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Maximum Likelihood Estimation for the Three Parameter Weibull Distribution Based on Censored Samples*

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This study develops maximum likelihood estimators for the three parameter Weibull distribution based on various left and right censored data situations. For the case of single censoring from the left and progressive censoring from the right the developed estimation procedure involves the simultaneous solution of two iterative equations compared to the arduous task of solving three simultaneous iterative equations as outlined by Harter and Moore [6]. For the two parameter Weibull [4], the above case results in an estimation procedure involving only one iterative equation. The asymptotic variance-covariance matrix of the estimators is given. Pivotal functions are provided whose distributions can be used for confidence interval estimation and for tests of hypotheses. An example is included which illustrates the estimation procedure involving left censored samples.

KEY WORDS

Weibull Distribution
Maximum Likelihood
Asymptotic Variance-Covariance Matrix
Pivotal Functions
Left and Right Censored Samples

1. BACKGROUND AND INTRODUCTION

This paper is concerned with maximum likelihood estimation of the parameters of the three parameter Weibull distribution given by

$$F(t; a, b, c) = 1 - \exp\left(-\left(\frac{t-c}{b}\right)^a\right);$$

$$t \geq c, \quad b > 0, \quad c > 0, \quad a > 0, \qquad (1)$$

for several types of available data situations.

In a typical life test, n specimens are placed under observation and as each failure occurs, the cumulated life is recorded. This is referred to as a "complete sample" test [3]. Various deviations from a "complete sample" test are considered in this paper. These are (i) single censoring from the right, (ii) progressive censoring from the right, (iii) single censoring from the left, (iv) progressive censoring from the left, and (v) combinations of (i) or (ii) with (iii) or (iv).

It is evident that (i) is a special case of (ii) and (i) will not be discussed here. While (iii) is a special case of (iv) it will be given special attention because of the form of the resulting parameter estimation equations and the special interest in this case.

Cohen [4] stated without proof that there is no difference in the form of the resulting maximum likelihood estimation equations obtained from Type I (fixed time) or Type II (fixed number of failures) censoring in cases (i) and (ii) above. This result, which Ringer and Sprinkle [9] questioned, is easily verified by directly comparing the Type I estimation equations (equations (18) and (19) of Cohen [4]) and the Type II estimation equations (equations (8) and (9) of Ringer and Sprinkle [9]). For cases (iii), (iv) and (v) above the resulting maximum likelihood estimating equations are the same for both Type I and Type II censoring even though intermediate steps in their derivation differ. That is, it is necessary to know only the time at which observation is started or stopped for each censored item. Therefore only Type I censoring will be considered in the remainder of this paper.

Leone et al [8] developed maximum likelihood estimators (MLEs), based on a complete sample, for the three parameter Weibull distribution. The MLE for the parameter b is given in [8] as a function of the parameters a, c, and the sample values. The MLEs

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for the parameters a and c are given in [8] as the iterative simultaneous solution of two equations involving the parameters a, c, and the sample values. Cohen [4] derived MLEs for the two parameter Weibull distribution ($c \equiv 0$) from samples singly censored from the right, and samples progressively (multiply) censored from the right. The MLE for the parameter b is given in [4] as a function of the parameter a and the sample values. The MLE for the parameter a is given in [4] as an iterative solution of a single equation involving the parameter a and the sample values. Harter and Moore [6] provided iterative procedures for joint maximum likelihood estimation (for complete and Type II singly censored samples) of the three parameters of the Weibull distribution. The type of censoring considered by Harter and Moore [6] is simultaneously right and left single Type II censoring. The MLEs for the parameters a, b, and c are given in [6] as the simultaneous iterative solution of three equations involving the three parameters and the sample values. Wingo [12] and Cohen [5] developed MLEs (for samples with Type I progressively censoring from the right) for the three parameter Weibull distribution. Cohen [5], also, introduced an interesting concept of modified MLEs which utilize the first order statistic.

