^c}$	Curtis et al. 1981; Larsen and Hann 1987; Wang and Hann 1988
[11] $H = 1.3 + a/\left(1 + b\,e^{-cD}\right)$	Pearl and Reed 1920
[12] $H = 1.3 + a\left(1 - e^{-bD}\right)^c$	Richards 1959
[13] $H = 1.3 + a\left(1 - e^{-bD^c}\right)$	Yang et al. 1978
[14] $H = 1.3 + a\,e^{-b \cdot e^{-cD}}$	Winsor 1932
[15] $H = \left\{y_1^b + \left(c^b - y_1^b\right)\left[1 - e^{-a(D-D_0)}\right] \div \left[1 - e^{-a(D_2-D_0)}\right]\right\}^{1/b}$	Schnute 1981
[16] $H = 1.3 + D^2/\left(a + bD + cD^2\right)$	Curtis 1967; Prodan 1968
[17] $H = 1.3 + aD^{bD^{-c}}$	Sibbesen 1981
[18] $H = 1.3 + a\,e^{b/(D+c)}$	Ratkowsky 1990
[19] $H = 1.3 + a/\left(1 + b^{-1}D^{-c}\right)$	Ratkowsky and Reedy 1986
[20] $H = 1.3 + a\left(1 - b\,e^{-cD}\right)^d$	Richards 1959

†H, total tree height (m); D, DBH (cm); a, b, c, d, parameters to be estimated; e, base of the natural logarithm ($\approx 2.718\,28$); 1.3 is a constant used to account that DBH is measured at 1.3 m above the ground. For eq. 15: $y_1 = 1.3$, $D_0 = 0.0$, $D_2 = 100.0$.

parameters and the standard errors of the estimates relative to the values obtained in the absence of weighting (Ratkowsky 1990). The interpretations of the weighted statistics are not as straightforward as those in the cases of unweighted statistics (Carroll and Ruppert 1988). However, comparison of the fit statistics for various functions can be made if the same weight is consistently used in all the function fittings and the same nonlinear least squares iteration procedure is used.

The use of the WLS requires a known weight. In many practical applications, however, this weight may not be readily available, so an estimate based on the results of an unweighted least squares fit is often necessary. Although there are many different procedures that are available for approximating the weight or implementing the generalized nonlinear

least squares techniques (Gallant 1987; Judge et al. 1988), a simpler procedure that is based on the analysis of the studentized residuals can be equally efficient.

Studentized residuals are the scaled version of residuals that are obtained by dividing each residual by its standard error. They are designed to take into account that unstandardized residuals have intrinsically unequal variances even though the theoretical error term is assumed to have constant variance (Draper and Smith 1981; Rawlings 1988; Neter et al. 1990). For a correctly identified function, when the assumptions of the regression analysis are met, the studentized residuals have zero mean and constant variance, and the plot of studentized residuals against the predicted values of the dependent variable will show a homogeneous band.

