Eur J Forest Res (2006) 125: 427–439
DOI 10.1007/s10342-006-0138-2

ORIGINAL PAPER

Ján Merganič · Hubert Sterba

Characterisation of diameter distribution using the Weibull function: method of moments

Received: 2 April 2006 / Accepted: 11 May 2006 / Published online: 28 June 2006

Abstract The shape of the diameter distribution is one of the important elements characterising forest stand structure. In this work we present the application of the method of moments for the estimation of the parameters of a 2-parameter Weibull function. Due to its properties, this function is often used for the description of the diameter distribution in forestry. The work analyses the properties of the Weibull function and its application to a data set representing natural (virgin forest Babia hora) and managed forests (Litschau and Forest School Enterprise of Technical University Zvolen). The parameters of the Weibull function are simply and reliably estimated from the basic stand variables, namely the mean diameter and the coefficient of variation of the diameters. The method is general and does not require specific parameterisation, e.g. for the individual tree species. The work also presents a new algorithm for the estimation of parameters in cases, where tree diameters are measured from a certain minimum recording limit. Based on this study, we suggest using the Weibull function in forest stands only for uni-modal diameter distribution with a mean diameter above 7 cm.

Keywords Weibull function · Diameter distribution · Moments · Complete data · Truncated data

Communicated by Hans Pretzsch

J. Merganič · H. Sterba
Department of Forest and Soil Sciences,
Institute of Forest Growth Research,
BOKU University of Natural Resources and Applied Life Sciences,
Peter-Jordan-Straße 82, 1190 Vienna, Austria

J. Merganič (✉)
Forest Research, Inventory and Monitoring—FORIM,
kpt. Nálepku 277/11, 073 01 Sobrance, Slovak Republic
E-mail: j.merganic@stonline.sk
Tel.: +421-904-355451

Introduction

Forest growth and yield models, such as Silva (Pretzsch 1992), Moses (Hasenauer 1994), Prognaus (Sterba et al. 1995) or Bwin (Nagel 1995), are becoming more and more important as they may form the core of decision support systems. They provide the user with forecasts of forest stand developments that may facilitate management decisions. These individual tree growth models need tree lists as input data, i.e. a list of the trees in a stand or a sample plot with at least the diameter at breast height, dbh, given. The distance dependent simulators Silva and Moses additionally need the spatial coordinates of the trees.

Very frequently, tree lists are not available, but rather some general description of the stand including its site index, density and mean diameter. In such cases, tree list needs to be generated from these measures. Such generators have been developed e.g. by Pretzsch (1993), Pommerening (2000) and several others. The value of these generators depends much on how reliably they can reproduce the dbh-distribution of differently structured stands. Commonly a certain kind of mathematical frequency distribution of breast height diameters is hypothesised, and the reliability of the generator depends on (1) how exactly this mathematical distribution fits the real dbh-distribution, (2) how accurate its parameters can be estimated, and (3) how accurately the frequency of trees within a given diameter interval can be predicted on the basis of summary stand characteristics. Within a stand other tree parameters, such as tree height, tree volume, tree assortment value etc. (Bailey and Dell 1973) and even the spatial distribution, are well correlated with dbh; therefore, an accurate generation of diameter distribution is crucial.

Many attempts have been made to describe diameter distributions mathematically: de Liocourt (1898) proposed a reversed J-shaped model for uneven-aged stands, which has been widely used in forestry (Meyer 1952, Leak 1965). For even-aged stands Schiffel (1904) proposed to use the Fekete-distribution, which depicts the dbh as a third degree polynomial of the cumulative frequency. Other distributions that have been successfully applied to the description of the dbh-distribution are, e.g., the Gram–Charlier series, the Pearl–Reed growth curve, Pearson's curve, the gamma-, beta- and log-normal distributions (Bailey and Dell 1973).

The Weibull function has become very popular for describing diameter distributions (Nagel and Biging 1995; Biging et al. 1994; Gerold 1988; Shiver 1988; Zutter et al. 1986; Shifley and Lentz 1985; Gadow 1984; Little 1983; Bailey and Dell 1973) because of its flexibility and ability to fit various distributions from the reversed J-shaped, through left-skewed and symmetrical distributions up to right-skewed distributions (Nagel and Biging 1995; Bailey and Dell 1973). That is also the reason why the Weibull function has earlier frequently been used to generate diameter distributions in forest growth simulators, e.g. Cactos (Wensel et al. 1986), Twigs (Miner et al. 1988), Silva (Pretzsch 1992), and Bwin (Nagel 1995).

In these applications, the parameters of the Weibull distribution are usually derived from stand characteristics using regression analysis. This technique can, however, cause bias in parameter estimates. Therefore, in this paper we present a more accurate, analytical method of parameter estimation. We will thus describe the estimation of the Weibull parameters using the method of moments, and, based on data collected in virgin and managed forests, show the suitability of this estimation method. In addition, we will consider the specific cases, where the measurement of diameters does not start from zero, but from a pre-defined recording limit (minimum diameter).

Data

Virgin forest Babia Hora

Babia Hora, with an elevation up to 1,725 m above sea level, is an isolated mountain massif belonging to the outer Western Carpathian mountain range lying in the northern part of Slovakia at the border to Poland. In 1926, a 118 ha national nature reserve was established to preserve the original forest ecosystems situated far from human settlements. In 1974, the reserve was enlarged and currently it covers an area of 504 ha (Korpel' 1989), where according to the Nature Protection Act no forest management is performed. The reserve encompasses the forest stands of Babia hora at an elevation from 1,100 m above sea level composed almost entirely of Norway spruce (*Picea abies* L. Karst.) with a small admixture of silver fir (*Abies alba* Mill.) and rowan (*Sorbus aucuparia* L.). The significance of these forest ecosystems was confirmed by a study of the genetic structure of spruce, which revealed that the spruce population of Babia hora can be regarded as a gene pool of the original populations of Norway spruce of the Western Carpathians (Gömöry 1988).

In the region of the Babia Hora national nature reserve, 57 permanent circular sample plots were established in 2002 (Merganič et al. 2003), each with an area of 0.05 ha (i.e. radius = 12.62 m). The plots are located at an elevation from 1,100 m above sea level to the timber line (approximately 1,500 m). They are equally distributed to four pre-defined elevation categories (below 1,260, 1,260–1,360, 1,360–1,460 m and above 1,460 m above sea level) and three development stages of virgin forests: stage of growth, maturity and breakdown as defined by Korpel' (1989). On each sample plot, every tree with a diameter at breast height above 7 cm was measured. Trees higher than 1.3 m but with dbh $\leq$ 7 cm were measured on the second concentric circle. Its radius was estimated directly in the field according to Šmelko (1968), who defined the optimum plot size as one that contains 15 to 25 trees. However, the size of this second concentric plot never exceeded the area of the first circle, i.e. its maximum radius was 12.62 m.

Managed forests in Litschau

In 1977, 22 permanent plots were established in mixed Norway spruce (*P. abies* L. Karst.)—Scots pine (*Pinus sylvestris* L.) stands in Seilern-Aspang's Forest enterprise, in the Austrian part of the Bohemian Massif. The elevation of these stands varies between 400 and 600 m. The predominant soil type is a podsol, sometimes with pseudogley dynamics. The management of this enterprise is presently shifting from a clear-cutting system towards a continuous-cover system with single-tree selection harvests. The plots were established and measured in 5-year intervals in order to parameterise the first distance dependent forest growth simulator for mixed-species stands (Sterba 1982). The stand age ranged from 15 to 110 years, and their dominant height from 5 to 31 m. The proportion of Norway spruce (by basal area) ranged from 27 to 100%. The plot size was between 200 and 1,600 m^2 depending on the dominant height at the time of establishment. Most of the plots were heavily damaged by snow breakage in the winter 1979/80. Thinning was performed in 1983, releasing individual trees that belonged to crown class "dominant" and "co-dominant", using Johann's (1982) way to define the neighbours being removed. All the older plots now comprise at least one additional layer of regeneration. At each of the remeasurements of the plots, individual tree co-ordinates, tree species, breast height diameter, tree height and height to the crown base were measured for every tree exceeding a height of 1.3 m; thus there was no diameter recording limit.