This paper develops an estimation procedure based on the simultaneous solution of three iterative equations which provides the MLEs for the three parameter Weibull distribution based on both left and right progressively censored samples. For the case of single censoring from the left and progressive censoring from the right the developed procedure is reduced to the simultaneous iterative solution of two equations. It should be noted that a different approach to simplifying the iterative estimation of the parameters has been taken by Wingo [11, 12], who uses what he calls "(constrained) modified quasilinearization" to decrease the number of iterations required for convergence. It appears that Wingo's method [12] could be applied to the results presented in this paper to further decrease the number of iterations required. For the two parameter (a, b) situation (the parameter c is assumed to be known) the above case results in a procedure involving only one iterative equation. [It should be pointed out that the use of MLEs assuming c unknown and assuming $c = 0$ can give disparate results even if c is actually equal to zero. This would be likely to happen when $c = 0$ and both a and b are large.] The asymptotic variance-covariance matrix of the estimators is given. Pivotal functions are provided whose distributions can be evaluated by use of Monte Carlo techniques. The distributions of these pivotal functions are independent of one or more of the unknown parameters and can be used for confidence interval estimation and for hypothesis testing. A numerical example is given which illustrates application of fitting censored residual strength data to the Weibull model.

2. Maximum Likelihood Estimation

In this section maximum likelihood techniques are used to develop estimators for the parameters of the three parameter Weibull distribution defined by (1). The most general case of a sample obtained from a combination of Type I both left and right progressive censoring is considered first; that is, a combination of (ii) and (iv) described in the previous section.

2.1 *Left and Right Progressive Censoring*

The density function for the three parameter Weibull distribution obtained by differentiating (1) is

$$f(t; a, b, c) = \frac{a}{b}\left(\frac{t-c}{b}\right)^{a-1} \exp\left(-\left(\frac{t-c}{b}\right)^{a}\right). \tag{2}$$

Let n items whose failure times have a three parameter Weibull distribution be placed on test. Associate with each sample a set of left and right censoring times which can be any real numbers greater than c. Let the n items under consideration be numbered so that items 1 through p respectively fail before the previously fixed times $t_1, t_2, \cdots, t_p$ (Type I progressive censoring from the left). Assume that items $p + 1$ through k are the only items which have their failure times exactly observed as t_{p+1}, $t_{p+2}, \cdots, t_k$. Also, assume that items $k + 1$ through n respectively are items which are taken from test, before failure, at the previously fixed times t_{k+1}, $t_{k+2}, \cdots, t_n$ (Type I progressive censoring from the right). The integers p and k ($0 \leq p < k \leq n$) are random variables. Note that this numbering of the items does not imply an ordering of times associated with each item.

The logarithm of the likelihood function for the above data situation is given by

$$\ln L = \ln\left[C \prod_{i=1}^{p} F(t_i) \prod_{i=p+1}^{k} f(t_i) \prod_{i=k+1}^{n} (1 - F(t_i)\right] \tag{3}$$

where

$$C = \binom{n}{p}\binom{n-p}{k-p}, \qquad F(t_i) = F(t_i\,; a, b, c),$$

$$f(t_i) = f(t_i\,; a, b, c).$$

By use of (1) and (2), equation (3) can be written as

$$\ln L = \ln C + \sum_{i=1}^{p} \ln F(t_i)$$

$$+ (k - p) \ln a - (k - p)a \ln b$$

$$+ (a-1) \sum_{i=p+1}^{k} \ln (t_i - c) - \sum_{i=p+1}^{n} \left(\frac{t_i - c}{b}\right)^a \quad (4)$$

Note that

$$\frac{\partial F(t)}{\partial a} = \left(\frac{t-c}{b}\right)^a \ln \left(\frac{t-c}{b}\right) R(t), \quad (5)$$

$$\frac{\partial F(t)}{\partial b} = -\frac{a}{b}\left(\frac{t-c}{b}\right)^a R(t), \quad (6)$$

$$\frac{\partial F(t)}{\partial c} = -\frac{a}{b}\left(\frac{t-c}{b}\right)^{a-1} R(t), \quad (7)$$

where $R(t) = 1 - F(t)$. On differentiating (4) with respect to a, b, and c in turn and equating to zero, the resulting estimation equations follow,