TABLE 4. Parameter estimations for two-parameter height–diameter functions

Function†	Parameter	Estimates for various species groups‡								
		1	2a	2b	3	4a	4b	5	6a	6b
[1]	a	1.7313	2.0196	1.3150	2.8211	1.9024	2.6947	1.2137	1.2469	1.1000
	b	0.7353	0.6899	0.8126	0.6056	0.6986	0.5871	0.8344	0.8163	0.7954
[2]	a	3.6042	3.4766	3.3789	3.3910	3.0097	3.3238	3.2087	3.4184	3.2256
	b	−16.1901	−13.8574	−12.6489	−10.1272	−7.5330	−10.9470	−11.3747	−14.3731	−14.1907
[3]	a	62.9784	51.4152	65.6462	39.9983	33.4618	37.0257	59.4777	58.3695	51.2611
	b	58.0915	43.2873	65.5679	24.7274	24.2608	26.0386	60.7484	59.0756	64.0364
[4]	a	38.8548	32.4692	37.9810	27.1294	21.3657	25.3302	34.1127	34.1281	29.9225
	b	0.0270	0.0349	0.0260	0.0549	0.0614	0.0512	0.0283	0.0285	0.0263
[5]	a	1.8737	1.6413	1.6840	1.1800	1.0601	1.3209	1.5986	1.8069	2.0261
	b	0.1519	0.1639	0.1666	0.1753	0.2089	0.1813	0.1814	0.1663	0.1805
[6]	a	35.2854	30.8991	27.5419	28.2674	18.3182	26.6049	22.7872	29.3762	23.8673
	b	−14.4531	−12.1948	−10.7183	−8.5907	−5.6927	−9.4854	−9.3829	−12.8412	−12.3567
[7]	a	0.2388	0.3048	0.1189	0.4509	0.2793	0.4305	0.0838	0.0953	0.0413*
	b	0.7350	0.6903	0.8126	0.6053	0.6986	0.5871	0.8347	0.8167	0.7955
[8]	a	3.8180	4.9317	2.0670	6.4194	3.2636	6.5487	1.5058	1.3123	1.8794
	b	0.5738	0.5487	0.6401	0.5349	0.6306	0.4507	0.6746	0.6418	0.4951
[9]	a	35.9867	31.6026	28.3882	28.9552	19.2299	27.1752	23.6995	29.9060	24.4681
	b	15.2897	13.0009	11.6357	9.3290	6.5500	10.1979	10.3221	13.5674	13.2207

NOTE: *, The asymptotic t-statistic for the parameter is not significant at the 0.05 α level.
†See Table 3 for the form of the function.
‡See Table 1 for species groups.
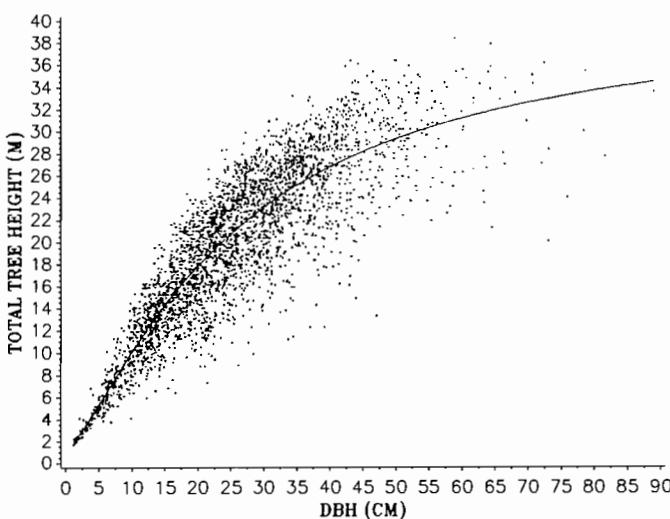
FIG. 1. Plot of total tree height against DBH for white spruce (*Picea glauca*). The curve was produced by $H = 1.3 + 39.3710/[1 + 1/(0.0130\text{DBH}^{1.3408})]$.

Figure 3 shows an example of the plot of studentized residuals against the predicted height for the modified logistic function [19] (Table 3) fitted to aspen data with unweighted nonlinear least squares. The plot reveals an obvious unequal error variance problem and suggests that a weighting factor in the form of $w_i = 1/\text{DBH}_i^k$ should achieve the desired equality of error variance. This function was then fitted with WLS using six alternative values for k ($k = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$). Among these alternative weights, the most homogeneous band of studentized residuals occurred with $k = 1.0$ (Fig. 4). In similar comparisons, $w_i = 1/\text{DBH}_i$ was also found to be most appropriate for all other species. Accordingly, this weighting factor was used in all remaining analyses. This

weight also agrees with the weight chosen by Larsen and Hann (1987), Wang and Hann (1988), and Farr *et al.* (1989) based on different procedures.

The fitting of the height–diameter functions for various species groups was accomplished using the PROC NLIN procedure on SAS software (SAS Institute Inc. 1985). The Gauss–Newton method as described in Gallant (1987) was applied, and multiple starting values for parameters were provided to ensure that the least squares solution was the global rather than the local solution.

