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Author(s): A. Clifford Cohen
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Maximum Likelihood Estimation in the Weibull Distribution Based On Complete and On Censored Samples

A. CLIFFORD COHEN*
The University of Georgia

This paper is concerned with the two-parameter Weibull distribution which is widely employed as a model in life testing. Maximum likelihood equations are derived for estimating the distribution parameters from (i) complete samples, (ii) singly censored samples and (iii) progressively (multiple) censored samples. Asymptotic variance-covariance matrices are given for each of these sample types. An illustrative example is included.

1. INTRODUCTION

Because of its versatility in fitting time-to-failure distributions of a rather extensive variety of complex mechanisms, the Weibull distribution has in recent years assumed a position of importance in the field of reliability and life testing. Various problems associated with this distribution have been considered by numerous writers, among whom are Dubey [3], Esary and Proschan [4], Jaech [5], Kao [6, 7, 8], Lehman [10], Leone, Rutenberg, and Topp [11], Lloyd and Lipow [12], Menon [13], Procassini and Romano [14], and Proschan [15]. A major deterrent to wider usage of the Weibull distribution has been the difficulty in estimating its parameters. Unfortunately, the calculations involved are not always simple.

This paper is concerned with maximum likelihood estimation in both complete and censored samples from the two-parameter Weibull distribution with density function

$$f(x) = (\gamma/\theta)x^{\gamma-1} \exp(-x^{\gamma}/\theta); \qquad x \geq 0, \quad \gamma > 0, \quad \theta > 0. \tag{1}$$

When $\gamma = 1$, this becomes the density function of the well-known one-parameter exponential distribution. The particular form in which (1) is written was chosen for the purpose of simplifying derivations of the maximum likelihood esti-

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The readers may be interested in noting the article entitled "Maximum-Likelihood Estimation of the Parameters of Gamma and Weibull Populations from Complete and from Censored Samples" by H. Leon Harter and Albert H. Moore which, appears in this same issue. Each of these papers was carried out independently of the other. (Editor's comment.)

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mating equations. As written here, though differing in form, (1) is equivalent to the two-parameter Weibull density function considered by Proschan [15], Menon [13] and others.

Estimating equations obtained here for complete samples are presented in a form which seems to be somewhat simpler to solve than equivalent equations given previously by other writers. Estimating equations for censored samples which arise naturally from life tests which are discontinued before all sample specimens fail, are presented in a form analogous to that employed for complete samples. Apparently censored samples from the Weibull distribution have received little if any previous attention in the literature. Asymptotic variances and covariances are given for the estimators obtained here from both complete and censored samples.

2. Complete Samples

Consider a random sample consisting of n observations when (1) is the applicable density function. The likelihood function of this sample is

$$L(x_1, \cdots x_n; \gamma, \theta) = \prod_{i=1}^{n} (\gamma/\theta)x_i^{\gamma-1} \exp(-x_i^{\gamma}/\theta). \tag{2}$$

On taking logarithms of (2), differentiating with respect to γ and θ in turn and equating to zero, we obtain the estimating equations

$$\left.\begin{aligned} \frac{\partial \ln L}{\partial \gamma} &= \frac{n}{\gamma} + \sum_1^n \ln x_i - \frac{1}{\theta}\sum_1^n x_i^{\gamma} \ln x_i = 0, \\ \frac{\partial \ln L}{\partial \theta} &= -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_1^n x_i^{\gamma} = 0. \end{aligned}\right\} \tag{3}$$

On eliminating θ between these two equations and simplifying, we have

$$\left[\frac{\sum_1^n x_i^{\gamma} \ln x_i}{\sum_1^n x_i^{\gamma}} - \frac{1}{\gamma}\right] = \frac{1}{n}\sum_1^n \ln x_i, \tag{4}$$

which may now be solved for the M. L. estimate $\hat{\gamma}$. This can be accomplished with the aid of standard iterative procedures, but in most instances a simple trial and error approach will suffice. Once two values γ_1 and γ_2 have been found within a sufficiently narrow interval such that $\gamma_1 < \gamma < \gamma_2$, linear interpolation will yield the required value.