$$\frac{\partial \ln L}{\partial a} = \sum_{i=1}^{p} \frac{\left(\frac{t_i - c}{b}\right)^a \ln \left(\frac{t_i - c}{b}\right) R(t_i)}{F(t_i)} + \frac{k-p}{a} - (k-p) \ln b + \sum_{i=p+1}^{k} \ln (t_i - c) - \sum_{i=p+1}^{n} \left(\frac{t_i - c}{b}\right)^a \ln \left(\frac{t_i - c}{b}\right) = 0 \quad (8)$$

$$\frac{\partial \ln L}{\partial b} = -\sum_{i=1}^{p} \frac{\frac{a}{b}\left(\frac{t_i - c}{b}\right)^a R(t_i)}{F(t_i)} - (k-p)\frac{a}{b} + \sum_{i=p+1}^{n} \frac{a}{b}\left(\frac{t_i - c}{b}\right)^a = 0 \quad (9)$$

$$\frac{\partial \ln L}{\partial c} = -\sum_{i=1}^{p} \frac{\frac{a}{b}\left(\frac{t_i - c}{b}\right)^{a-1} R(t_i)}{F(t_i)} - \sum_{i=p+1}^{k} \frac{a-1}{t_i - c} + \sum_{i=p+1}^{n} \frac{a}{b}\left(\frac{t_i - c}{b}\right)^{a-1} = 0 \quad (10)$$

Equations (8), (9), and (10) are nonlinear in the three parameters and can be solved iteratively by the methods described by Harter and Moore [6] or Wingo [12].

If $t_1 = t_2 = \cdots = t_p = t_f$ and $t_{k+1} = t_{k+2} = \cdots = t_n = t_l$, where t_f and t_l are the first and last order statistics based on the observations t_{p+1}, t_{p+2}, $\cdots$, t_k, then Type II sampling is evident with p items censored from the left and $n - k$ items censored from the right. For this case equations (8), (9), and (10) agree respectively with equations (3.4), (3.3), and (3.5) given by Harter and Moore [6] (The second term of (3.3) [6] should have a division sign preceding theta as the authors have noted in the erratum).

2.2 *Left Single and Right Progressive Censoring*

Special interest is centered on the case of left single and right progressive censoring because of both its practical importance (see examples below) and the form of the resulting estimation equations.

In this case t_1, t_2, $\cdots$, t_p are all assumed to be equal to the predetermined fixed time t_0. For this case equation (9) can be rewritten as

$$p\left(\frac{t_0 - c}{b}\right)^a \frac{R(t_0)}{F(t_0)} = \sum_{i=p+1}^{n} \left(\frac{t_i - c}{b}\right)^a - (k-p) \quad (11)$$

Upon substituting (11) into the first term of (8) and simplifying, the MLE of b can be expressed in terms of a and c as

$$b = \left[\frac{\ln (t_0 - c) \sum_{i=p+1}^{n} (t_i - c)^a - \sum_{i=p+1}^{n} (t_i - c)^a \ln (t_i - c)}{(k-p) \ln (t_0 - c) - \frac{k-p}{a} - \sum_{i=p+1}^{k} \ln (t_i - c)}\right]^{1/a} \quad (12)$$

Now equation (11) can be rewritten as

$$\sum_{i=p+1}^{n} (t_i - c)^a = p(t_0 - c)^a \frac{R(t_0)}{F(t_0)} + (k-p) b^a \quad (13)$$

Upon substituting (12) into (13) and solving for the parameter a in the second term of the denominator of (12), the parameter a can be expressed as

$$a = \frac{(k-p)\left[\sum_{i=p+1}^{n} (t_i - c)^a - \Psi\right]}{(k-p)\left[\sum_{i=p+1}^{n} (t_i - c)^a \ln (t_i - c) - \Psi \ln (t_0 - c)\right] - \left[\sum_{i=p+1}^{k} \ln (t_i - c)\right]\left[\sum_{i=p+1}^{n} (t_i - c)^a - \Psi\right]} \quad (14)$$

where

$$\Psi = p(t_0 - c)^a \frac{R(t_0)}{F(t_0)}. \quad (15)$$

A suitable expression [obtained from equation (10)] for the parameter c is given for $a \leq 1$ by

$$c = t_{p+1} \quad (16)$$

and for $a > 1$ by

$$c = t_{p+1} - \frac{a-1}{\sum_{i=p+1}^{n} \frac{a}{b}\left(\frac{t_i - c}{b}\right)^{a-1} - \sum_{i=p+1}^{k} \frac{a-1}{t_i - c} - p\,\frac{a}{b}\left(\frac{t_0 - c}{b}\right)^{a-1} \frac{R(t_0)}{F(t_0)}} \tag{17}$$

where, without loss of generality, t_{p+1} can be considered as the first order statistic of the set t_{p+1}, $t_{p+2}, \cdots, t_k$.

