Mathematical Look at Ocean Dynamics

Susan Allen

UBC

2013

Outline

The Equations *Review from Day 1*

Making Boussinesq assumption (inertial density constant), incompressible assumption (really good for water).
Conservation of Volume

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Density

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = K\nabla^2\rho$$

where (u, v, w) is the velocity, (x, y, z) are east,north and upward, ρ is density and K is the diffusion of density. We really should diffuse the two components of ocean density: salinity and temperature, separately. Here we will take the diffusivity to be that of temperature.

Density, three Components

We will separate the density into three components:

- ρ_o a constant. Typical ocean densities are 1025 kg m^{-3}.
- $\rho_*(z)$ the background stratification. We usually write this in terms of the buoyancy frequency,

$$N^2 = \frac{-g}{\rho_o}\frac{\partial \rho_*}{\partial z}$$

- $\rho'(x, y, z, t)$ the fluctuating part

So that

$$\rho = \rho_o + \rho_* + \rho'$$

Acknowledge Turbulence of Ocean

The ocean is turbulent:

$$Re = UL/\nu$$

$U \approx 0.3\text{m s}^{-1}$, $L \approx 100$ km, $\nu = 1 \times 10^{-6}\text{m}^2\text{s}^{-1}$, so $Re = 3 \times 10^{10}$
So we should neglect molecular effects and include instead turbulent processes as discussed on Tuesday. But in the ocean horizontal scales are much bigger than vertical scales and so turbulenet eddies are bigger and faster in the horizontal. So

$$\nu_{TH} \approx K_{TH} \gg \nu_{TV} \approx K_{TV}$$

Conservation of Mass, Density Equation with Split Density, Turbulent Diffusion

Conservation of Volume

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Density

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} - \frac{\rho_o N^2}{g} w + w\frac{\partial \rho'}{\partial z} = K_{TH}\nabla_H^2 \rho' + K_{TV}\frac{\partial^2 \rho'}{\partial z^2}$$

where $\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The Equations *Review from Day 1*

Momentum Equations: East, North and Vertical
Time Rate of Change + Advection + Coriolis = Pressure Gradient + Buoyancy (vertical only) + Friction

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv + f_R w = -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_R u = -\frac{1}{\rho_o}\frac{\partial p}{\partial z} - \frac{\rho'}{\rho_o}g + \nu\nabla^2 w$$

where p is pressure, $f = 2\Omega\sin\phi$ and $f_R = 2\Omega\cos\phi$ and ν is the kinematic viscosity.

Split the Coriolis Terms

A number of important processes in the ocean (and atmosphere) depend on the variation of the Coriolis parameter (f) with latitude. To a first approximation we can write

$$f = f_o + \beta y$$

and

$$f_R = f_{Ro} - \beta_R y$$

where $f_o = 2\Omega \sin \phi_o$ and $f_R = 2\Omega \cos \phi_o$ for some ϕ_o in the middle of our domain.
and $\beta = 2\Omega \cos \phi_o a^{-1}$ and $\beta_R = 2\Omega \sin \phi_o a^{-1}$ where a is the radius of the earth.

Momentum Equations with Beta-Approximation, Turbulent Viscosity

Momentum Equations: East, North and Vertical

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - f_o v - \beta y v + f_{RO} w - \beta_R y w$$
$$= -\frac{1}{\rho_o}\frac{\partial p}{\partial x} + \nu_{TH}\nabla_H^2 u + \nu_{TV}\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + f_o u + \beta y u = -\frac{1}{\rho_o}\frac{\partial p}{\partial y} + \nu_{TH}\nabla_H^2 v + \nu_{TV}\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_{RO} u + \beta_R y u =$$
$$-\frac{1}{\rho_o}\frac{\partial p}{\partial z} - \frac{\rho'}{\rho_o} g + \nu_{TH}\nabla_H^2 w + \nu_{TV}\frac{\partial^2 w}{\partial z^2}$$

Ocean Scales

To determine which terms in the equations are really important, we want to scale the terms. So for each of our five variables we will write

$$q = Q\tilde{q}$$

where q is one of u, v, w, p, ρ' and $\tilde{q}$ is a nondimensional variable of order 1 and Q is the size including units of q. R is the scale for ρ'
Here are scales for the Ocean

U	0.3 m s^{-1}
V == U	0.3 m s^{-1}
W	10^{-4} s^{-1}
R	0.3 kg m^{-3}
P	3×10^{3} Nm^{-2}

Ocean Scales

To complete the scaling we also need space and time scales and sizes for the parameters in the equations.