Managed forests of the Technical University Zvolen

The third set of data used in this study originates from the permanent inventory plots (PIP) established in 1977 with the aim of examining the representativity of forest inventory sample plots (Šmelko 1979). The PIP plots are located in forest stands with a structure typical for the forests of the Forest School Enterprise (FSE) of the Technical University Zvolen. This enterprise has a total area of 8,044 ha, out of which 7,744 ha are covered by forests. Within the forested area, 32% is identified as management forest, 7% as protection forest, and the rest as the forests for special purposes (e.g. for recreation, research, hunting). All the PIP plots are situated in the forest stands that are subjected to forest management according to the approved forest management plans, i.e. management operations on these plots are not influenced by research activities. The area of the PIP plots ranges from 3 to 7 ha. For the purpose of this work, 9 permanent inventory plots were used, each of them representing a specific stand structure. Together they cover the whole range of species composition from very diverse stands (PIP 8 and 1) to pure homogenous stands (PIP 3).

All trees on each PIP were numbered and their position within the plot is described by their co-ordinates. For every tree, besides tree species, the following basic biometrical characteristics were determined: diameter at breast height (dbh), tree height, and tree volume. Minimum recording dbh was 2.0 cm (PIP No. 5), 7.0 cm (PIP No. 1, 3, 4 and 7), and 8.0 cm in stands No. 2, 6, 8 and 9.

Methods

Weibull function

The Weibull distribution is named after G.W. Weibull, who in 1939 developed a statistical theory of the strength of materials based on the probability distribution proposed by Fischer and Tippett in 1928. The name Weibull distribution was first used by Mann (1967, 1968 cif. Bailey and Dell 1973). The complete three-parameter probability density function (pdf) is defined as follows:

$$f(x) = \frac{c}{b} \cdot \left(\frac{x-\alpha}{b}\right)^{c-1} \cdot \mathrm{e}^{-\left(\frac{x-\alpha}{b}\right)^{c}} \tag{1}$$

where: $f(x)$ is density of trees with the size x ($0 \leq x$), α is a location parameter, b is a scale parameter, and c is a shape (slope) parameter.

The shape parameter c is a dimensionless number. If $c < 1$, the function has a reversed J shape. In the case when $c = 1$, the function becomes the exponential distribution. For $1 < c < 3.6$, the Weibull density function is mound shaped and left-skewed, whereby when c equals 2, the curve results in the Rayleigh-distribution, which is a special case of the χ^2-distribution. If the c parameter reaches the value 3.6, the coefficient of skewness of the pdf function approaches 0 meaning that the shape of the Weibull function approximates the shape of the normal distribution. For $c > 3.6$ the distribution is right-skewed.

The scale parameter b has the same dimension as the variable x, i.e. in the case of the dbh-distribution as the breast height diameter. Its value determines the location of the peak of the pdf curve. Thus, if c is constant an increase in this parameter stretches out the pdf curve and decreases its height. However, the parameter b does not influence the overall shape or behaviour of the distribution.

The location parameter α indicates the minimum value of x and thus, the position of the distribution along the x-axis. The value of α does not affect the shape of the pdf function. In diameter distribution functions, this parameter is equal to the minimum tree diameter, i.e. if $\alpha = 0$, the minimum diameter is 0 cm and the distribution starts at $x = 0$.

Applying the condition $\alpha = 0$ to Eq. 1 we obtain the two-parameter Weibull probability density function, which will be used subsequently. It is given by:

$$f(x) = \frac{c}{b} \cdot \left(\frac{x}{b}\right)^{c-1} \cdot \mathrm{e}^{-\left(\frac{x}{b}\right)^{c}} \tag{2}$$

Consequently, the cumulative distribution function (cdf) is defined as follows:

$$F(x) = 1 - \mathrm{e}^{-\left(\frac{x}{b}\right)^{c}} \tag{3}$$

Setting $x = b$ leads to the cumulative probability corresponding to the scale parameter b, $F(b) \approx 0.63$. Thus, the parameter b can be interpreted as the 63rd percentile of the diameter distribution, i.e. about 63% of trees have the diameter smaller than b (Bailey and Dell 1973).

The inverse function of the Weibull-cdf is given by:

$$x = b \cdot [-\ln(1 - F(x))]^{\frac{1}{c}} \tag{4}$$

This function is used for generating a tree diameter, whereby $F(x)$ is replaced by a random number from a uniform distribution on the interval (0;1).

In forestry, inventory data are frequently incomplete, i.e. they usually do not include observations below a certain minimum diameter. In such cases we talk about the truncated or censored data on the left end. For these cases, an extra parameter (T) needs to be included in the Weibull-function and hence, Eqs. 2, 3, and 4 look as follows (Nagel and Biging 1995):

$$f_T(x) = \frac{c}{b} \cdot \left(\frac{x}{b}\right)^{c-1} \cdot \mathrm{e}^{\left(\frac{T^c - x^c}{b^c}\right)} \tag{5}$$

$$F_T(x) = 1 - \mathrm{e}^{\left(\frac{T}{b}\right)^{c}} \mathrm{e}^{-\left(\frac{x}{b}\right)^{c}} \tag{6}$$

$$x_T = b \cdot \left[\left(\frac{T}{b}\right)^{c} - \ln(1 - F_T(x))\right]^{\frac{1}{c}} \tag{7}$$

Estimation of Weibull parameters using the method of moments

Analytical methods for estimating the parameters of the Weibull distribution include the least square and maximum likelihood methods, or the estimation from particular percentiles, or—only scarcely—from the moments (Al-Fawzan 2000).

In forestry applications, which use the Weibull function for the description of diameter (or other) distributions, most authors estimate the parameters using the maximum likelihood (Zutter et al. 1986; Gadow 1984; Little 1983; Bailey and Dell 1973) or percentile methods (Gerold 1988). The method of moments is used infrequently and often only when the intention is to compare the applicability of selected analytical methods for the estimation of Weibull-parameters (Kangas and Maltamo 2000; Maltamo et al. 2000; Shiver 1988; Nanang 1998; Ueno and Ôsawa 1987; Zarnoch and Dell 1985; Shifley and Lentz 1985). Few examples which deal with the algorithm used to calculate the Weibull parameters by the method of moments can be found in Burk and Newberry (1984) and Ek et al. (1975).

The method of moments is, due to its simplicity and accuracy of parameter estimation, described as the best method by several authors, e.g. Al-Fawzan (2000), Nanang (1998), Ueno and Ôsawa (1987), Shifley and Lentz (1985). Moreover, this method is convenient as the parameter calculation can be done on the basis of the mean and variance of the dataset. Thus, the needed moments can be estimated from the samples in an unbiased way, whereas the estimation of maximum dbh, which is in some cases used for the estimation of the Weibull parameters (Nagel and Biging 1995), is increasingly biased with decreasing sample size.

The basic moment equation of the complete two-parameter Weibull function is (Weisstein 2003):

$$M(r) = b^r \cdot \Gamma\left(1 + \frac{r}{c}\right) \tag{8}$$

with: r being the order of the moment M, b and c are the parameters of the Weibull function, and Γ () is the Gamma function.