By use of (12) the estimation equations (16), (17) and (14) can be solved for the joint iterative estimates of the parameters a and c. This joint solution for a and c can be used in (12) to obtain the MLE for the parameter b. The iterative methods discussed by Harter and Moore [6] or Wingo [12] can be used to solve the nonlinear estimation equations (14) and (17) if we recall that c is constrained to lie between zero and t_{p+1}.

It should be noted that (12), (14), and (17) can be applied to the Type II censoring problem of Harter and Moore [6] discussed earlier in order to reduce the arduous task of iterating three estimation equations to that of iterating two equations and evaluating the third.

2.2.1 Parameter a Known

Since the parameter estimation equation (12) is based on (8), it is not valid for estimating the parameter b when the parameter a is known. For this case, if (11) is substituted into (10), then the MLE of b can be obtained in terms of the unknown parameter c as

$$b = \left[\frac{\sum_{i=p+1}^{n} (t_i - c)^{a-1} - \frac{1}{t_0 - c} \sum_{i=p+1}^{n} (t_i - c)^a}{\frac{1}{a} \sum_{i=p+1}^{k} \frac{a-1}{t_i - c} - \frac{k-p}{t_0 - c}}\right]^{1/a}. \tag{18}$$

Equation (17) is still valid for evaluating iteratively the MLE of the parameter c for $a > 1$; if $a \leq 1$, then the MLE of c is t_{p+1}.

2.2.2 Parameters a and c Known

When the parameters a and c are known, the estimation equation for the parameter b is given by

$$b = \left[\frac{\sum_{i=p+1}^{n} (t_i - c)^a - p(t_0 - c)^a \frac{R(t_0)}{F(t_0)}}{k - p}\right]^{1/a} \tag{19}$$

This equation must be solved iteratively to find the MLE of b.

2.3 Right Progressive Censoring

For the right progressive censoring case, p is assumed to be zero, hence $t_0 = c$. When these values are substituted into the estimation equations (12), (14), and (17), indeterminate forms occur. Therefore, if we apply L'Hospital's rule with respect to t_0 and then evaluate the resulting expression at $p = 0$ and $t_0 = c$, equations (12), (14) and (17) become respectively

$$b = \left[\frac{\sum_{i=1}^{n} (t_i - c)^a}{k}\right]^{1/a}, \tag{20}$$

$$a = \frac{k \sum_{i=1}^{n} (t_i - c)^a}{k \sum_{i=1}^{n} (t_i - c)^a \ln (t_i - c) - \sum_{i=1}^{n} (t_i - c)^a \sum_{i=1}^{k} \ln (t_i - c)}, \tag{21}$$

and for $a > 1$

$$c = t_1 - \frac{a-1}{\sum_{i=1}^{n} \frac{a}{b}\left(\frac{t_i - c}{b}\right)^{a-1} - \sum_{i=2}^{k} \frac{a-1}{t_i - c}}, \tag{22}$$

for $a \leq 1$

$$c = t_1. \tag{23}$$

Special cases of the estimation equations (20), (21), and (22) are well known results. For example, (20) and (21) with $c = 0$ are given by Cohen [4], and (20), (21), and (22) with $k = n$ (complete sample) can be obtained from the results of Leone et al. [6]. The results of this section can also be derived by the appropriate redefinition of the scale parameter b from the work of Wingo [12] and Cohen [5].

If the parameter a is known (20), (22) and (23) are still valid. If the parameters a and c are known, then (19) also reduces to (20).