g	10 m s^{-2}
f_o	10^{-4} s^{-1}
f_{Ro}	10^{-4} s^{-1}
β	$2 \times 10^{-11} \text{ (ms)}^{-1}$
β_R	$2 \times 10^{-11} \text{ (ms)}^{-1}$
ρ_o	10^3 kg m^{-3}
$\lvert N^2 \rvert$	10^{-4}s^{-2}

T	3×10^5 s
X	10^5 m
Y == X	10^5 m
Z	1000 m

ν_{TH}	$10^2\text{m}^2\text{s}^{-1}$
$K_{TH} == \nu_{TH}$	$10^2\text{m}^2\text{s}^{-1}$
ν_{TV}	$10^{-1}\text{m}^2\text{s}^{-1}$
$K_{TV} == \nu_{TV}$	$10^{-1}\text{m}^2\text{s}^{-1}$

(Exercise: Calculate the Size of Each Term in Each of the Five Equations)

Non-dimensional Numbers

Because the horizontal Coriolis terms (fu, fv) are large terms in the equation and easy to estimate, we use the size of these terms to normalize the others. We introduce four non-dimensional numbers.

- Rossby Number : $Ro = U/(fL)$
- temporal Rossby Number : $Ro_t = 1/(fT)$
- Ekman Number : $Ek = \nu_{TV}/(fZ^2)$
- Beta Number : $Beta = \beta X/f$

(Exercise: Calculate the Size of Each of the Non-dimensional numbers)
(Exercise: Make the horizontal momentum equations non-dimensional by dividing through by the scale for the Coriolis terms. Insert the four non-dimensional numbers as appropriate)

The Basic Balance

Looking for the main balance, the 90% solution, we get

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} = 0 \tag{1}$$

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{TU}{L}\left(\tilde{u}\frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \tilde{v}\frac{\partial \tilde{\rho}}{\partial \tilde{y}}\right) - \frac{\rho_o TW|N^2|}{gR}\tilde{N}^2\tilde{w} = 0 \tag{2}$$

$$\tilde{v} = \frac{P}{\rho_o f_o UX}\frac{\partial \tilde{p}}{\partial \tilde{x}} \tag{3}$$

$$\tilde{u} = \frac{-P}{\rho_o f_o UX}\frac{\partial \tilde{p}}{\partial \tilde{y}} \tag{4}$$

$$\frac{\partial \tilde{p}}{\partial \tilde{z}} = -\frac{gRZ}{P}\tilde{\rho} \tag{5}$$

where we can see that $P \approx \rho_o f_o UX$, $R \approx P/gZ$ and $W = gR/(\rho_o T|N^2|)$. We will define these as our new scales for P, R and W, respectively.

Degeneracy

Exercise: Show that (3)+(4) implies (1)

These equations are degenerate.

If we know everything else, (2) gives the vertical velocity.

Hydrostatic and Geostrophic (with Dimensions)

Hydrostatic: Pressure = weight of water above
(perturbation pressure = preturbation of weight of water above)

$$\frac{\partial p}{\partial z} = -\rho' g$$

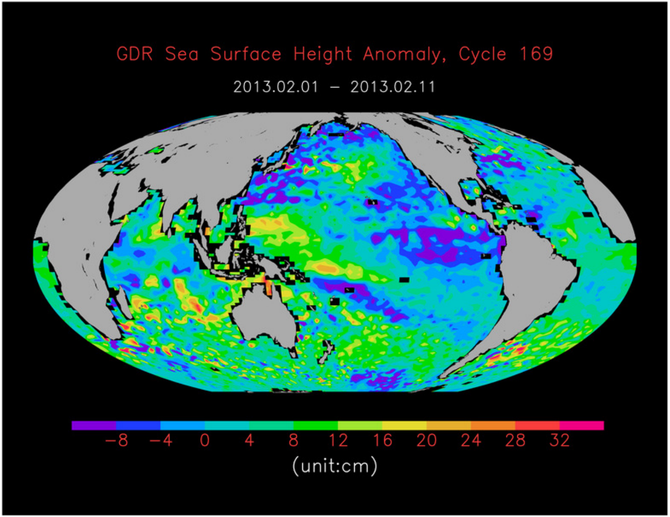
Geostrophic : Horz. Pressure Gradient = Coriolis Force

$$f_o v = \frac{1}{\rho_o}\frac{\partial p}{\partial x}$$

$$f_o u = \frac{-1}{\rho_o}\frac{\partial p}{\partial y}$$

Implications of Geostrophy

Geostrophy implies the pressure is a streamfunction.