The moment equation of the left-truncated function is given by:

$$M(r) = \int_T^\infty x^r \cdot \left(\left\{ 0 \quad \text{if} \quad 0 \leqslant x.\right) = \int_T^\infty x^r \cdot f_T(x)\mathrm{d}x \tag{9}$$

As given in Weisstein (2003), the statistical characteristics of the function, i.e. arithmetic and quadratic mean (AM, QM), standard deviation of arithmetic and quadratic mean (SD_AM, SD_QM), and coefficient of variation of arithmetic and quadratic mean (CV_AM, CV_QM), can be calculated from the moments using the following equations:

$$\text{AM} = M(1) \tag{10}$$

$$\text{QM} = \sqrt{M(2)} \tag{11}$$

$$\text{SD\_AM} = \sqrt{M(2) - (M(1))^2} \tag{12}$$

$$\text{SD\_QM} = \sqrt{M(2) - 2 \cdot \sqrt{M(2)} \cdot M(1) + M(2)} \tag{13}$$

$$\text{CV\_AM} = \frac{\text{SD\_AM}}{\text{AM}} \cdot 100 \text{or} \text{CV\_QM} = \frac{\text{SD\_QM}}{\text{QM}} \cdot 100 \tag{14}$$

Using the same mathematical principles we derived the moment equations for the other two statistical characteristics, skewness and kurtosis:

$$\text{Skewness} = \frac{M(3) - M(1) \cdot M(2) + 2 \cdot (M(1))^3}{\left(\sqrt{M(2) - (M(1))^2}\right)^3} \tag{15}$$

$$\text{Kurtosis} = \frac{M(4) - 4 \cdot M(1) \cdot M(3) + 6 \cdot (M(1))^2 \cdot M(2) - 3 \cdot (M(1))^4}{\left(\sqrt{M(2) - (M(1))^2}\right)^4} \tag{16}$$

Complete Weibull function

Substituting SD_AM and AM, or SD_QM and QM in Eq. 14 for Eqs. 10 and 11 or 12 and 13, respectively, and the respective moments for Eq. 8 shows, that the coefficient of variation depends only on the parameter c:

$$\text{CV\_AM} = \left[\frac{\sqrt{\Gamma\left(\frac{c+2}{c}\right) - \left[\Gamma\left(\frac{c+1}{c}\right)\right]^2}}{\Gamma\left(\frac{c+1}{c}\right)}\right] \cdot 100 \text{ or}$$

$$\text{CV\_QM} = \left[\frac{\sqrt{2 \cdot \Gamma\left(\frac{c+2}{c}\right) - 2 \cdot \sqrt{\Gamma\left(\frac{c+2}{c}\right)} \cdot \Gamma\left(\frac{c+1}{c}\right)}}{\sqrt{\Gamma\left(\frac{c+2}{c}\right)}}\right] \cdot 100 \tag{17}$$

The relationship between the parameter c and the coefficient of variation is depicted in Fig. 1a.

Similarly, simplifying Eqs. 15 and (16) reveals that skewness and kurtosis of the Weibull function are also independent of the parameter b.

By solving Eq. 17 iteratively for c until the observed coefficient of variation is achieved, the Weibull parameter c can be found. Once c is estimated, the parameter b can be calculated by rearranging the moment Eq. 8 with $r = 1$ in the case when AM is used, and $r = 2$ if QM is applied:

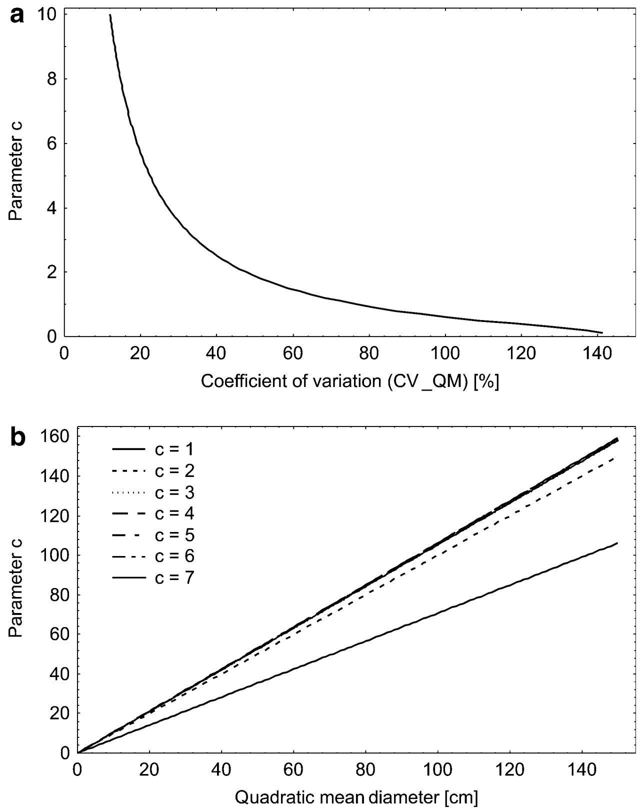

Fig. 1 Relationships between the parameter c and coefficient of variation CV_QM of the complete Weibull function (**a**), and between the parameter b and quadratic mean for different values of the parameter c(**b**)

$$b = \frac{\mathrm{AM}}{\Gamma\left(\frac{c+1}{c}\right)} \text{or} b = \frac{\mathrm{QM}}{\sqrt{\Gamma\left(\frac{c+2}{c}\right)}} \tag{16}$$

Figure 1b shows, how b depends on c and quadratic mean.

Truncated Weibull function

When data are truncated, it is incorrect to use the complete Weibull function for their description (Zutter et al. 1986). Instead, it is required to use the truncated form of the Weibull function (Eq. 5). The estimation of the parameters of the truncated function is, however, more complicated than for the complete function, as the values of coefficient of variation, skewness and kurtosis depend also on the values of the parameter b. An example of the relation between the parameter b and the coefficient of variation CV_QM is given in Fig. 2.

The algorithm for the estimation of the parameters of the truncated Weibull function consists of the following steps:

1. Determine the point of truncation T. In many cases, T is not equal to the lowest measured diameter, but the diameter, from which the measurements were performed. This value should be given either in the manual for the field measurements or in the lists from the field measurements.
2. Calculate the mean and the coefficient of variation from the measured data.
3. Estimate the initial value of the parameter c using Eq. 17.
4. Estimate the initial value of the parameter b using Eq. 18.
5. Iteratively change b and c until we obtain the modelled mean and the coefficient of variance calculated using the moment Eq. 9 that approximate the observed mean and coefficient of variation most (Fig. 3). The change of the parameters b and c, i.e. the iteration steps, can be set by a user with respect to his demands on the accuracy of the estimation. In this study the steps were set to 0.1 and 0.01 for the parameters b and c, respectively. These values were proved sufficient for practical use.

Although the algorithm for the parameter estimation of the truncated function seems to be rather cumbersome, it is very useful because (1) most forestry data are truncated, and (2) if the estimated values of the parameters c and b for the truncated Weibull distribution are used in the complete function, the distribution

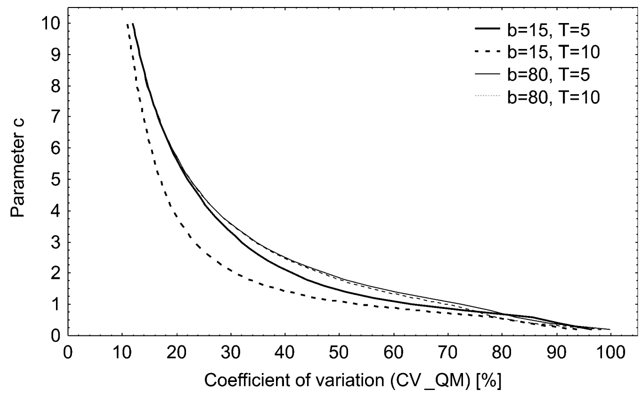

Fig. 2 Influence of the parameter b and the minimum recording limit T on the relationship between the parameter c and the coefficient of variation CV_QM for the truncated Weibull function

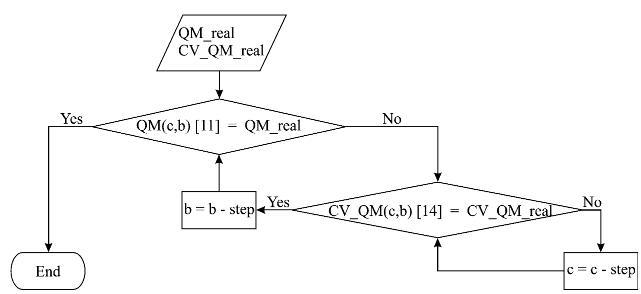

Fig. 3 Iteration process for estimating the parameters of truncated Weibull function by the method of moments from quadratic mean and its coefficient of variance (Step 5 in the algorithm presented in Truncated Weibull function). The iteration steps, i.e. the change of the parameters b and c, were set to 0.1 and 0.01, respectively

of the unmeasured data, i.e. the data below the point of truncation, can be estimated.