3. Variances and Covariances of Estimates

The asymptotic variance-covariance matrix of the MLEs for the parameters a, b, and c is obtained by inverting the information matrix with elements which are the negative expected values of the second order derivatives of the logarithm of the likelihood function. If we follow the approach of Cohen [4] the expected values will be approximated by MLEs. Hence, the approximate variance-covariance symmetric matrix is given by

$$\left(-\frac{\partial^2 \ln L}{\partial P_i\, \partial P_j}\right)^{-1}_{ij}\Bigg|_{\text{MLEs}} \doteq \begin{bmatrix} V(\hat{a}) & \text{Cov}(\hat{a}, \hat{b}) & \text{Cov}(\hat{a}, \hat{c}) \\ \text{Cov}(\hat{a}, \hat{b}) & V(\hat{b}) & \text{Cov}(\hat{b}, \hat{c}) \\ \text{Cov}(\hat{a}, \hat{c}) & \text{Cov}(\hat{b}, \hat{c}) & V(\hat{c}) \end{bmatrix} \tag{24}$$

where $P_1 = a$, $P_2 = b$, and $P_3 = c$.

Appropriate differentiation of equations (8), (9), and (10) gives the elements of the information matrix on the left side of (24). Let

$$y_i = \left(\frac{t_i - c}{b}\right)^a, \tag{25}$$

$$R(y_i) = 1 - F(y_i) = \exp(-y_i), \tag{26}$$

and

$$\Phi(y_i) = 1 - y_i - y_i \frac{R(y_i)}{F(y_i)}. \tag{27}$$

Then,

$$a^2 \frac{\partial^2 \ln L}{\partial a^2} = \sum_{i=1}^{p} \left\{ y_i \ln^2 (y_i) \frac{R(y_i)}{F(y_i)} \Phi(y_i) \right\} - (k - p) - \sum_{i=p+1}^{n} y_i \ln^2 (y_i) \tag{28}$$

$$b \frac{\partial^2 \ln L}{\partial a\, \partial b} = -\sum_{i=1}^{p} \left\{ y_i \frac{R(y_i)}{F(y_i)} + \left[y_i \ln (y_i) \frac{R(y_i)}{F(y_i)} \right] \Phi(y_i) \right\} - (k - p) + \sum_{i=p+1}^{n} y_i (\ln (y_i) + 1) \tag{29}$$

$$b \frac{\partial^2 \ln L}{\partial a\, \partial c} = -\sum_{i=1}^{p} \left\{ y_i^{(a-1)/a} \frac{R(y_i)}{F(y_i)} + \left[y_i^{(a-1)/a} \ln (y_i) \frac{R(y_i)}{F(y_i)} \right] \Phi(y_i) \right\} - \sum_{i=p+1}^{k} y^{-1/a} + \sum_{i=p+1}^{n} y^{(a-1)/a} (\ln (y_i) + 1) \tag{30}$$

$$\frac{b^2}{a} \frac{\partial^2 \ln L}{\partial b^2} = \sum_{i=1}^{p} \left\{ -y_i \frac{R(y_i)}{F(y_i)} + \left[a y_i \frac{R(y_i)}{F(y_i)} \right] \Phi(y_i) \right\} + (k - p) - \sum_{i=p+1}^{n} y_i (a + 1) \tag{31}$$

$$\frac{b^2}{a^2} \frac{\partial^2 \ln L}{\partial c\, \partial b} = \sum_{i=1}^{p} y_i^{(a-1)/a} \frac{R(y_i)}{F(y_i)} \Phi(y_i) - \sum_{i=p+1}^{n} y_i^{(a-1)/a} \tag{32}$$

$$b^2 \frac{\partial^2 \ln L}{\partial c^2} = a \sum_{i=1}^{p} [y_i]^{(a-2)/a} \frac{R(y_i)}{F(y_i)} (a\Phi(y_i) - 1) - \sum_{i=p+1}^{k} (a - 1) y_i^{-2/a} - a(a - 1) \sum_{i=p+1}^{n} y^{(a-2)/a} \tag{33}$$

As noted by Cohen [4] equation (24) is valid in a strict sense only for large samples, but it may be relied upon to provide reasonable approximations to estimate variances and covariances for moderate size samples where the bias is small.