With $\hat{\gamma}$ thus determined, θ is estimated from the second equation of (3) as

$$\hat{\theta} = \sum_1^n x_i^{\hat{\gamma}}/n. \tag{5}$$

The symbol (^) is employed here to distinguish M. L. estimators from the parameters being estimated.

3. Singly Censored Samples

In a typical life test, N specimens are placed under observation and as each failure occurs, the time is noted. Finally at some pre-determined fixed time x_0

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or after some pre-determined fixed number of sample specimens fail, the test is terminated. In both of these cases the data collected consist of observations $x_1, x_2, \cdots x_n$ plus the information that $(N - n)$ specimens survived beyond the time of termination, x_0 in the former case, and x_n in the latter. Consistent with standard terminology as employed in [1], when x_0 is fixed and n is thus a random variable, censoring is said to be of *type I*. When n is fixed and the time of termination x_n is a random variable, censoring is said to be of *type II*.

In both *type I* and *type II* censoring, the likelihood function may be written as

$$L = \frac{N!}{(N-n)!}\left[\prod_{i=1}^{n} (\gamma/\theta)x_i^{\gamma-1} \exp(-x_i^\gamma/\theta)\right]\cdot[1 - F(x_T)]^{N-n}, \tag{6}$$

where in *type I* censoring, the time of termination $x_T = x_0$, and in *type II* censoring $x_T = x_n$. The distribution function $F(x)$ follows from (1) as

$$\begin{aligned} F(x) &= \int_0^x \gamma t^{\gamma-1} \exp(-t^\gamma/\theta)\, dt/\theta \\ &= 1 - \exp(-x^\gamma/\theta). \end{aligned} \tag{7}$$

Accordingly $\ln L$ follows from (6) and (7) as

$$\begin{aligned} \ln L = n \ln \gamma - n \ln \theta + (\gamma - 1) \sum_1^n \ln x_i - (1/\theta) \sum_1^n x_i^\gamma \\ - [(N-n)/\theta]x_T^\gamma + \text{const.} \end{aligned} \tag{8}$$

On differentiating (8) with respect to γ and θ in turn and equating to zero, we obtain the estimating equations

$$\left.\begin{aligned} \frac{\partial \ln L}{\partial \gamma} &= \frac{n}{\gamma} + \sum_1^n \ln x_i - \frac{1}{\theta} \sum\nolimits^* x_i^\gamma \ln x_i = 0, \\ \frac{\partial \ln L}{\partial \theta} &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum\nolimits^* x_i^\gamma = 0, \end{aligned}\right\} \tag{9}$$

where $\sum^*$ signifies that the summation extends over the entire sample with the $(N - n)$ survivors assigned the value x_T; i.e. x_0 or x_n. In particular

$$\left.\begin{aligned} \sum\nolimits^* x_i^\gamma \ln x_i &= \sum_1^n x_i^\gamma \ln x_i + (N-n)x_T^\gamma \ln x_T, \\ \sum\nolimits^* x_i^\gamma &= \sum_1^n x_i^\gamma + (N-n)x_T^\gamma. \end{aligned}\right\} \tag{10}$$

In the form given above, estimating equations (9) are fully analogous with equations (3) for complete samples, and on eliminating θ between the two equations of (9), we have

$$\left[\frac{\sum^* x_i^\gamma \ln x_i}{\sum^* x_i^\gamma} - \frac{1}{\gamma}\right] = \frac{1}{n} \sum_1^n \ln x_i, \tag{11}$$

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to be solved for $\hat{\gamma}$ employing the same techniques suggested for use in solving equation (4) in the case of a complete sample. With $\hat{\gamma}$ thus determined, it then follows from the second equation of (9) that

$$\hat{\theta} = \sum\nolimits^{*} x_i^{\hat{\gamma}}/n. \tag{12}$$

4. Progressively Censored Samples

In many life testing situations, the initial censoring results in withdrawal of only a portion of the survivors, with some remaining on test and therefore continuing under observation until ultimate failure or until a subsequent stage of censoring is performed. For sufficiently large samples, censoring may be progressive through several stages. Progressive censoring in connection with the normal and the exponential distributions was considered in an earlier paper [2].