Evaluation of Weibull function to describe the diameter distributions of forest stands

The above-stated methodology was applied to observed data in order to test the appropriateness of the Weibull function for the description of diameter distributions. The necessary statistical characteristics, i.e. quadratic mean QM and coefficient of variation CV_QM, were calculated from the tree diameters observed on each plot. These observed statistical characteristics were used to estimate the parameters c and b from Eqs. 17 and 18, respectively. The calculated parameters of the Weibull function were applied to Eqs. 2 or 5 to obtain the modelled distribution curve. To enable visual comparison of the observed distribution and the Weibull pdf, the observed breast height diameters were divided into diameter classes.

Afterwards, the maximum difference between the cumulated frequencies of the observed and modelled Weibull cdf was tested by the Kolmogorov–Smirnov test at two significance levels 0.05 and 0.01. The modelled Weibull cdf was calculated from Eqs. 3 or 6. To complete the analysis, minimum and maximum diameters were calculated from the modelled inverse Weibull cdf, whereby the probability $F(x)$ or $F_T(x)$ of these values was estimated from the sampling size, i.e. from the number of trees in each specific case. Moreover, the algorithm for estimating the parameters of the Weibull function in the case of the truncated data (Fig. 3) was tested on the data from the managed forests in Litschau.

Results

Virgin forest Babia Hora

The analysis was performed on 12 variants of diameter distributions, which were derived by the aggregation of the data from the sample plots situated in the same elevation category and development stage. As shown in Table 1, the modelled quadratic mean QM and coefficient of variation CV_QM are identical with the statistics calculated from the observed data. According to Kolmogorov–Smirnov tests, only 3 distributions out of the 12 analysed variants differed significantly from the modelled Weibull distributions. Two of these belong to the development stage of breakdown, which is characterised by a bimodal distribution. The Weibull function is not able to simulate this kind of distribution. However, this disadvantage could be eliminated when the bimodal distribution is divided into two unimodal distributions representing certain layers of the forest stand, and for each part the parameters of Weibull function may be estimated using the method of moments.

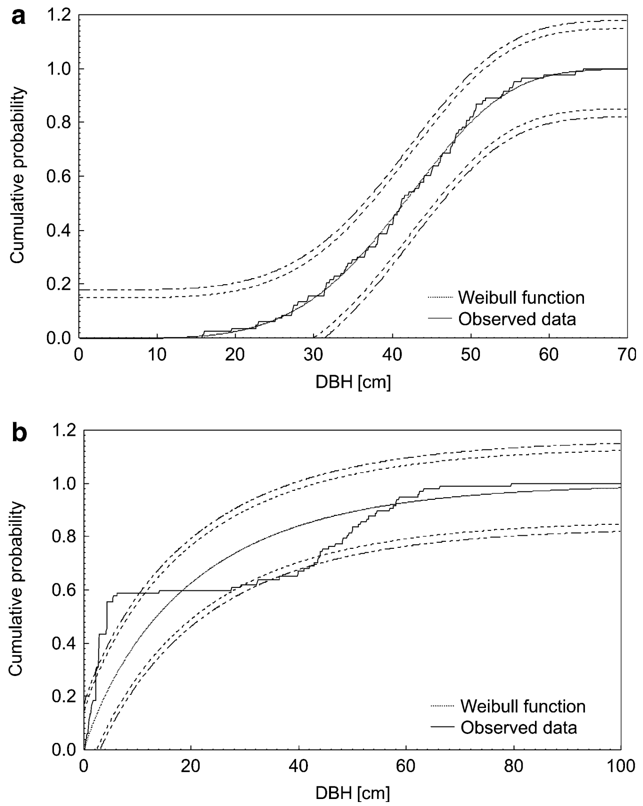

Fig. 4 Kolmogorov–Smirnov test of the consistence of the observed diameter distribution with the modelled Weibull distribution for the best (**a**) and the worst (**b**) fit of Weibull function, where the dashed line represents the critical value of Kolmogorov-Smirnov test at the 5%, and dashed and dotted line represent the critical value at the 1% level

Comparing the observed minimum and maximum diameters with values calculated from the Weibull function shows that the Weibull function is able to represent the tails of the observed diameter distribution well (Table 1). The differences between the observed and modelled diameters are of course higher in the cases, when the curve of the Weibull function deviates more from the observed diameter distribution. The results of the frequency analysis for the cases, when the values of Kolmogorov-Smirnov test were the lowest and the highest are shown in Fig. 4a, b, respectively.

Managed forests Litschau

On the data from Litschau, we analysed if the Weibull function is appropriate to describe diameter distributions of management forests. In this case, an analysed variant was represented by an observed diameter distribution of a particular tree species on a certain permanent inventory plot in a particular year.

The number of Norway spruce trees was sufficient on all 23 plots during all remeasurements. Therefore, all 115 diameter distributions could be evaluated for this species. The number of trees per variant varied between 27

Table 1 Mathematical–statistical description of the observed and modelled diameter distribution and statistical test of the differences between them