Several regularity conditions on the original distribution $f(t; a, b, c)$ are necessary to insure that the MLEs have an approximate multivariate normal distribution for large samples with variance-covariance matrix (24). Leibniz's Theorem can be used to show that the regularity condition of interchanging the operation of integration with respect to t and differentiation with respect to the parameter c is permissible only for values of the parameter $a > 2$. Also the regularity condition that the variable $(\partial^2/\partial c^2)(\ln f)$ must have finite mean and variance is satisfied only if $a > 4$ ($a > 2$ provides a finite mean).

Asymptotic variances and covariances are given by Harter and Moore [7] for samples of size n, with proportion q_1 and q_2 of the sample values censored from below and above, respectively; and by Wingo [12] for right progressively censored samples. When all three Weibull parameters are unknown, Monte Carlo results (see [7]) indicate that convergence of (24) is slow.

4. Pivotal Functions and Estimation

In the discussion which follows, $\hat{a}_{abc}$, $\hat{b}_{abc}$, and $\hat{c}_{abc}$ are used to denote the MLEs of the parameters a, b, and c, when in fact sampling is from a three parameter Weibull distribution with these parameters. That is, $\hat{a}_{a10}$ is the MLE of the parameter a when sampling from a Weibull distribution with parameters $a = a$, $b = 1$, and $c = 0$.

References [1] and [10] give pivotal functions for the two parameter (a, b) Weibull distribution for use in confidence interval estimation and tests of hypotheses. A pivotal function transforms a given MLE estimator into a new form which is distributed independently of certain population parameters. In addition, pivotal functions are sought such that their distributions are the same as those obtained using some function of the MLEs derived from a "standardized" distribution with respect to the parameters which are not involved in the pivotal functions. The principal advantage of working with pivotal functions is that their distributions are independent of one or more of the unknown parameters and can be obtained by a Monte Carlo evaluation of the "standardized" distribution. The resulting forms of many pivotal functions are such that their distributions can be used for confidence interval estimation and for tests of hypotheses.

Table I contains pivotal functions equated to their equivalent "standardized" forms for the MLEs of the three parameter Weibull distribution. It is readily apparent which parameters are not involved in the pivotal function and the form of the "standardized" distribution. For example, the pivotal function $\hat{a}_{abc} \cong \hat{a}_{a10}$ is distributed independently of the parameters b and c and the standardized Weibull distribution has parameters $a = a$, $b = 1$, and $c = 0$. The pivotal functions given in Table I can be derived by use of the approach of Thoman et al. [10], and will not be shown here. The pivotal functions given in Table I are valid for all types of censoring considered here. It must be noted that the distribution of these pivotal functions are directly dependent on the sample size and type of censoring used. The reader is also referred to Thoman et al. [10] for an example of the use of pivotal functions in testing statistical hypotheses.

For the case where the parameter a is known, it is apparent from Table I that exact confidence limits for the parameters b and c could be obtained if the distributions of the two pivotal functions $\hat{b}_{a10}$ and $\hat{c}_{a10}/\hat{b}_{a10}$ were developed by use of Monte Carlo methods. When both parameters a and b are known the distribution of c_{a10} can be used to obtain exact confidence limits for the parameter c.

Monte Carlo methods can be used to obtain the γ-percentiles, $\hat{a}_\gamma$, for each member of the family of cumulative distributions $H(\hat{a}_{a10}\,;\,a)$ of the pivotal function $\hat{a}_{a10}$ for given values of a; that is, $\gamma = H(\hat{a}_{a10} = \hat{a}_\gamma\,;\,a)$. Therefore, if one uses the resulting plot of a vs. $\hat{a}_\gamma$, a $\gamma - \%$ confidence bound for the parameter a can be determined by finding the value of a corresponding to $\hat{a}_\gamma$ equal to the observed $\hat{a}_{abc}$.

5. Discussion and Example

Numerous examples which fit the data censoring situation discussed in section 1 could be described. Cohen [3] describes situations where both Type I and Type II censoring from the left or right can occur.