Suppose that censoring occurs progressively in k stages at times T_i where $T_i > T_{i-1}$, $i = 1, 2 \cdots k$, and that at the ith stage of censoring r_i sample specimens selected randomly from the survivors at time T_i are removed (censored) from further observation. If we let N designate the total sample size as in section 3, and n the number of specimens which fail and therefore provide completely determined life spans, it follows that

$$N = n + \sum_1^k r_i\,. \tag{13}$$

In type I progressive censoring where the T_i are fixed, the likelihood function may be written as

$$L = C \prod_{i=1}^{n} f(x_i) \prod_{i=1}^{k} [1 - F(T_i)]^{r_i}, \tag{14}$$

where C is a constant, $f(x)$ is the density function, and $F(x)$ is the distribution function.

With $f(x)$ given by (1) and $F(x)$ by (7), the logarithm of the likelihood function becomes

$$\ln L = n \ln \gamma - n \ln \theta + (\gamma - 1) \sum_1^n \ln x_i - (1/\theta) \sum_1^n x_i^{\gamma} - (1/\theta) \sum_1^k r_i T_i^{\gamma} + \ln C. \tag{15}$$

On differentiating (15) with respect to γ and θ in turn and equating to zero, the resulting estimating equations follow as

$$\left.\begin{aligned} \frac{\partial \ln L}{\partial \gamma} &= \frac{n}{\gamma} + \sum_1^n \ln x_i - \frac{1}{\theta} \sum\nolimits^{**} x_i^{\gamma} \ln x_i = 0, \\ \frac{\partial \ln L}{\partial \theta} &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum\nolimits^{**} x_i^{\gamma} = 0, \end{aligned}\right\} \tag{16}$$

where $\sum^{**}$ signifies summation over the entire sample with the r_i observations censored at time T_i assigned the value $x_i = T_i$. More specifically

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$$\left.\begin{aligned}\sum{}^{**} x_i^\gamma \ln x_i &= \sum_1^n x_i^\gamma \ln x_i + \sum_1^k r_i T_i^\gamma \ln T_i\,,\\ \sum{}^{**} x_i^\gamma &= \sum_1^n x_i^\gamma + \sum_1^k r_i T_i^\gamma\,.\end{aligned}\right\} \tag{17}$$

In the above form, estimating equations (16) are fully analogous with equations (3) for complete samples and with equations (9) for singly censored samples. In fact equations (3) and (9) may be considered as special cases of equations (16). Accordingly on eliminating θ between the two equations of (16), we have

$$\left[\frac{\sum^{**} x_i^\gamma \ln x_i}{\sum^{**} x_i^\gamma} - \frac{1}{\gamma}\right] = \frac{1}{n}\sum_1^n \ln x_i\,, \tag{18}$$

to be solved for $\hat{\gamma}$ in the same manner as that suggested for solving (4) in the case of a complete sample and for solving (11) in the case of a singly censored sample. With $\hat{\gamma}$ thus determined, it follows from the second equation of (16) that

$$\hat{\theta} = \sum{}^{**} x_i^\gamma/n. \tag{19}$$

As noted in [2], intermediate steps in the derivation of estimating equations for type II progressively censored samples differ from corresponding steps in the case of type I samples. The end result, however, is the same in both cases, and the estimating equations given here are applicable for both sample types. It is necessary only to keep in mind that the times T_i are the times at which withdrawals are actually made.