Results and discussion

Three different criteria were selected for judging the performance of the height–diameter functions: (*i*) the asymptotic t-statistics of the parameters, (*ii*) the weighted mean squared error (MSE) of the model, and (*iii*) the plot of studentized residuals against the predicted height. For any appropriate height–diameter function, the asymptotic t-statistic for each coefficient should be significant, and the model MSE should be small. The studentized residual plot should show approximately homogeneous variance over the full range of predicted values. Any other pattern may indicate bias, unequal variation, or other problems such as outliers or poor model specification.

Tables 4, 5, and 6 show the least squares estimates of the parameters. The associated asymptotic t-statistics for testing the null hypothesis that each parameter is zero (or in some models, one) are calculated, and the insignificant parameters are marked. The weighted MSE are summarized in Table 7. Although not reported here, coefficient of determination (R^2) values for the fitted functions on weighted observations ranged from 0.70 to 0.92, with the average being about 0.85.

Results in Table 4 show that for the two-parameter functions, [1] to [9], with the exception of parameter a in function [7] for species group 6b, all the t-statistics for the parameters of the functions are significant at the 0.05 α level.

TABLE 5. Parameter estimations for three-parameter height–diameter functions

Function[†]	Parameter	Estimates for various species groups[‡]								
		1	2a	2b	3	4a	4b	5	6a	6b
[10]	a	4.3207	4.2512	6.1440	3.8984	6.1541	4.3133	4.6202	4.0034	4.4488
	b	−6.5426	−5.7514	−6.6024	−4.7580	−5.8482	−4.5425	−5.6452	−6.4430	−6.0225
	c	−0.4872	−0.4588	−0.2204	−0.5182	−0.1778*	−0.3614	−0.3577	−0.5375	−0.3793
[11]	a	26.0850	23.7434	21.8863	22.5297	16.9311	21.5241	17.0593	19.2315	17.3308
	b	8.5482	5.9593	8.5656	5.9461	5.7035	5.0012	8.5954	15.9742	9.7975
	c	0.1339	0.1311	0.1612	0.1704	0.1996	0.1404	0.2063	0.2204	0.1703
[12]	a	32.0363	29.4214	31.7252	25.7461	25.3245	26.0462	25.0216	23.6894	22.3239
	b	0.0456	0.0457	0.0376	0.0669	0.0409*	0.0464	0.0518	0.0724	0.0522
	c	1.2974	1.1381	1.1150	1.1308	0.8779	0.9465	1.2004	1.6232	1.3270
[13]	a	31.0481	29.0401	29.8908	25.4088	26.2522	26.1321	24.5127	22.4771	20.8982
	b	0.0209	0.0318	0.0269	0.0486	0.0579	0.0535	0.0308	0.0179	0.0219
	c	1.1973	1.0902	1.1061	1.0892	0.9017	0.9659	1.1361	1.3905	1.2490
[14]	a	27.8725	25.2831	24.1320	23.5467	18.4726	22.6368	18.8367	20.9530	19.0959
	b	2.8490	2.4343	2.7151	2.3800	2.2367	2.1570	2.8446	3.6061	2.9034
	c	0.0848	0.0873	0.0943	0.1152	0.1235	0.0951	0.1247	0.1259	0.0988
[15]	a	0.0494	0.0466	0.0450	0.0696	0.0382*	0.0464	0.0536	0.0929	0.0685
	b	0.6387	0.8289	0.7717	0.8151	1.2179	1.0716	0.7411	0.2072	0.4335*
	c	32.4840	30.3314	30.5534	26.8357	26.5976	27.0745	26.1924	23.8101	21.9696
[16]	a	2.6944	1.4431	0.3504*	0.8408	−0.2324*	0.0038*	1.2706	4.4024	2.4627*
	b	0.6514	0.6806	0.9442	0.4951	0.7813	0.7027	0.8044	0.4670	0.9370
	c	0.0214	0.0233	0.0168	0.0284	0.0273	0.0270	0.0246	0.0311	0.0273
[17]	a	36.8921	28.4645	39.5300	26.1702	22.7752	22.9433	20.8584	27.8154	35.2386
	b	−13.0405	−16.5206	−8.3474	−13.1935	−7.4274	−20.9985	−14.1796	−15.7403	−8.1497
	c	1.3051	1.5168	1.1040	1.5795	1.3156	1.7680	1.5780	1.4637	1.0545
[18]	a	43.4552	38.6721	43.7438	33.6553	31.0846	33.2971	31.7946	34.2258	33.0533
	b	−24.1871	−21.4197	−28.1548	−14.5592	−18.4473	−18.4014	−18.5302	−18.7186	−25.2112
	c	5.0167	5.0827	7.3227	3.5766	5.8302	5.5088	4.0490	3.1265	5.8787
[19]	a	39.3710	37.5445	46.1750	31.3194	41.9635*	34.4682	32.8728	27.6307	28.4451
	b	0.0130	0.0203	0.0174	0.0328	0.0365	0.0369	0.0204	0.0109	0.0146
	c	1.3408	1.2169	1.1253	1.2487	0.9155	1.0589	1.2307	1.5829	1.3299