A Type I sample progressively censored from the right frequently occurs in fatigue testing where specimens are suspended if they do not fail before

Table 1—*Pivotal functions for the Weibull distribution*

PARAMETERS KNOWN	PARAMETERS UNKNOWN	PIVOTAL FUNCTIONS
None	a,b,c	$\hat{a}_{abc} \cong \hat{a}_{a10}$; * $\frac{\hat{b}_{acb}}{b} \cong \hat{b}_{a10}$; ** $\frac{\hat{c}_{abc}-c}{\hat{b}_{abc}} \cong \frac{\hat{c}_{a10}}{\hat{b}_{a10}}$ **
a	b,c ***	$\frac{\hat{b}_{abc}}{b} \cong \hat{b}_{a10}$; $\frac{\hat{c}_{abc}-c}{\hat{b}_{abc}} \cong \frac{\hat{c}_{a10}}{\hat{b}_{a10}}$
b	a,c	$\hat{a}_{abc} \cong \hat{a}_{a10}$; $\frac{\hat{c}_{abc}-c}{b} \cong \hat{c}_{a10}$ **
c (assume c=0)	a,b ***	$\frac{\hat{a}_{ab0}}{a} \cong \hat{a}_{110}$; $\left(\frac{\hat{b}_{ab0}}{b}\right)^{\hat{a}_{ab0}} \cong \hat{b}_{110}^{\hat{a}_{110}}$
b,c (assume c=0)	a	$\frac{\hat{a}_{ab0}}{a} \cong \hat{a}_{110}$
a,c (assume c=0)	b	$\left(\frac{\hat{b}_{ab0}}{b}\right)^{a} \cong \hat{b}_{110}$
a,b	c	$\frac{\hat{c}_{abc}-c}{b} \cong \hat{c}_{a10}$

* The quantities on each side of the $\cong$ sign have the same distribution.

** The distribution of these pivotal functions (which are given for completeness and to save others the task of their derivation) can be approximated by using $a=\hat{a}_{abc}$ and hence approximate confidence intervals for b and c obtained by Monte Carlo methods; e.g., $\hat{b}_{\hat{a}10}$ is used for $\hat{b}_{a10}$.

*** Some percentage points for the distributions of these pivotal functions are expected to be included in The Weibull Distribution and Related Topics by C. E. Antle and L. J. Bain, Pennsylvania State University Press (In preparation). Also see [2] and [10].

specified amounts of fatigue testing time. These suspended items are called fatigue "runouts". Frequently these suspended fatigue items are tested in static strength at various levels of cumulative fatigue time to evaluate residual static strength characteristics. A Type I sample singly censored from the left can occur in residual static strength tests where some specimens fail in fatigue before they accumulate sufficient fatigue life to be tested statically for residual strength. In this case all that can be said about those specimens which failed in fatigue is that their residual strength is less than the maximum running load in the fatigue spectrum. In this case the items which fail in fatigue are called residual strength "under runs".

Table II contains fatigue and four-lifetime residual strength data. The left censored (at 5000 pounds) residual strength data were developed by use of a two parameter Weibull random number generator with shape parameter $a = 10$ and scale parameter $b = 5800$. The four-lifetime censored fatigue data are given to account for the seven residual strength specimens which would have failed in fatigue if the fatigue loading had gone to 5000 pounds.

The data in Table II can be analyzed in several different ways assuming that both residual strength and fatigue life are distributed in accordance with the two parameter Weibull distribution. Equations (12) and (14) can be used with $c = 0$ to evaluate the MLEs for the Weibull distribution which was assumed to produce the residual strength data of Table II. The solution of equation (14) can be obtained graphically as the intersection of the line $g(a) = a$ and the line $g(a)$ equal to the right hand side of equation (14). These lines are shown graphically