5. Variances and Covariances of Estimates

The asymptotic variance-covariance matrix of $(\hat{\gamma}, \hat{\theta})$ is obtained by inverting the information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood functions. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood estimates. Accordingly, we have as the approximate variance-covariance matrix

$$\begin{bmatrix} -\left.\dfrac{\partial^2 \ln L}{\partial\gamma^2}\right|_{\hat{\gamma},\hat{\theta}} & -\left.\dfrac{\partial^2 \ln L}{\partial\gamma\,\partial\theta}\right|_{\hat{\gamma},\hat{\theta}} \\ -\left.\dfrac{\partial^2 \ln L}{\partial\theta\,\partial\gamma}\right|_{\hat{\gamma},\hat{\theta}} & -\left.\dfrac{\partial^2 \ln L}{\partial\theta^2}\right|_{\hat{\gamma},\hat{\theta}}\end{bmatrix}^{-1} \doteq \begin{bmatrix} V(\gamma) & \mathrm{Cov}\,(\hat{\gamma}, \hat{\theta}) \\ \mathrm{Cov}\,(\hat{\gamma}, \hat{\theta}) & V(\hat{\theta})\end{bmatrix}. \tag{20}$$

The elements of the information matrix on the left side of (20) follow by differentiating (3) for complete samples, (9) for singly censored samples, and (16) for progressively censored samples. We thereby obtain

For Complete Samples

$$\left.\begin{aligned} -\left.\frac{\partial^2 \ln L}{\partial\gamma^2}\right|_{\hat{\gamma},\hat{\theta}} &= \frac{n}{\hat{\gamma}^2} + \frac{1}{\hat{\theta}}\sum_1^n x_i^{\hat{\gamma}}(\ln x_i)^2,\\ -\left.\frac{\partial^2 \ln L}{\partial\gamma\,\partial\theta}\right|_{\hat{\gamma},\hat{\theta}} &= -\left.\frac{\partial^2 \ln L}{\partial\theta,\,\partial\gamma}\right|_{\hat{\gamma},\hat{\theta}} = -\frac{1}{\hat{\theta}^2}\sum_1^n x_i^{\hat{\gamma}} \ln x_i\,,\\ -\left.\frac{\partial^2 \ln L}{\partial\theta^2}\right|_{\hat{\gamma},\hat{\theta}} &= -\frac{n}{\hat{\theta}^2} + \frac{2}{\hat{\theta}^3}\sum_1^n x_i^{\hat{\gamma}}\,,\end{aligned}\right\} \tag{21}$$

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For Singly Censored Samples

$$\left.\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \gamma^2}\bigg|_{\hat\gamma,\hat\theta} &= \frac{n}{\hat\gamma^2} + \frac{1}{\hat\theta}\sum{}^{*} x_i^{\hat\gamma}(\ln x_i)^2, \\
-\frac{\partial^2 \ln L}{\partial \gamma\,\partial \theta}\bigg|_{\hat\gamma,\hat\theta} &= -\frac{\partial^2 \ln L}{\partial \theta\,\partial \gamma}\bigg|_{\hat\gamma,\hat\theta} = -\frac{1}{\hat\theta^2}\sum{}^{*} x_i^{\hat\gamma} \ln x_i\,, \\
-\frac{\partial^2 \ln L}{\partial \theta^2}\bigg|_{\hat\gamma,\hat\theta} &= -\frac{n}{\hat\theta^2} + \frac{2}{\hat\theta^3}\sum{}^{*} x_i^{\hat\gamma}\,,
\end{aligned}\right\} \tag{22}$$

For Progressively Censored Samples

$$\left.\begin{aligned}
-\frac{\partial^2 \ln L}{\partial \gamma^2}\bigg|_{\hat\gamma,\hat\theta} &= \frac{n}{\hat\gamma^2} + \frac{1}{\hat\theta}\sum{}^{**} x_i^{\hat\gamma}(\ln x_i)^2, \\
-\frac{\partial^2 \ln L}{\partial \gamma\,\partial \theta}\bigg|_{\hat\gamma,\hat\theta} &= -\frac{\partial^2 \ln L}{\partial \theta\,\partial \gamma}\bigg|_{\hat\gamma,\hat\theta} = \frac{1}{\hat\theta^2}\sum{}^{**} x_i^{\hat\gamma} \ln x_i\,, \\
-\frac{\partial^2 \ln L}{\partial \theta^2}\bigg|_{\hat\gamma,\hat\theta} &= -\frac{n}{\hat\theta^2} + \frac{2}{\hat\theta^3}\sum{}^{**} x_i^{\hat\gamma}\,,
\end{aligned}\right\} \tag{23}$$

