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NOTES

Maximum-Likelihood Estimation of the Parameters of Gamma and Weibull Populations from Complete and from Censored Samples

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Iterative procedures are given for joint maximum-likelihood estimation, from complete and censored samples, of the three parameters of Gamma and of Weibull populations. For each of these populations, the likelihood function is written down, and the three maximum-likelihood equations are obtained. In each case, simultaneous solution of these three equations would yield joint maximum-likelihood estimators for the three parameters. The iterative procedures proposed to solve the equations are applicable to the most general case, in which all three parameters are unknown, and also to special cases in which any one or any two of the parameters are known. Numerical examples are worked out in which the parameters are estimated from the first m failure times in simulated life tests of n items ($m \leq n$), using data drawn from Gamma and Weibull populations, each with two different values of the shape parameter.

1. INTRODUCTION

A large number of authors have considered estimation of the parameters of Gamma (Pearson Type III) and Weibull populations by the method of moments, the method of maximum likelihood, and other methods. Fisher [4] showed that for estimating the parameters of a Pearson Type III population, except when it closely approximates normality, the method of moments is inefficient, and recommended use of the method of maximum likelihood. Among those who have applied this method to the parameters of Gamma populations are Masuyama and Kuroiwa [11], Des Raj [13], Chapman [1], Greenwood and Durand [5], Mickey, Mundle, Walker, and Glinski [12] and Wilk, Gnanadesikan, and Huyett [15]. Among those who have studied maximum-likelihood estimation of the parameters of Weibull populations are Kao [7], [8], Leone, Rutenberg, and Topp [10], Dubey [3], Lehman [9], and Ravenis [14]. In this paper we give maximum-likelihood estimators, for the most general case (all three parameters unknown), based on the first m order statistics of a sample of size $n(m \leq n)$, together with a completely computerized iterative procedure and a workable modification for use when the usual procedure breaks down (location parameter unknown or known to be equal to first order statistic and estimate of shape parameter ≤ 1). The mathematical formulation will be given for the Gamma population in Section 2 and for the Weibull population in Section 3. The iterative estimation procedures will be described in Section 4, numerical examples will be worked out in Section 5, and some concluding remarks will be made in Section 6.

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The readers may be interested in noting the article entitled "Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and On Censored Samples" by A. Clifford Cohen, which appears in this same issue. Each of these papers was carried out independently of the other. (Editor's comment.)

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2. Gamma Population—Mathematical Formulation

The probability density function of the random variable X having a Gamma distribution with location parameter $c \geq 0$, scale parameter θ, and shape parameter α is given by

$$f(x; c, \theta, \alpha) = [1/\Gamma(\alpha)\theta][(x - c)/\theta]^{\alpha-1} \cdot \exp[-(x - c)/\theta], \qquad \theta, \alpha > 0, \qquad x \geq c \geq 0. \tag{2.1}$$

The natural logarithm of the likelihood function of the $(m - r)$ order statistics $X_{r+1}, X_{r+2}, \cdots, X_m$ of a sample of size n is given by

$$\begin{aligned} L_r = {} & \ln n! - \ln (n - m)! - \ln r! - n \ln \Gamma(\alpha) \\ & - (m - r)\alpha \ln \theta + (\alpha - 1) \sum_{i=r+1}^{m} \ln (x_i - c) - \sum_{i=r+1}^{m} (x_i - c)/\theta \\ & + (n - m) \ln [\Gamma(\alpha) - \Gamma_{(x_m - c)/\theta}(\alpha)] + r \ln [\Gamma_{(x_{r+1} - c)/\theta}(\alpha)]. \end{aligned} \tag{2.2}$$