NOTE: *, The asymptotic *t*-statistic for the parameter is not significant at the 0.05 α level.
†See Table 3 for the form of the function.
‡See Table 1 for species groups.

The weighted MSE results of the two-parameter functions shown in Table 7 indicate that functions [3], [4], and [5] have lower MSE values compared with the others, with function [4] generally giving the most satisfactory results. Function [8] has very poor performance and large MSE values. Examination of the plots of studentized residuals for function [8] shows biased height estimates for all species groups when DBH is small. The performance of the remaining two-parameter functions is roughly the same and can be regarded as intermediate.

Judged from the plots of studentized residuals and the weighted MSE values, the three-parameter functions, [10] to [19], generally perform better than the two-parameter functions. Parameter *a* in function [16] shows several insignificant *t*-statistics (Table 5). The parameter estimates for the remaining functions are generally satisfactory, with a few exceptions of insignificant *t*-statistics in functions [10], [12], [15], and [19] for species group 4a and parameter *b* in function [15] for species group 6b. Insignificant *t*-statistics were generally associated with small data sets. In terms of the weighted MSE values for three-parameter functions (Table 7), functions [12], [13], [15], [18], and [19] generally give lower values. Functions [10] and [14] give rather similar results and can also be regarded as satisfactory. Function [17] has large MSE values, and the plots of studentized residuals

show biased estimates when DBH is small. Occasionally, function [11] fits the data well, but in general it performs poorly.

Although the four-parameter function, [20], fits the data well when the sample size is large (such as for species groups 1 and 3), the function fails to converge for species groups 2b and 6b, and in fitting for species group 4a, has resulted in insignificant *t*-statistics for parameters *b*, *c*, and *d* (Table 6). Several additional four-parameter functions (included Bailey's 1980 function) that were fitted, but not reported here, also suggested that they might perform well for large samples; however, insignificant *t*-statistics occurred frequently, and in many cases, failed to coverage or converged at the local rather than the global minimum when the sample size was small. The gain of using the four-parameter function may not be substantial. Depending on the choice of the initial values of the parameters and the size of the samples, the fittings of the four-parameter functions may also be rather time-consuming.