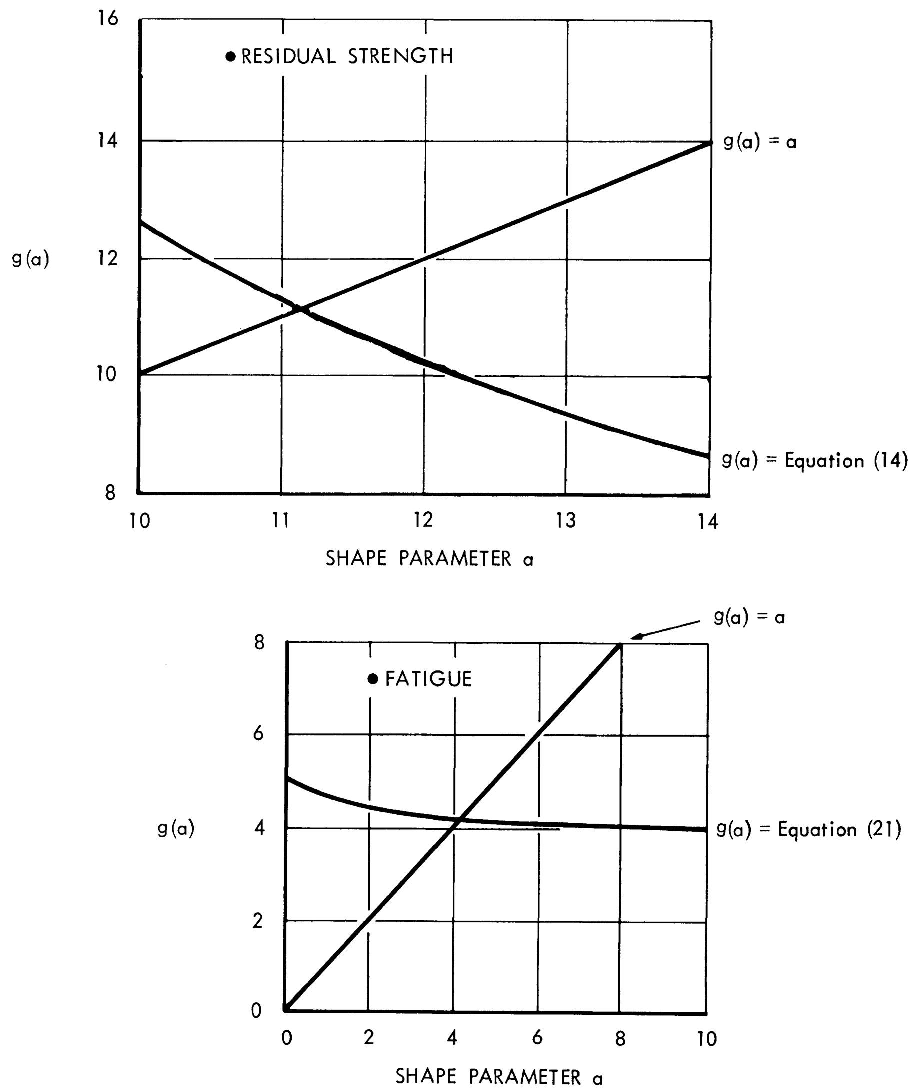

FIGURE 1—Shape parameter for residual strength/fatigue life—graphical solution for Table I data

TABLE 2—*Four-lifetime residual strength/fatigue lifetime data*

Table II FOUR-LIFETIME RESIDUAL STRENGTH/ FATIGUE LIFETIME DATA

Residual Strength* (Four Lifetimes)			Fatigue Life
5000 lb	5318	5807	2.22 Lifetimes
5000	5384	5825	2.43
5000	5389	5935	3.17
5000	5479	5979	3.41
5000	5604	6020	3.51
5000	5613	6141	3.54
5000	5653	6150	3.73
5179	5689	6162	(23 "runouts" at 4 lifetimes)
5209	5785	6376	
5219	5802	6435	

* The first seven residual strengths are considered to be censored from the left at 5000 lb

in Figure 1. An iterative computer procedure was also used to solve equation (14) giving $a = 11.1353$. This value of the shape parameter a was substituted into equation (12) giving $b = 5737$ as the value of the scale parameter.

The data in Table II can alternately be used to evaluate the fatigue characteristics of the data. In this case there are seven fatigue failures and twenty-three fatigue "runouts" at four lifetimes, i.e., Type I single censoring from the right. In this situation equations (23) and (24) are used to evaluate the MLEs for the scale and shape parameters of the Weibull distribution from which the lifetime data are assumed to follow. The estimate of the shape parameter was first determined from equation (21) with $c = 0$ by use of an iterative procedure (see Figure 1 for the graphical solution) to be $a = 4.1860$. This value was used in equation (20), giving $b = 5.47$.

In this example the parameter c was constrained to equal zero due to the physical consideration that failure-time or strength data do not appear to exhibit thresholds. As previously mentioned, assuming c unknown and assuming $c = 0$ can give very disparate results even if c is actually equal to zero.

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