where as in the case of estimating equations (9), $\sum^{*}$ signifies summation over the entire sample with the $(N - n)$ survivors assigned the value x_T, and as in the case of estimating equations (16), $\sum^{**}$ signifies summation over the entire sample with the r_i observations censored at times T_i assigned the values $x_i = T_i$.

Although the foregoing results are valid in a strict sense only for large samples, they may be relied upon to provide reasonable approximations to estimate variances and covariances for moderate size samples. In small samples, it must be recognized that errors due to bias sometimes greatly exceed the errors induced by large estimate variances. This is an area which requires further investigation with respect to the Weibull distribution. Perhaps some linear combination of order statistics will ultimately provide unbiassed estimates that may be preferred over the estimates considered here. At least the maximum likelihood estimates are consistent and we are thereby assured that the bias diminishes as the sample size becomes large.

6. A First Approximation to the Shape Parameter

The coefficient of variation of the Weibull distribution is a function of the shape parameter γ alone. Therefore with the aid of a suitable graph or table of this function, it is easy to obtain a good first approximation to the maximum likelihood estimate $\hat\gamma$ by equating the first two moments of a complete sample to corresponding population (theoretical) moments.

The kth non-central moment readily follows from (1) as

$$\mu_k' = \theta^{k/\gamma}\Gamma[(k/\gamma) + 1], \tag{24}$$

where Γ signifies the gamma function

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}\,dx, \qquad (m > 0). \tag{25}$$

Accordingly, when the variance is divided by the square of the mean, we obtain the following expression which is a function of γ only

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$$\frac{V(x)}{\mu_1'^2} = \frac{\Gamma[(2/\gamma) + 1] - \Gamma^2[(1/\gamma) + 1]}{\Gamma^2[(1/\gamma) + 1]}, \tag{26}$$

where $V(x) = \mu_2 = \mu_2' - \mu_1'^2$.

On taking square roots of (26), we have for the *coefficient of variation*

$$\mathrm{CV} = \frac{\sqrt{\Gamma[(2/\gamma) + 1] - \Gamma^2[(1/\gamma) + 1]}}{\Gamma[(1/\gamma) + 1]}. \tag{27}$$

Following is an abridged table of the coefficient of variation and of its square with γ as the argument. At some future date it might be advisable to prepare a more extensive table at closer intervals of the argument as a further aid in obtaining moment estimates of γ in practical applications.

TABLE 1

The Weibull Coefficient of Variation as a Function of the Shape Parameter

γ	$V(x)/\mu_1'^2$	Coefficient of Variation
1/3	19	4.3589
1/2	5	2.2361
0.8	1.5904	1.2611
1.0	1	1
5/4	0.6480	0.8050
5/3	0.3801	0.6165
2.0	0.2732	0.5227
3.0	0.1323	0.3637
4.0	0.0787	0.2805

The graphs of Figure 1 were plotted using values from this table. With the sample quantity $s^2/\bar{x}^2$ equated to $V(x)/\mu_1'^2$, the moment estimate γ^* might be read from Figure 1 with an accuracy of at least one decimal and perhaps two. The value thus read should provide a satisfactory first approximation for use in iterating to the maximum likelihood estimate. For many purposes, the moment estimate thus read will be of such accuracy that no further improvement through iteration is necessary. In such cases the moment estimate of θ follows from (24) with $k = 1$ and with μ_1' equated to $\bar{x}$, as

$$\theta^* = \{\bar{x}/\Gamma[(1/\gamma^*) + 1]\}^{\gamma^*}. \tag{28}$$

The stars (*) serve to distinguish moment estimates from maximum likelihood estimates denoted (^) and from the parameters being estimated.