In terms of the fit of the functions for each species group, several functions may give similar results and perform nearly equally well. However, judging from the weighted MSE values, the asymptotic *t*-statistics of the parameters, and the principle of parsimony, the following functions are most appropriate for each species group taken independently of the

TABLE 6. Parameter estimations for the four-parameter height–diameter function

Function†	Parameter	Estimates for various species groups‡								
		1	2a	2b	3	4a	4b	5	6a	6b
[20]	a	32.5525	30.8722	24.4874**	25.4676	20.7813	25.2716	31.3035	23.3678	17.8206**
	b	1.0200	1.0413	0.2528**	0.9687	0.8247*	0.9574	1.0334	0.9716	0.0314**
	c	0.0428	0.0383	0.0878**	0.0709	0.0767*	0.0530	0.0298	0.0766	0.1069**
	d	1.2034	0.9570	9.4899**	1.2419	1.5810*	1.1025	0.8964	1.7781	90.9244**

NOTE: *, The asymptotic t-statistic for the parameter is not significant at the 0.05 α level; **, covergence is not obtained.
†See Table 3 for the form of the function.
‡See Table 1 for species groups.
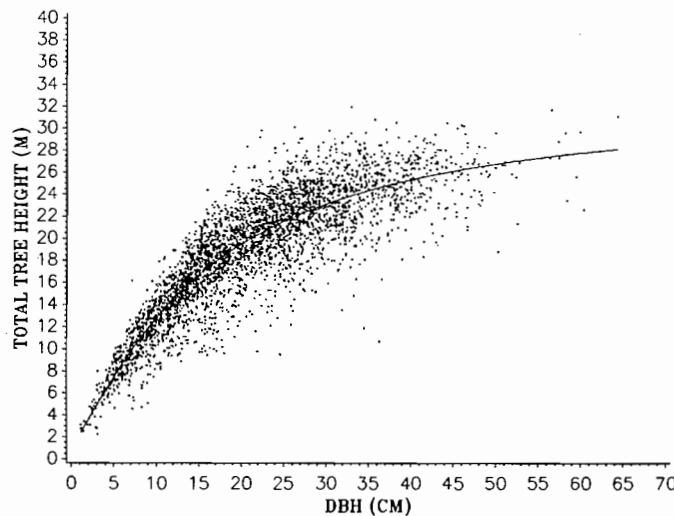
FIG. 2. Plot of total tree height against DBH for aspen (*Populus tremuloides*). The curve was produced by $H = 1.3 + 31.3194/[1 + 1/(0.0328DBH^{1.2487})]$.
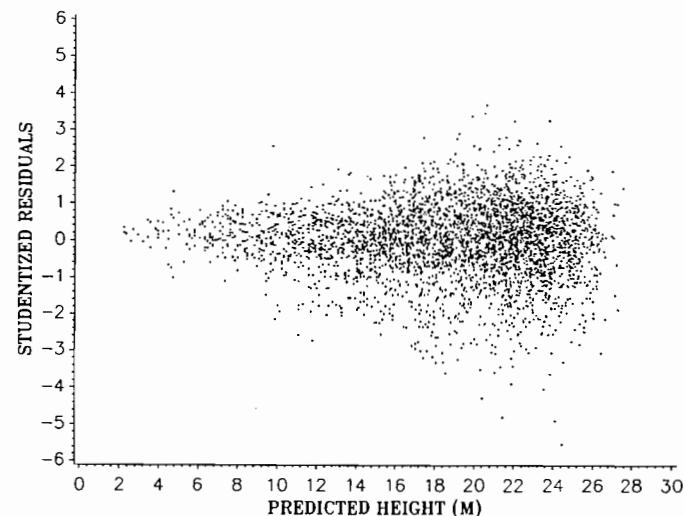

FIG. 3. The plot of studentized residuals against the predicted height for aspen (*Populus tremuloides*). The studentized residuals were obtained by fitting function [19] without weighting.

others: (*i*) the Chapman–Richards function [12] for species group 1; (*ii*) the fractional function [16] for species group 2a; (*iii*) the Gompertz function [14] for species groups 2b and 6b; (*iv*) the Weibull function [13] and the modified Schnute function [15] for species group 3; (*v*) the two-parameter Michaelis–Menten function [3] for species group 4a; (*vi*) the Mitscherlich function [4] for species group 4b; (*vii*) the modified exponential function [10] for species group 5; (*viii*) the modified logistic-type function [19] for species group 6a.