Stage elevation categories	Growth				Maturity				Breakdown			
	below 1,260 m	1,260 to 1,360 m	1,360 to 1,460 m	above 1,460 m	below 1,260 m	1,260 to 1,360 m	1,360 to 1,460 m	above 1,460 m	below 1,260 m	1,260 to 1,360 m	1,360 to 1,460 m	above 1,460 m
Observed data												
AM	16.09	12.93	17.81	8.11	46.62	46.12	41.00	23.17	21.42	26.64	37.64	21.07
QM	23.32	17.57	22.85	11.95	50.22	49.00	42.26	25.61	31.93	35.53	39.96	24.33
SD_AM	16.92	11.92	14.34	8.79	18.80	16.67	10.33	10.94	23.81	23.66	13.58	12.22
SD_QM	18.40	12.80	15.20	9.60	19.14	16.92	10.41	11.21	26.05	25.30	13.78	12.65
CV_AM	105.14	92.22	80.51	108.39	40.32	36.14	25.19	47.21	111.17	88.82	36.08	57.98
CV_QM	78.91	72.82	66.55	80.29	38.12	34.52	24.62	43.78	81.58	71.21	34.49	51.98
Skewness	1.91	2.02	1.41	1.83	− 0.44	− 0.01	− 0.23	0.42	0.63	0.31	0.62	0.81
Kurtosis	5.88	7.56	5.01	5.97	3.47	2.97	2.76	3.53	1.72	1.64	4.94	3.26
D_{MIN}	0.50	0.50	0.90	0.30	0.30	7.40	15.80	1.40	0.30	0.60	2.20	1.20
D_{MAX}	81.00	65.30	73.30	44.70	100.00	89.00	64.50	62.60	79.60	76.50	82.00	58.60
N	262	350	193	358	78	73	83	115	97	75	45	127
Weibull function												
Parameter b	15.695	13.319	19.092	7.801	52.388	51.590	44.906	26.140	20.219	27.674	42.076	23.654
Parameter c	0.950	1.084	1.248	0.923	2.669	3.017	4.502	2.238	0.898	1.124	3.021	1.781
AM	16.06	12.91	17.79	8.10	46.57	46.08	40.98	23.15	21.31	26.52	37.58	21.05
QM	23.32	17.57	22.85	11.95	50.22	49.00	42.26	25.61	31.93	35.53	39.96	24.33
SD_AM	16.91	11.92	14.34	8.79	18.79	16.66	10.33	10.94	23.78	23.64	13.58	12.21
SD_QM	18.40	12.80	15.20	9.60	19.14	16.92	10.41	11.21	26.05	25.30	13.78	12.65
CV_AM	105.28	92.30	80.60	108.51	40.36	36.16	25.20	47.24	111.62	89.14	36.12	58.03
CV_QM	78.91	72.82	66.55	80.29	38.12	34.52	24.62	43.78	81.58	71.21	34.49	51.98
Skewness	2.16	1.77	1.43	2.26	0.29	0.16	− 0.18	0.49	2.35	1.68	0.16	0.79
Kurtosis	10.11	7.57	5.82	10.85	2.80	2.73	2.81	3.01	11.61	7.05	2.73	3.59
D_{MIN}	0.04	0.06	0.28	0.01	10.27	12.47	16.85	3.14	0.12	0.60	11.98	1.56
D_{MAX}	95.63	67.99	72.21	53.24	90.93	83.60	62.46	52.41	110.04	101.69	65.50	57.35
Difference and Kolmogorov–Smirnov test												
ΔD_{MIN}	− 0.46	− 0.44	− 0.62	− 0.29	9.97	5.07	1.05	1.74	− 0.18	0.00	9.78	0.36
ΔD_{MAX}	14.63	2.69	− 1.09	8.54	− 9.07	− 5.40	− 2.04	− 10.19	30.44	25.19	− 16.50	− 1.25
K-St	0.127**	0.071	0.048	0.071	0.126	0.066	0.043	0.070	0.332**	0.214**	0.106	0.042

AM is the arithmetic mean diameter (cm), *QM* quadratic mean diameter (cm), *SD* standard deviation, *CV* coefficient of variation (%), D_{MIN} minimum diameter at breast height (cm), D_{MAX} maximum diameter at breast height (cm), *N* number of observations, Δ difference and *K-St* is the value of the Kolmogorov–Smirnov statistics
Significance level: *95%, **99%

and 1,093, whereby in 30.4% of the distributions the number was less than 50. The difference between the observed distribution and the distribution modelled with the Weibull function was found to be statistically significant at the 1% level in 21 variants, i.e. in 18.2% cases from the total 115 analysed variants (Table 2). Out of the significant cases, 12 variants, i.e. more than half, could be characterised as bimodal distributions suggesting that in these forests Norway spruce occurred in more than one layer.

The extent of the dataset for this tree species allowed us to perform also other analyses, which could explain the significant differences between the observed and modelled diameter distributions. In the first analysis, we tested the differences in mean diameter between the three artificial groups defined according to the results of Kolmogorov–Smirnov test: insignificant result, significant at the 5% level, and significant at the 1% level. Note that statistically significant variants with evident bimodal diameter distribution were excluded from this analysis.

This analysis revealed that the mean diameter of the forest stand strongly influences the shape of the diameter distribution. Young forest stands, characterised by a small quadratic mean diameter QM cannot be described by the Weibull function properly. According to our result, the Weibull function is with 99% probability not suitable for the description of diameter distributions of the forest stands with quadratic mean diameters below 6–7 cm.

Nevertheless, the interpretation of this result must be done with caution. The result of the test is influenced by the size of the dataset, in our case by the number of trees, from which the frequency analysis was performed. Therefore, although statistically significant differences between the observed and modelled distributions were detected mostly in the forest stands with a small mean diameter, these stands are composed of a larger number of trees which on one side decreases the error, but on the other hand causes that small differences become significant.

The results of this analysis fit well with the time trend in the development of the sample plots, since statistically significant differences between the observed and the modelled diameter distributions usually occurred at the beginning of the experiment (young forests or old forests stands with a high proportion of upper layer), or at the end of the experiment, when due to the applied singletree harvesting management vertical stand structure starts to differentiate.

Similarly, we analysed the differences between the three groups with regard to the coefficient of variation of diameters. We found that statistically significant differences between the observed and modelled diameter distributions can be detected if the observed variability of tree diameters in a stand is high, typically above 50%.

The analysed plots consisted of more tree species. However, due to the small number of trees of a particular species on a plot it was not possible to analyse all possible variants. Therefore, we selected only those variants, which included more than 30 individuals of each species. Thus, we obtained 30 variants representing Scots pine, common beech (*Fagus sylvatica* L.) and other broadleaves. Just as for Norway spruce, the estimation of the Weibull function using the method of moments was in most cases successful. Only four modelled diameter distributions were significantly (at the 1% level) different from the observed distributions (Table 2), whereby all significant variants are characterised by bimodal distribution.

The algorithm for the estimation of the parameters of the Weibull function for the truncated data (Truncated Weibull function) was applied to 93 diameter distributions with QM above 10 cm and to a minimum number of trees 30 individuals. The goal of this analysis was to document that the proposed method of the parameters estimation is correct. From the truncated data only the parameters of the truncated Weibull function should be estimated. The obtained values should coincide with the parameters of the complete Weibull function estimated from the complete data. If, however, truncated data are used to estimate the parameters of the complete Weibull function, the estimated values of the parameters contain a systematic error causing a significant bias in the function statistics. Its magnitude depends on the value of T, the data integrity below T, and the distance between

Table 2 Results of Kolmogorov–Smirnov test about the identity of the observed and modelled diameter distribution separately for Norway spruce, Scots pine, common beech and other broadleaved species

Tree species	Significance level			Total
	Not significant	Significant at 5%	Significant at 1%	
	Number of diameter distributions/relative proportion from the total number per species (%)			
Norway spruce	86	8	21	115
	74.78	6.96	18.26	
Scots pine	19	2	2	23
	82.61	8.70	8.70	
Common beech	3	0	0	3
	100.00	0.00	0.00	
Other broadleaves	1	0	2	3
	33.33	0.00	66.67	
Total	109	10	25	144
	75.69	6.94	17.37	100

QM and T. In younger and middle old forest stands the bias can reach 1.5 to 3 cm, in the cases where diameters were measured from the border characterising commercial timber.

Hence, within this analysis we tested the coincidence of the estimated parameters of the truncated and the complete Weibull function. For this aim, the minimum recording limit T was set to 7 cm meaning that from each distribution trees with diameters below 7 cm were excluded. In the next step, the algorithmus for the estimation of the parameters of the truncated Weibull function was applied to these truncated data. The results of the analysis were strongly influenced by the original shape of the distribution function, i.e. if the observed complete distribution was significantly different from the complete modelled Weibull function, the values of the parameters estimated for the truncated Weibull function were also significantly different from the complete modelled Weibull function. This happened mainly for bimodal distributions, which were observed in 22 cases out of the total 93 distributions. After excluding these cases we found that from the remaining 71 distributions in 59 cases (83%), the estimated parameters of the truncated Weibull function deviated only slightly from their original values calculated from the complete data. In these cases, the differences of the parameters b and c between the complete and the truncated functions caused changes of QM and CV_QM of less than 0.5 cm and 5%, respectively. In the remaining cases the truncation of the data caused a more significant change to the shape of the distribution or to the bias in the estimation due to the small number of trees that characterised the diameter distribution.