NOAA

Special Case: Density Constant

(Exercise: Consider the implications of geostrophic, homogeneous flow, particularly if the bottom of the ocean is not flat) https://ocw.mit.edu/courses/12-003-atmosphere-ocean-and-climate-dynamics-fall-2008/pages/labs/lab6/

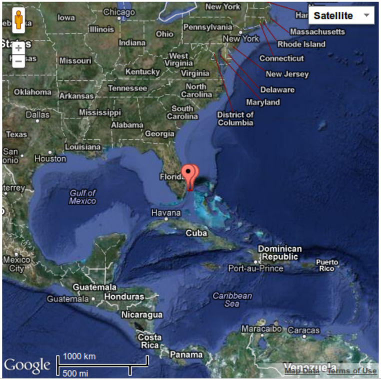
Thermal Wind Equations

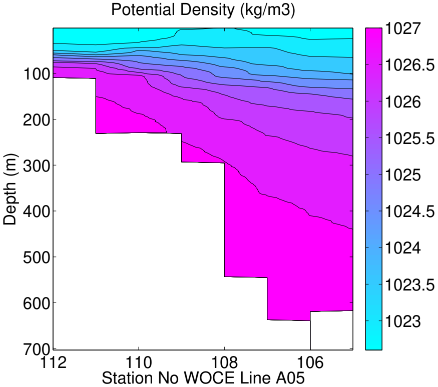
(Exercise:Assuming geostrophy and hydrostatic, determine the vertical shear of the horizontal velocity. That is, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$.)

From these equations we can find the velocity at any depth, provided we know it at one depth.

Computer Room Exercise #1

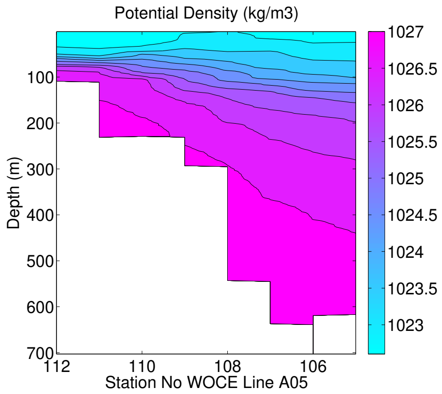
Use provided code to calculate the velocity through the Florida Straits in balance with the measured density field. Consider different levels of no motion and a "known" surface height difference.

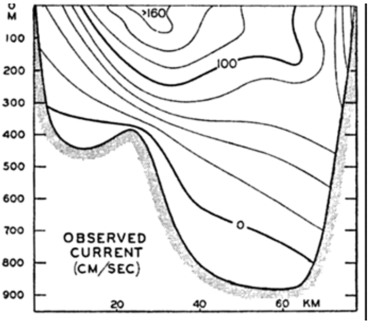




Left: Google Maps Right: SEA using data from WOCE

Thermal Wind Calculation





Left: SEA w data from WOCE, Right: Sverdrup et al, 1942. Note that latter is different time and perhaps different cross-section and hand-contoured

More than Geostrophy

- The geostrophic equations are degenerate. We need more terms!
- The next order terms in the momentum equations are those with Ro and then Ro_t
- *Beta* is smaller but important for larger scale processes.

Momentum Equations including Next Order Terms, Non-dimensional

Tildes dropped

$$Ro_t \frac{\partial u}{\partial t} + Ro \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) - v - Beta\, yv = -\frac{\partial p}{\partial x}$$

$$Ro_t \frac{\partial v}{\partial t} + Ro \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + u + Beta\, yu = -\frac{\partial p}{\partial y}$$

$$\frac{\partial p}{\partial z} = -\rho$$

No other terms are significant within an order of 30 in the vertical momentum equation.

The Assumption

We will write $u = u_g + \epsilon u_a$ and $v = v_g + \epsilon v_a$ where u_g, v_g is the geostrophic velocity, that exactly balances the pressure gradient term. And we will formally assume that all of Ro, Ro_t and $Beta$ are of size ϵ.

(Exercise: Substitute the expansion for u and v and the approximation for the non-dimensional numbers. Verify that the equations are solved to order one. Find the order ϵ equation. Solve for v_a and u_a.)