Conclusions and recommendations

This comparison of nonlinear height–diameter functions show that depending on the sample sizes and the species groups, many functions perform well in describing the height–diameter relationships for major Alberta tree species. The choice of a particular function may depend on the relative ease of achieving convergence to a solution, the function's mathematical properties, and its biological interpretation. Although any function may be considered superior or inferior in a particular situation, in general, the functions discussed below are recommended for use since they often give relatively lower MSE values, significant asymptotic t-statistics, and satisfactory plots of studentized residuals against the predicted values of the dependent variable. Any one of these functions could be used when the same model form is desir-

able for several species. The recommended functions also have the flexibility to assume various shapes with different parameter values and produce satisfactory curves under most circumstances. All the curves assume biologically reasonable shapes that prevent the unrealistic height predictions in the cases of extrapolating the functions beyond the range of the original data.

Function [12]: $H = 1.3 + a(1 - e^{-bD})^c$

This three-parameter Chapman–Richards function has been used extensively in describing the height–age relationships. The results shown in this analysis indicate that the function is also well suited for modelling height–diameter relationships. One limiting form of the function, function [14], also gives satisfactory fits, especially when the sample size is relatively small, such as the fits for species groups 2b, 4a, and 6b. However, function [14] may not fit as well as either the Weibull-type function or the Chapman–Richards function when the sample size is large. A cautionary note for the Chapman–Richards function is that it approaches the asymptote too quickly when the dependent variable is only weakly related to the independent variable.

Function [13]: $H = 1.3 + a(1 - e^{-bD^c})$

This Weibull-type function is consistently among the best height–diameter functions. It is interesting to see that in fitting species group 4a data, the three- or four-parameter Chapman–

TABLE 7. Comparison of nonlinear height–diameter functions: weighted mean squared errors

Function[†]	Weighted MSE for various species groups[‡]								
	1	2a	2b	3	4a	4b	5	6a	6b
[1]	0.5082	0.3886	0.2938	0.3863	0.3326	0.3677	0.2685	0.3366	0.3359
[2]	0.4675	0.3800	0.3564	0.3370	0.3781	0.3687	0.2702	0.2577	0.3373
[3]	0.4596	0.3702	0.2770	0.3265	0.3257(1)	0.3465	0.2571	0.2865	0.3171
[4]	0.4539	0.3686	0.2748	0.3189(5)	0.3261(2)	0.3454(1)	0.2566	0.2813	0.3147
[5]	0.4443	0.3685	0.2976	0.3218	0.3557	0.3542	0.2547	0.2466	0.3124
[6]	0.4832	0.3891	0.4049	0.3599	0.4268	0.3828	0.2876	0.2701	0.3584
[7]	0.5082	0.3886	0.2938	0.3863	0.3326	0.3677	0.2685	0.3366	0.3359*
[8]	0.5841	0.4210	0.3170	0.4778	0.3539	0.4112	0.2843	0.3894	0.3588
[9]	0.4751	0.3842	0.3801	0.3477	0.4009	0.3756	0.2784	0.2642	0.3474
[10]	0.4459	0.3668(2)	0.2804	0.3234	0.3295*	0.3500	0.2502(1)	0.2516	0.3158
[11]	0.4935	0.3877	0.2701(2)	0.3400	0.3314	0.3598	0.2915	0.2865	0.3035(2)
[12]	0.4424(2)	0.3674(4)	0.2729	0.3166(2)	0.3268*	0.3458(2)	0.2534(5)	0.2449(2)	0.3089(4)
[13]	0.4426(3)	0.3675(5)	0.2723(5)	0.3165(1)	0.3271(4)	0.3459(3)	0.2539	0.2458(5)	0.3080(3)
[14]	0.4597	0.3761	0.2648(1)	0.3236	0.3267(3)	0.3504	0.2694	0.2533	0.3024(1)
[15]	0.4430(5)	0.3676	0.2717(4)	0.3165(1)	0.3272*	0.3459(3)	0.2544	0.2463	0.3069*
[16]	0.4435	0.3667(1)	0.2767*	0.3200	0.3275*	0.3472*	0.2510(3)	0.2460	0.3131*
[17]	0.4658	0.4033	0.2857	0.3795	0.3596	0.4192	0.3014	0.2568	0.3089(4)
[18]	0.4433	0.3677	0.2708(3)	0.3181(4)	0.3273(5)	0.3463(5)	0.2540	0.2455(4)	0.3090(5)
[19]	0.4427(4)	0.3671(3)	0.2747	0.3180(3)	0.3279*	0.3469	0.2525(4)	0.2445(1)	0.3100
[20]	0.4423(1)	0.3668(2)	0.2662*	0.3165(1)	0.3288*	0.3462(4)	0.2507(2)	0.2453(3)	0.3100*