The maximum-likelihood equations are obtained by equating to zero the partial derivatives of L_r with respect to each of the three parameters, which are given by

$$\begin{aligned} \partial L_r/\partial\theta = {} & -(m - r)\alpha/\theta + \sum_{i=r+1}^{m} (x_i - c)/\theta^2 + (n - m)(x_m - c)^{\alpha} \\ & \cdot \exp[-(x_m - c)/\theta]/\theta^{\alpha+1}[\Gamma(\alpha) - \Gamma_{(x_m - c)/\theta}(\alpha)] - r(x_{r+1} - c)^{\alpha} \\ & \cdot \exp[-(x_{r+1} - c)/\theta]/\theta^{\alpha+1}\Gamma_{(x_{r+1} - c)/\theta}(\alpha), \end{aligned} \tag{2.3}$$

$$\begin{aligned} \partial L_r/\partial\alpha = {} & -(m - r) \ln \theta + \sum_{i=r+1}^{m} \ln (x_i - c) - n\Gamma'(\alpha)/\Gamma(\alpha) + (n - m) \\ & \cdot [\Gamma'(\alpha) - \Gamma'_{(x_m - c)/\theta}(\alpha)]/[\Gamma(\alpha) - \Gamma_{(x_m - c)/\theta}(\alpha)] \\ & + r\Gamma'_{(x_{r+1} - c)/\theta}(\alpha)/\Gamma_{(x_{r+1} - c)/\theta}(\alpha), \end{aligned} \tag{2.4}$$

$$\begin{aligned} \partial L_r/\partial c = {} & (1 - \alpha) \sum_{i=r+1}^{m} (x_i - c)^{-1} + (m - r)/\theta + (n - m)(x_m - c)^{\alpha-1} \\ & \cdot \exp[-(x_m - c)/\theta]/\theta^{\alpha}[\Gamma(\alpha) - \Gamma_{(x_m - c)/\theta}(\alpha)] - r(x_{r+1} - c)^{\alpha-1} \\ & \cdot \exp[-(x_{r+1} - c)/\theta]/\theta^{\alpha}\Gamma_{(x_{r+1} - c)/\theta}(\alpha), \end{aligned} \tag{2.5}$$

where the primes in (2.4) indicate differentiation with respect to α.

3. Weibull Population—Mathematical Formulation

The probability density function of the random variable X having a Weibull distribution with location parameter $c \geq 0$, scale parameter θ, and shape parameter K is given by

$$f(x; c, \theta, K) = [K(x - c)^{K-1}/\theta^K] \exp\{-[(x - c)/\theta]^K\}, \quad \theta, K > 0, \quad x \geq c \geq 0. \tag{3.1}$$

The natural logarithm of the likelihood function of the $(m - r)$ order statistics $X_{r+1}, X_{r+2}, \cdots, X_m$, of a sample of size n is given by

$$\begin{aligned} L_r = {} & \ln n! - \ln (n - m)! - \ln r! + (m - r)(\ln K - K \ln \theta) \\ & + (K - 1) \sum_{i=r+1}^{m} \ln (x_i - c) - \sum_{i=r+1}^{m} [(x_i - c)/\theta]^K \\ & - (n - m)[(x_m - c)/\theta]^K + r \ln \{1 - \exp[-(x_{r+1} - c)^K/\theta^K]\}. \end{aligned} \tag{3.2}$$

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The maximum-likelihood equations are obtained by equating to zero the partial derivatives of L_r with respect to each of the three parameters, which are given by

$$\begin{aligned}\partial L_r/\partial\theta = {} & -K(m-r)/\theta + K\sum_{i=r+1}^{m}(x_i-c)^K\ \theta^{K+1} \\ & + K(n-m)(x_m-c)^K/\theta^{K+1} - Kr(x_{r+1}-c)^K \\ & \cdot\exp\left[-(x_{r+1}-c)^K/\theta^K\right]/\theta^{K+1}\{1-\exp\left[-(x_{r+1}-c)^K/\theta^K\right]\},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\partial L_r/\partial K = {} & (m-r)(1/K-\ln\theta) \\ & + \sum_{i=r+1}^{m}\ln(x_i-c) - \sum_{i=r+1}^{m}[(x_i-c)/\theta]^K\ln[(x_i-c)/\theta] \\ & - (n-m)[(x_m-c)/\theta]^K\ln[(x_m-c)/\theta] + r(x_{r+1}-c)^K\ln[(x_{r+1}-c)/\theta] \\ & \cdot\exp\{-[(x_{r+1}-c)/\theta]^K\}/\theta^K\{1-\exp[-(x_{r+1}-c)^K/\theta^K]\},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\partial L_r/\partial c = {} & (1-K)\sum_{i=r+1}^{m}(x_i-c)^{-1} + K\theta^{-K}\sum_{i=r+1}^{m}(x_i-c)^{K-1} \\ & + (n-m)K\theta^{-K}(x_m-c)^{K-1} - Kr(x_{r+1}-c)^{K-1} \\ & \cdot\exp[-(x_{r+1}-c)^K/\theta^K]/\theta^K\{1-\exp[-(x_{r+1}-c)^K/\theta^K]\}.\end{aligned} \tag{3.5}$$