Kendall [9, page 233] has shown that the approximate asymptotic sampling variance of the coefficient of variation (valid for large samples) is

$$V(\mathrm{CV}) \doteq (\mathrm{CV})^2/2n. \tag{29}$$

This result can be used in conjuction with the CV graph of Figure 1 to provide an indication of the extent of the sampling variation in estimates of γ. For example, suppose $n = 50$, and suppose further that $\gamma = 1$; i.e. our distribution is in fact the exponential distribution. When $\gamma = 1$, then CV $= 1$ and from (29)

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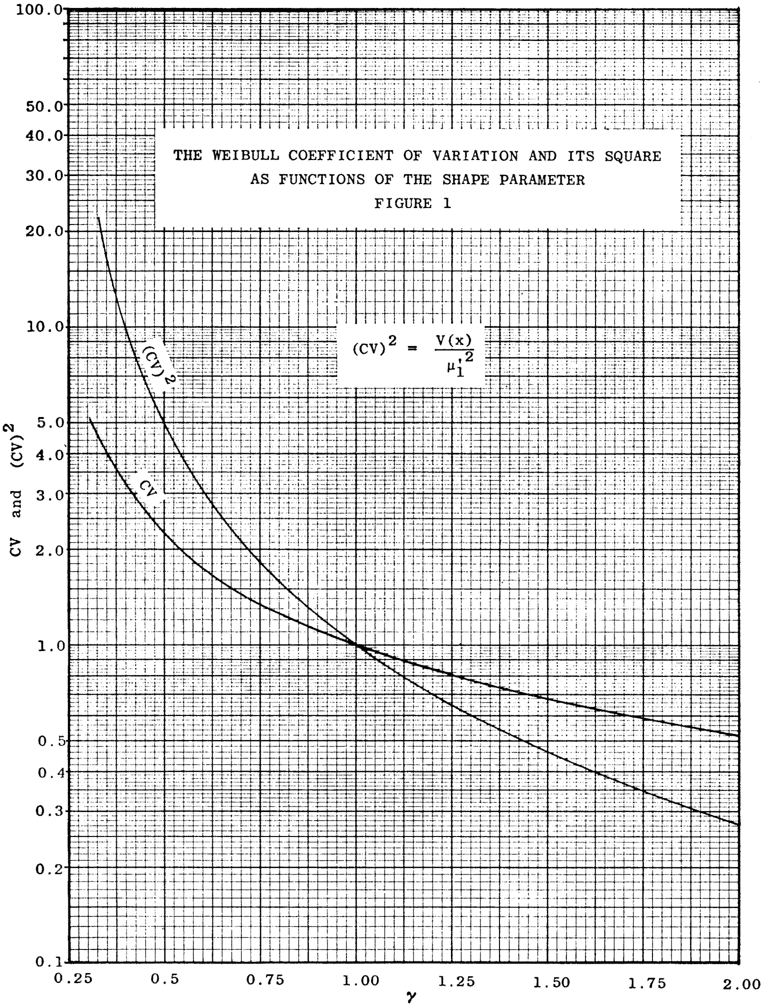

$$V(\mathrm{CV}) \doteq 0.01.$$

Accordingly $\sigma^*_{\mathrm{CV}} = 0.1$, and there is approximately a 95% probability that the sample value of CV will be in the interval .8 to 1.2. As can be shown from the graph, this corresponds to an interval of from .83 to 1.25 for the estimate γ^*; i.e.

$$.83 < \gamma^* < 1.25,$$

when the sample size is 50 and actually $\gamma = 1$.

This result tends to justify the natural reluctance which one might have toward abandoning the much simpler exponential distribution as a model for a life distribution in favor of the Weibull distribution on the sole evidence of a single small sample.

7. An Illustrative Example

A sample given by Menon [13] has been selected to illustrate the practical application of results obtained here. Data for this sample are given below.