NOTE: *, The MSE values are not compared because of insignificant t-statistic(s) or the failure of convergence.
†See Table 3 for the form of the function.
‡The smallest five MSE values for each species group, with ranks 1 (smallest) to 5 in parentheses.
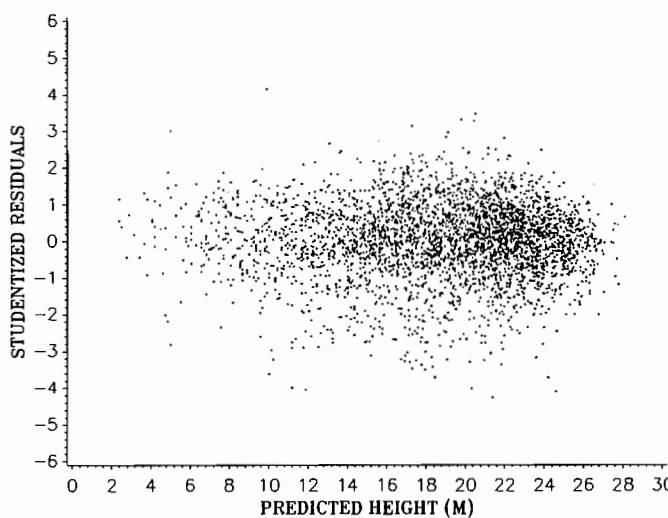

FIG. 4. The plot of studentized residuals against the predicted height for aspen (*Populus tremuloides*). The studentized residuals were obtained by fitting function [19] with weight $w_i = 1/DBH_i$.

Richards function fails to produce a significant t-statistic for parameter b. However, the Weibull function performs better and gives all significant t-statistics for the parameters.

Function [19]: $H = 1.3 + a/(1 + b^{-1}D^{-c})$

Although termed as the modified logistic-type function, the function is quite different from the commonly used logistic function (such as function [11]). It accommodates many shapes that are commonly described by other sigmoidal functions. The function fits the height–diameter relationship well and is consistently among the best height–diameter functions. As examples, the fits of the function for white spruce and aspen are shown in Figs. 1 and 2. The plot of studentized residuals against the predicted height for aspen is shown in Fig. 4. It is clear that the function appropriately fits the data.

Function [18]: $H = 1.3 + a\ e^{b/(D+c)}$

This exponential-type function is particularly well suited for deciduous species. However, it might slightly overestimate height for large-diameter trees.

Function [15]:

$$H = \left\{ y_l^b + \left(c^b - y_l^b \right)\left[1 - e^{-a(D - D_0)} \right] / \left[1 - e^{-a(D_2 - D_0)} \right] \right\}^{1/b}$$

This modified Schnute function (with origin set at $D = 0$, $H = 1.3$) was shown to fit the height–diameter relationships reasonably well. With the versatility of this function and its abilities to describe various biological shapes, and the relatively easy parameter estimations and interpretations, further application and evaluation of the function should prove useful.

It should be straightforward to extend the functions analyzed in this study to model other forestry relationships such as volume–age, height–age, and basal area – age functions. The parameter estimates in Tables 4, 5, and 6, if appropriately scaled, might be useful as the initial values in new applications.

Acknowledgements

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