4. Iterative Estimation Procedure

Iterative procedures have been developed for finding the joint maximum-likelihood estimators of the parameters of Gamma and Weibull populations from complete or censored samples. These involve estimating the three parameters, one at a time, in the cyclic order θ, α [or K], and c, omitting any assumed to be known. Assuming that the first m order statistics of a sample of size $n(m \leq n)$ are known. one starts by setting $r = 0$ (no censoring from below). One then chooses initial estimates for the unknown parameters.

At each step, the rule of false position (iterative linear interpolation) is used to determine the value (if any) of the parameter then being estimated which satisfies the appropriate likelihood equation, in which the latest estimates (or known values) of the other two parameters have been substituted. Positive values $\hat{\theta}$ and $\hat{\alpha}$ [or $\hat{K}$] can always be found in this way. In estimating c, however, one may find that no value of c in the permissible interval $0 \leq c \leq x_1$ satisfies the likelihood equation (2.5) [or (3.5)]. In such cases, the likelihood function in that interval is either monotone decreasing, so that $\hat{c} = 0$, or monotone increasing, so that $\hat{c} = x_1$. The latter situation occurs when $\hat{\alpha} \leq 1$ [or $\hat{K} \leq 1$], since then the right-hand side of equation (2.5) [or (3.5)], for $r = 0$, contains only positive terms. Once that has occurred, it is impossible to continue estimation with $r = 0$, since some of the terms in the likelihood equations become infinite, so it is necessary to censor the smallest observation x_1 and any others equal to it (r observations in all). Subsequently, x_1 plays no role in the estimation procedure except as an upper bound on $\hat{c}$. Iteration continues until the results of successive steps agree to within some assigned tolerance.

5. Numerical Examples

As illustrations, consider the simulated life tests, each on forty components, summarized in Tables 1 and 2. We shall suppose that the "data" in Table 1 represent observed failure times (in hours). Actually, they were obtained by random sampling from Gamma and Weibull populations, each with shape parameters 2 and 3. For each set of data, the appropriate iterative estimation procedure was carried out for $m = 10(10)40$ in the following cases: (1) all parameters unknown; (2) any two parameters unknown; and (3) any one parameter unknown. The resulting estimates are shown in Table 2.

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TABLE 1

Simulated Life Test Data—Hours to Failure of Ordered Random Samples of 40 Items

G2—Gamma Population ($\theta = 100, \alpha = 2, c = 30$)									
47	56	58	64	77	79	89	128	131	142
144	149	163	166	175	176	184	184	188	190
191	204	216	227	241	250	256	256	261	273
282	283	286	297	299	338	352	353	357	495
G3—Gamma Population ($\theta = 50, \alpha = 3, c = 20$)									
25	54	56	67	69	79	91	102	108	109
113	126	132	134	139	143	153	156	156	166
174	178	181	182	194	198	202	202	217	231
236	246	246	251	263	272	276	343	352	392
W2—Weibull Population ($\theta = 100, K = 2, c = 10$)									
15	20	27	42	42	43	44	46	64	65
65	68	68	71	74	75	75	76	77	78
92	95	100	102	102	112	113	116	117	124
124	126	127	134	149	152	153	161	168	205
W3—Weibull Population ($\theta = 100, K = 3, c = 20$)									
40.9	52.2	53.2	59.4	60.0	66.8	77.3	78.0	79.7	81.1
81.4	85.4	86.0	86.3	87.4	88.5	89.9	92.4	93.0	93.2
108.7	109.3	111.6	113.1	114.2	117.7	121.6	121.9	127.6	128.0
129.7	130.8	134.1	137.5	139.2	140.3	153.0	153.8	183.3	185.1