In the following, we examined how the minimum recording limit T influences the value of quadratic mean QM. The analysis was performed for three T values: 3, 5, and 7, with 71 distributions, out of which 63 represented Norway spruce and 8 Scots pine. The influence of the T value on the average difference was examined with the analysis of variance for the following variants, while the null hypothesis was that the average difference between the modelled and the real value of QM is equal to 0 if:

1. The difference is calculated from the data within the interval $T \rightarrow \infty$,
2. The difference between the modelled and the real value of QM is calculated from the data within the interval $0 \rightarrow T$. Here, we analysed two cases distinguished on the basis of the information about the occurrence of trees below diameter T (which is of Boolean value, 0—they do not occur, 1—they occur):

(a) First, we analysed all the distributions regardless of the information about the occurrence of trees below diameter T;

(b) Second, the observed distributions with no trees below diameter T were excluded, i.e. only filled distributions were analysed (those where the Boolean value was equal to 1).

3. The difference between the modelled and the real value of QM is calculated from the data within the interval $0 \rightarrow \infty$, in the cases (a) and (b) as defined earlier.

In all three cases, the modelled QM was determined from the parameters of the truncated Weibull function for the particular interval using Eq. (11).

The results of the first tested variant revealed that T values do not have a significant influence on the values of QM. Positive values of average differences indicate that the model gives systematically higher QM than the observed QM. However, the absolute value of the difference is very small (below 0.035), which in forestry applications is negligible. This bias can be eliminated by decreasing the convergence factor in the algorithm (Truncated Weibull function). Tree species seems to have an effect on the value of average difference, too, while the bias of light demanding species (Scots pine) was only two-third of the bias of shade bearers (Norway spruce). However, this result is related to the distance between T and QM. As the truncation limit T approaches the value of QM, estimating the parameters of the Weibull function becomes more difficult and a smaller step of convergence factor is required. In the analysed data, Scots pine has higher QM values, i.e. its diameter distributions are moved to the right, whereas Norway spruce represents thinner stands, where the bias is greater.

The analysis of the variant 2(a) revealed that T value has a significant effect on the values of modelled QM within the interval $0 \rightarrow T$, and that the average differences are positively biased. Similarly to the first variant, the tree species was found to influence model bias significantly. While for Norway spruce the bias magnitude was approximately 2 cm regardless of the T value, the average differences of Scots pine were close to the values of the truncation limit T, and increased with increasing T. This is caused by higher QM values for the whole data range (i.e. $0 \rightarrow \infty$) and by the absence of trees with diameter below T.

However, if only filled diameter distributions within the interval $0 \rightarrow T$ were analysed, i.e. the cases where the observed QM was greater than 0 (variant 2b), T value did not have a significant influence on the average difference between the observed and modelled QM. Although the absolute values of average differences were always below zero indicating that the model underestimates QM, they were not significantly different from 0. We assume that reducing the convergence factor would result in a more accurate result.

The test of the third variant documents that the proposed method for the estimation of the parameters of the Weibull function for the truncated data is correct and well-founded. The average difference between the modelled and observed QM was in all tested cases very close to zero. The significant average difference was found only for Norway spruce and $T = 7$ in both analysed cases 3(a) and 3(b) due to the small distance

Table 3 Mathematical–statistical description of the observed and modelled diameter distribution and the statistical test of the differences between them for the selected managed forests SFE TU Zvolen

Tree species	Forest stands	Minimum recording limit T	Number of trees	Observed data				Weibull function						Difference		Kolmogorov–Smirnov test
				QM	CV_QM	D_{MIN}	D_{MAX}	Parameter b	Parameter c	QM	CV_QM	D_{MIN}	D_{MAX}	ΔD_{MIN}	ΔD_{MAX}	
1	6	8.0	2151	29.79	24.76	8.90	60.50	31.62	4.42	29.79	24.77	8.34	50.16	− 0.56	− 10.34	0.043**
2	1	7.0	120	18.17	36.55	9.30	41.10	17.47	2.23	18.17	36.58	7.20	35.65	− 2.10	− 5.45	0.108
2	3	7.0	184	20.81	54.13	8.00	62.90	10.54	0.98	20.83	54.16	7.06	63.81	− 0.94	0.91	0.090
2	6	8.0	1105	28.32	24.82	12.60	54.00	30.05	4.39	28.32	24.82	8.50	46.81	− 4.10	− 7.19	0.065**
2	2	8.0	261	30.43	39.74	9.00	59.20	30.47	2.28	30.43	39.80	8.28	64.84	− 0.72	5.64	0.060
20	4	7.0	4850	18.80	30.22	7.10	39.80	19.50	3.18	18.80	30.24	7.01	38.24	− 0.09	− 1.56	0.031**
20	3	7.0	871	24.80	34.30	7.90	88.30	25.69	2.83	24.80	34.35	7.11	50.55	− 0.79	− 37.75	0.036
20	8	8.0	1475	26.22	24.68	8.20	52.00	27.81	4.40	26.22	24.68	8.28	43.70	0.08	− 8.30	0.025
20	7	7.0	1769	26.37	26.08	8.20	57.50	27.98	4.15	26.37	26.08	7.28	45.46	− 0.92	− 12.04	0.034*
20	9	8.0	1178	32.37	23.95	8.30	65.00	34.38	4.61	32.38	23.96	8.98	52.53	0.68	− 12.47	0.032
20	1	7.0	692	33.67	21.08	11.90	57.70	35.71	5.37	33.67	21.08	10.77	50.66	− 1.13	− 7.04	0.069**
21	3	7.0	943	24.12	49.32	6.80	78.00	19.48	1.44	24.11	49.39	7.02	75.56	0.22	− 2.44	0.027
21	1	7.0	1001	29.65	36.38	7.40	64.10	30.66	2.68	29.65	36.42	7.13	63.13	− 0.27	− 0.97	0.042
22	3	7.0	136	13.92	41.19	7.00	40.00	6.36	1.06	13.93	41.21	7.04	34.37	0.04	− 5.63	0.097
22	8	8.0	3413	15.02	30.09	8.10	34.10	13.71	2.34	15.03	30.08	8.00	34.14	− 0.10	0.04	0.021
22	7	7.0	3832	15.47	31.97	7.70	72.80	15.11	2.55	15.47	31.97	7.01	34.77	− 0.69	− 38.03	0.055**
22	2	8.0	731	17.67	38.38	2.00	55.80	14.23	1.68	17.68	38.39	8.02	45.29	6.02	− 10.51	0.071**
22	1	7.0	147	18.06	31.43	9.50	39.30	18.49	2.93	18.07	31.47	7.27	32.13	− 2.23	− 7.17	0.102
22	9	8.0	2894	18.26	33.79	8.00	46.10	17.49	2.35	18.27	33.82	8.01	42.71	0.01	− 3.39	0.038**
50	8	8.0	194	22.14	33.49	8.30	41.30	22.51	2.72	22.14	33.53	8.25	41.62	− 0.05	0.32	0.048
50	7	7.0	240	22.94	33.19	8.30	41.80	23.79	2.93	22.94	33.22	7.34	42.63	− 0.96	0.83	0.033
50	9	8.0	175	28.92	33.72	8.70	55.10	30.08	2.92	28.92	33.77	8.69	52.90	− 0.01	− 2.20	0.071

Tree species: 1-Spruce, 2-Fir, 20-Oak, 21-Beech, 22-Hornbeam, 50-Cherry

QM is the quadratic mean diameter (cm), *CV* coefficient of variation (%), D_{MIN} minimum diameter at breast height (cm), D_{MAX} maximum diameter at breast height (cm) and Δ the difference