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x	$\ln x$	x	$\ln x$	x	$\ln x$	x	$\ln x$
0.806	−0.216	57.628	4.054	1.550	0.438	7.057	1.954
0.664	−0.410	1.033	0.032	9.098	2.208	2.046	0.716
0.345	−1.064	3.532	1.262	0.470	−0.754	0.185	−1.686
0.001	−6.824	0.970	−0.030	0.505	−0.684	0.435	−0.832
0.469	−0.758	0.071	−2.640	0.030	−3.506	1.550	0.438

According to Menon, this sample is from a population in which $\gamma = .5$, and $\theta = \sqrt{e} \doteq 1.649$. These data are summarized as: $n = 20$, $\sum_1^{20} x_i = 88.445$, $\sum_1^{20} x_i^2 = 3479.170201$, and $\sum_1^{20} \ln x_i = -8.302$. It follows that $\bar{x} = 4.42225$, $s^2 = 154.402$, $(\text{CV})^2 = 7.895$, and $\text{CV} = 2.810$. Reading from the charts of Figure 1, we have as the moment estimate and first approximation to the maximum likelihood estimate of the shape parameter, $\gamma^* = 0.43$. To obtain the maximum likelihood estimate we decide to try interpolating between $\gamma = .4$ and $\gamma = .5$ using estimating equation (4). We subsequently calculate $\sum_1^{20} x_i^{0.4} = 23.580$, $\sum_1^{20} x_i^{0.4} \ln x_i = 27.661$, $\sum_1^{20} x_i^{0.5} = 27.007$, and $\sum_1^{20} x_i^{0.5} \ln x_i = 41.637$. On writing equation (4) in the form

$$K(\gamma) = \frac{\sum_1^n x_i^\gamma \ln x_i}{\sum_1^n x_i^\gamma} - \frac{1}{\gamma} - \frac{1}{n} \sum_1^n \ln x_i = 0,$$

we calculate $K(.4) = -0.9118$ and $K(.5) = -0.0432$. Since both of these values are negative, we now calculate $\sum_1^{20} x_i^{0.6} = 32.086$, $\sum_1^{20} x_i^{0.6} \ln x_i = 61.018$ and it follows that $K(.6) = 0.6501$. To obtain the required estimate, we interpolate linearly as shown below.

γ	$K(\gamma)$
.500	−0.0432
.506	0
.600	0.6501

Thus we have $\hat{\gamma} = 0.506$. To obtain $\hat{\theta}$, we calculate $\sum_1^{20} x_i^{0.506} = 27.261$ and from (5)

$$\hat{\theta} = 27.261/20 = 1.363.$$

For comparison, the maximum likelihood estimates are listed below with corresponding moment estimates, and estimates based on Menon's results, along with the population values. The moment estimate of θ was calculated using equation (28).

	Population Values	Moment Estimates	Menon's Estimates	M.L. Estimates
θ	1.649	1.23	1.40	1.363
γ	.5000	.43	.57	.506

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In order to evaluate the variance-covariance matrix (20), we first obtain the additional summations $\sum_1^{20} x_i^{0.506} \ln x_i = 42.611596$ and $\sum_1^{20} x_i^{0.506} (\ln x_i)^2 = 166.254404$. We then evaluate the partials of (21), and the desired matrix follows as

$$\begin{bmatrix} 200.09 & -22.94 \\ -22.94 & 10.77 \end{bmatrix}^{-1} = \begin{bmatrix} 0.007 & 0.014 \\ 0.014 & 0.123 \end{bmatrix}.$$

Accordingly, $V(\hat{\gamma}) \doteq 0.007$, $V(\hat{\theta}) \doteq 0.123$, and $\mathrm{Cov}\,(\hat{\gamma}, \hat{\theta}) = 0.014$. The correlation coefficient between estimates follows as

$$\rho_{\hat{\gamma},\hat{\theta}} = \frac{\mathrm{Cov}\,(\hat{\gamma}, \hat{\theta})}{\sqrt{V(\hat{\gamma}) \cdot V(\hat{\theta})}} = 0.48.$$

Calculations for censored samples are essentially the same as for the complete sample illustrated here and are not likely to present any unusual computational difficulties.

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