TABLE 2

Estimates of Parameters from First m Order Statistics of Samples in Table 1

	m	$\hat\theta$	$\hat\theta\|\alpha$	$\hat\theta\|c$	$\hat\theta\|\alpha, c$	$\hat\alpha$	$\hat\alpha\|\theta$	$\hat\alpha\|c$	$\hat\alpha\|\theta, c$	$\hat c$	$\hat c\|\theta$	$\hat c\|\alpha$	$\hat c\|\theta, \alpha$
G2	10	548.5	119.7	215.8	112.4	0.790	2.173	1.363	2.056	47.00	23.74	23.83	30.88
	20	68.0	98.6	103.2	99.1	3.203	1.931	1.940	1.980	0.00	32.90	30.73	30.17
	30	98.6	95.6	92.9	96.7	1.930	1.899	2.072	1.964	33.37	34.04	31.82	30.04
	40	53.9	86.3	72.8	88.8	3.851	1.776	2.441	1.900	0.00	37.90	35.00	29.53
G3	10	66.5	62.2	125.7	50.6	2.813	3.538	1.665	2.879	4.61	0.00	2.21	11.84
	20	57.9	62.0	92.5	53.8	3.227	3.611	1.952	3.020	0.00	0.00	2.77	13.08
	30	53.0	59.4	75.3	53.6	3.432	3.599	2.206	3.039	0.00	0.00	4.92	13.23
	40	45.6	55.7	60.8	51.8	3.844	3.549	2.555	3.007	0.00	0.00	8.19	13.09
	m	$\hat\theta$	$\hat\theta\|K$	$\hat\theta\|c$	$\hat\theta\|K, c$	$\hat K$	$\hat K\|\theta$	$\hat K\|c$	$\hat K\|\theta, c$	$\hat c$	$\hat c\|\theta$	$\hat c\|K$	$\hat c\|\theta, K$
W2	10	143.2	115.2	136.6	101.3	1.257	1.586	1.372	1.704	12.05	11.72	2.87	6.76
	20	90.8	87.1	83.8	84.8	2.787	2.484	2.091	1.779	0.00	0.00	8.30	4.14
	30	98.7	100.9	96.3	96.1	1.880	1.911	1.780	1.770	7.66	6.87	5.45	5.80
	40	101.8	96.2	92.8	93.3	2.199	2.156	1.945	1.988	2.48	3.75	6.79	5.23
W3	10	95.6	91.1	93.4	92.7	1.612	1.515	2.954	2.696	37.40	38.07	21.04	16.43
	20	100.6	76.0	81.7	84.8	5.226	5.199	3.747	2.699	0.00	0.54	27.25	11.67
	30	82.3	99.5	95.8	95.7	2.143	2.844	2.719	2.686	33.12	16.88	16.31	15.94
	40	83.5	101.6	95.5	97.0	2.330	2.844	2.715	2.775	30.80	16.69	14.86	16.18

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The iterative estimation procedures for the parameters of Gamma and Weibull populations were programmed in FORTRAN and run on the IBM 7094 computer. Most cases required only a few seconds of machine time, and even the most difficult case took only about 2 minutes.

5. Concluding Remarks

Mickey, Mundle, Walker, and Glinski [12] have pointed out that maximum-likelihood estimation of the parameters of a Gamma population is regular in the sense defined by Cramér [2, p. 479] if and only if the value of the shape parameter is greater than two. It is a well known fact that the same is true for a Weibull population. The main contributions of the present paper are iterative estimation procedures which provide maximum-likelihood estimates of the parameters of Gamma and Weibull populations. The results obtained in the numerical examples are surprisingly good even in cases in which estimation is non-regular, except when one is obviously asking for too much information from too few observations, the extreme case being the estimation of all three parameters from the first 10 of 40 observations. The asymptotic variances and covariances of the estimators are given in another paper by Harter and Moore [6].

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