Significance level: *95%, **99%

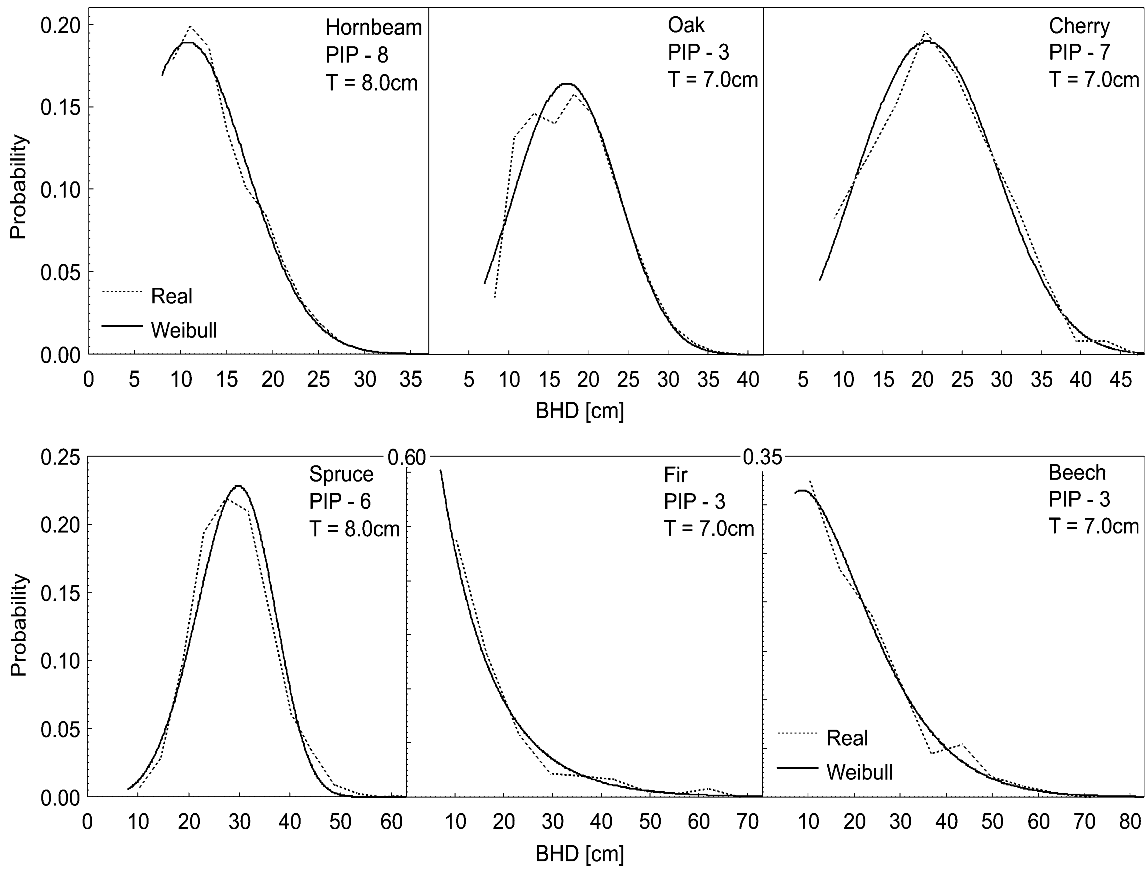

Fig. 5 Example of the estimation of six different diameter distributions for six tree species that occurred within the selected managed forests SFE TU Zvolen

between T and QM and the convergence factor. However, in forestry applications such values of average differences are negligible.

Managed forests SFE TU Zvolen

For a sample of selected forest stands (PIP) we applied the method of moments to the truncated data. The point of truncation was $T = 7.0$ cm for forest stands No. 1, 3, 4 and 7, $T = 8.0$ cm in stands No. 2, 6, 8 and 9, and $T = 2.0$ cm for stand No. 5. From the collected data 22 different diameter distributions were created, whereby each variant described a single tree species. Since the forest stands are of a larger size, the diameter distributions are based on a relatively large sample of 120 to 4,850 trees. It emerged that the differences between the observed and modelled distributions were often significant because of the large number of trees (8 cases out of 22 tested variants), although the Kolmogorov–Smirnov values were low (Table 3). From the results we also note that the differences between the observed and the modelled values of D_{MIN} and D_{MAX} decrease with an increasing number of trees. Figure 5 documents the flexibility of the Weibull model for describing various shapes of distributions on an example of six different diameter distributions. Overall, the estimation of diameter distribution using the Weibull function seems to work well also in this case.

Discussion and conclusions

The literature review confirmed that the method of moments is one of the most accurate methods for estimating the Weibull function parameters (Al-Fawzan 2000; Nanang 1998; Ueno and Ôsawa 1987; Shifley and Lentz 1985). Moreover, as it was shown in this work as well as in Shifley and Lentz (1985), the advantage of the method of moments is its minimal data requirement. To create the diameter distribution it is sufficient to know the mean tree diameter and the coefficient of variation of the diameters. Our analyses also showed that the presented procedure yields distributions with summary statitistics that are in agreement with the observed ones (Tables 1, 3). Therefore, it should not happen that the actual mean diameter that enters the model as an input variable will differ significantly from the mean diameter calculated from generated data. Other techniques of parameters estimation, e.g. regression methods that are based on the relation between the parameters of the Weibull function and selected stand characteristics (most frequently mean diameter, maximum or minimum diameter, mean height, etc.), can lead to biased estimates, since the relationships of these variables mainly with the parameter c are very loose (Nagel and Biging 1995; Biging et al. 1994; Little 1983; Van Laar and Mosandl 1989; Clutter and Belcher 1978).

In addition, using the method of moments makes it easier to parameterise the distribution function. Taking into account the statistical character of the observed variables, we have to be conscious of the fact that the real values of the mean diameter and the coefficient of variation are only estimated, since they are calculated from the samples. Nevertheless these estimates are always better than those of a maximum dbh. Data representing the whole population are seldom available. Therefore, the observed values are usually affected by sampling error. In some cases, where the data are somehow distorted or where the sampling estimate is not sufficiently consistent, the Weibull function will not behave logically. For example, during the simulation of diameters we can obtain illogical (very high) values. If we use a very small probability $F(x)$ or $F_T(x)$, that corresponds the simulation of 10,000,000 trees, in Eqs. 4 or 7, the modelled maximum diameter should not exceed the double of the observed maximum diameter. If the result from Eqs. 4 or 7 is outside the predefined range, such value should be excluded and a new value should be generated. However, this happens very rarely and does not affect statistical characteristics of the function. An advantage of this method over other estimation techniques is that it is universal and does not require special parameterisations, e.g. for individual tree species (Table 3; Fig. 5). For example, Nagel and Biging (1995), who used the regression method, estimated the parameters of the Weibull function for each tree species separately. The work also presents a new algorithm for parameters estimation in cases, where tree diameters are measured from a certain recording limit. As Zutter et al. (1986) pointed out, the application of the complete Weibull distribution to describe truncated data can cause great systematic errors in the parameters. This bias is increasing if the value of QM is close to the pre-defined recording limit.

The Weibull function is not suitable for the description of diameter distribution in young forest stands. A similar finding was reported by Nanang (1998), who analysed the suitability of several different types of distributions for the description of diameter distributions. The author found that for the definition of the diameter distribution in young forests a log-normal distribution is more appropriate. Based on our experience as well as on the results of Nanang (1998) we suggest using the Weibull function only when the mean diameter is greater than 7 cm.

The information obtained can be applied with great advantage to modelling of the forest stand structure. Among the existing individual-tree forest growth simulators, the only one using the method of moments is according to our knowledge the growth simulator TWIGS (Miner et al. 1988) in its sub-model TREE-GEN. This generator, however, utilises the three-parameter Weibull function and does not allow modelling diameter distributions if tree diameters are measured from a certain pre-defined recording limit. From our point of view, the two-parameter Weibull function is more suitable than the three-parameter function because logically the function should start from 0. This assumption should be valid also in very old forest stands. In such cases, although the function begins from 0, it is very improbable that during the diameter simulation we get a diameter close to 0.

Acknowledgements This work was carried out during the research stay of J. Merganič at the BOKU University of Natural Resources and Applied Life Sciences in Vienna, financed by ÖAD during 2002–2003. We are very grateful to Prof. Šmelko for providing us with the data from permanent inventory plots of FSE TU Zvolen for the presented analyses. We also thank Robert Williams and two anonymous reviewers for their comments, which helped us to improve the quality of the presented paper.

References

Al-Fawzan MA (2000) Methods for estimating the parameters of the Weibull distribution. InterStat, statistics on the Internet. Online in Internet: URL:http://www.ip.statjournals.net:2002/InterStat/ARTICLES/2000/abstracts/O00001.html-ssi [Cited 2006-03-10]

Bailey RL, Dell TR (1973) Quantifying diameter distributions with the Weibull function. For Sci 19:97–104

Biging GS, Robards TA, Van Deusen PC, Turnblom EC (1994) The predictive models and procedures used in the Forest Stand Generator (STAG), Hilgardia

Burk TE, Newberry JD (1984) A simple algorithm for moment-based recovery of Weibull distribution parameters. For Sci 30:329–332

Clutter JL, Belcher DM (1978) Yield of site prepared slash pine plantations in the lower coast plain of Georgia and Florida. In: Fries J, Burkhart H, Max T (eds) Growth models for long term forecasting of timber yields. School of Forest and Wildlife Resources, VPI and State University. Publ. No. FWS-1-78, pp 53–70

Ek AR, Issos JN, Bailey RL (1975) Solving for Weibull diameter distribution parameters to obtain specified mean diameters. For Sci 21:290–292

von Gadow K (1984) Die Erfassung von Durchmesserverteilungen in gleichaltrigen Kiefernbeständen. Forstw Cbl 103:360–374

Gerold D (1988) Beschreibung der Durchmesserstruktur und ihrer Entwicklung mit Hilfe der Weibullverteilung. Wissenschaftliche Zeitschrift der Technischen Universität Dresden 37(6):221–224

Gömöry D (1988) Influence of management on the genetic structure of the populations of Norway spruce (Picea abies Karst.) in Slovakia. PhD Thesis. Faculty of Forestry, Technical University Zvolen

Hasenauer H (1994) Ein Einzelbaumwachstumssimulator für ungleichaltrige Fichten-Kiefer- und Buchen-Fichtenmischbestände. Forstl. Schriftenreihe Univ. f. Bodenkultur. 8. Öster. Ges. F. Waldökosystemforschung und experimentelle Baumforschung an der Univ. f. Bodenkultur, Wien

Johann K (1982) Der "A-Wert"—ein objektiver Parameter zur Bestimmung der Freistellungsstärke von Zentralbäumen. Deutscher Verband Forstlicher Forschungsanstalten, Sektion Ertragskunde. Weibersbrunn, pp 146–158

Kangas A, Maltamo M (2000) Performance of percentile based diameter distribution prediction and Weibull method in independent data sets. Silva Fenn 34(4):381–398

Korpel' Š (1989) Virgin forests of Slovakia. Veda, Slovenská Akadémia Vied, Bratislava (in Slovak)

Leak WB (1965) The J-shaped probability distribution. For Sci 11: 405–409

Liocourt F de (1898) De l'aménagement des sapineres. Bull Soc Forest Franche-Comté et Belfort 4:396–409

Little SN (1983) Weibull diameter distributions for mixed stands of western conifers. Can J For Res 13:85–88

Maltamo M, Kangas A, Uuttera J, Torniainen T, Saramäki J (2000) Comparison of percentile based prediction methods and the Weibull distribution in describing the diameter distribution of heterogeneous Scots pine stands. For Ecol Manage 133(3):263–274

Merganič J, Vorčák J, Merganičová K, Ďurský J, Miková A, Škvarenina J, Tucek J, Mindáš J (2003) Diversity monitoring of mountain forests in the North Orava region. EFRA, Tvrdošín. Online in Internet: URL: http://www.efra.sk/projekty.php (in Slovak) [Cited 2006-03-10]

Meyer HA (1952) Structure, growth, and drain in balanced uneven-aged forests. J For 50:85–92

Miner CL, Walters NR, Belli ML (1988) A guide to the TWIGS program for the North Central United States. Gen Tech Rep NC-125. U.S. Department of Agriculture, Forest Service, North Central Forest Experiment Station, St. Paul

Nagel J (1995) BWERT: programm zur Bestandesbewertung und zur Prognose der Bestandesentwicklung. DFFA, Sektion Ertragskunde, Jahrestagung Joachimsthal, pp 184–198

Nagel J, Biging GS (1995) Schätzung der Weibullfunktion zur Generierung von Durchmesserverteilungen. Allg Forst-uJ-Ztg 166(9–10):185–189

Nanang DM (1998) Suitability of the Normal, Log-normal and Weibull distributions for fitting diameter distributions of neem plantations in Northern Ghana. For Ecol Manage 103(1):1–7

Pommerening A (2000) Methoden zur Reproduktion und Fortschreibung einzelner konzentrischer Probekreise von Betriebs- und Landeswaldinventuren. Forstarchiv 71(5):190–199

Pretzsch H (1992) Konzeption und Konstruktion von Wuchsmodellen für Rein- und Mischbestände. Forstl Forschungsberichte München, pp 115–358

Pretzsch H (1993) Analyse und Reproduktion räumlicher Bestandesstrukturen - Versuche mit dem Strukturgenerator STRUGEN. Schriften aus der Forstlichen Fakultät der Universität Göttingen und der Niedersächsischen Forstlichen Versuchsanstalt, 114, 87 pp

Schiffel A (1904) Wuchsgesetze normaler Fichtenbestände. Mitt Forstl Versuchswesen Österreichs. XXIX

Shifley SR, Lentz E (1985) Quick estimation of the three-parameter Weibull to describe tree size distributions. For Ecol Manage 12:195–203

Shiver BD (1988) Sample sizes and estimation methods for the Weibull distribution for unthinned slash pine plantation diameter distributions. For Sci 34(3):809–814

Šmelko Š (1968) Mathematic—statistic inventory of forest stand volume. Slovenská Akadémia Vied, Bratislava (in Slovak with German summary)

Šmelko Š (1979) Experimental plots for an evaluation of representativeness of forest inventory plots. Zborník prác Lesníckej fakulty VŠLD Zvolen, 21, Bratislava, Príroda, pp 153–182 (in Slovak with German abstract)

Sterba H (1982) Single stem models from inventory data with temporary plots. Mitt Forstl Bundesversuchsanstalt Wien 147:87–102

Sterba H, Moser M, Hasenauer H, Monserud RA (1995) PROGNAUS ein abstandsunabhängiger Waldwachstumssimulator für ungleichaltrige Mischbestände. Deutscher Verband Forstlicher Forschungsanstalten, Sektion Ertragskunde. Eberswald bei Berlin, pp 173–183

Ueno Y, Osawa Y (1987) The applicability of the Weibull and the expanded Weilbull distributions. J Jpn For Soc 69(1):24–28

Van Laar A, Mosandl R (1989) Durchmesserverteilung in jungen Eichenbeständen. Allg Forst-uJ-Ztg 160(9,10):189–194

Weisstein EW (2003) Weibull distribution. From MathWorld—a Wolfram Web Resource. Online in Internet: URL: http://www.mathworld.wolfram.com/WeibullDistribution.html [Cited 2006-03-10]

Wensel LC, Daugherty PJ, Meerschaert WJ (1986) CACTOS user's guide: the California conifer timber output simulator. Bulletin 1920. Division of Agriculture Science, University of California, Berkeley

Zarnoch SJ, Dell TR (1985) An evaluation of percentile and maximum likelihood estimators of Weibull parameters. For Sci 31(1):260–268

Zutter BR, Oderwald RG, Murphy PA, Farrar RMJr (1986) Characterizing diameter distributions with modified data types and forms of the Weibull distribution. For Sci 32(1):37–48