Conservation of Volume, Density, Vertical Momentum

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{Ro_t}{Bu}\frac{\partial w}{\partial z} = 0$$

where we have used the scale equivalences on page 12 and $Bu = |N^2|Z^2/f_o^2X^2$ is order 1. (Exercise: verify that Bu is order 1 and that this equation is exactly solved at order 1)

$$\frac{\partial \rho}{\partial t} + \frac{Ro}{Ro_t}\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) - \tilde{N}^2 w = 0$$

(Exercise: Taking Bu as 1, substituting the expansion for u and v, eliminate w between these equations and using the hydrostatic equation, eliminate ρ)

Merging

In front of you, you have equations for u_a, v_a in terms of the u_g, v_g and p from the horizontal momentum equations and for the horizontal divergence of u_a, v_a in terms of the u_g, v_g and p from conservation of volume, density equation and hydrostatic equation.
You could now, differentiate the first two and substitute into the third to get an equation only in u_g, v_g and p. Then substituting the geostrophic velocties written in terms of the pressure, you get one equation in p.

$$\left[\frac{\partial}{\partial t} + \frac{\partial p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial}{\partial x}\right]\left[\nabla_h^2 p + \frac{\partial}{\partial z}\left(\frac{1}{\tilde{N}^2}\frac{\partial p}{\partial z}\right)\right] + \frac{\partial p}{\partial x} = 0$$

where $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

What is this?

(Exercise:

- Show that the first bracket is equivalent to the Lagrangian derivative (d/dt) using advection by the geostrophic velocity.
- Show that $\nabla_h^2 p$ is the relative vorticity $\partial v/\partial x - \partial u/\partial y$.
- Show that the second term in the second bracket is the vertical expansion of the distance between lines of constant density.

The last term on the right-hand side is the northward advection v_g changing the Coriolis parameter.)

With our Non-dimensional Numbers

If we had included our non-dimensional numbers (letting them all be of order ϵ but not necessarily equal) we would have gotten:

$$\left[Ro_t\frac{\partial}{\partial t}+Ro\left(\frac{\partial p}{\partial x}\frac{\partial}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\right)\right]\left[\nabla_h^2 p+\frac{\partial}{\partial z}\left(\frac{1}{Bu\tilde{N}^2}\frac{\partial p}{\partial z}\right)\right]+Beta\frac{\partial p}{\partial x}=0$$

This equation is separable: $p(x,y,z,t)=\Pi(z)\bar{p}(x,y,t)$
(Exercise: Separate this equation)

Vertical Part of Equation

The vertical part can be written

$$\frac{\partial}{\partial z}\left(\frac{1}{Bu\,\tilde{N}^2}\frac{\partial \Pi}{\partial z}\right) + \alpha^2 \Pi = 0$$

If we assume a flat bottom and that the vertical velocity at the surface of the ocean is also zero our boundary conditions are:

$$w = 0, \text{ at } z = 0 \text{ and } z = -\tilde{H}$$

Need to move to w

Normal Mode Equation

From page 25 we see that

$$\tilde{N}^2 w = \left[\frac{\partial}{\partial t} + \frac{Ro}{Ro_t}\left(\frac{\partial p}{\partial x}\frac{\partial}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial}{\partial x}\right)\right]\rho$$

and hydrostatic gives

$$\frac{\partial p}{\partial z} = -\rho$$

So the vertical component of w,

$$w_e = \frac{-1}{\tilde{N}^2}\frac{\partial \Pi}{\partial z}$$

Exercise: show that this implies the equation for w_e is:

$$\frac{\partial^2 w_e}{\partial z^2} + \alpha^2 Bu\, \tilde{N}^2 w_e = 0$$

Computer Room Exercise #2

Use provided code to find the vertical normal modes for different stratifications.

- How do vertical changes in stratification change the shape of the modes?
- Explain why the density and vertical velocity modes look like they do, relative to each other.
- Relate the density mode and the horizontal velocity modes.

One more Mode

By assuming the upper boundary condition is $w = 0$ we eliminated the gravest mode.
For this mode we need to consider the surface moving.

$$w = \frac{d\eta}{dt} \,\mathrm{at}\; z = \eta \approx 0$$

But if the surface is higher, the pressure is higher.

$$p = \rho_o g \eta \,\mathrm{at}\; z = 0$$

Combining and applying our scaling:

$$w_e = \frac{|N^2| Z}{g} \Pi \,\mathrm{at}\; z = 0$$

Note that our scaling is out as $w_e \ll \Pi$

Boundary Conditions in w_e

Now we had

$$w_e = \frac{-1}{\tilde{N}^2}\frac{\partial \Pi}{\partial z}$$

Differentiate and substitute the differential equation for Π gives:

$$\frac{\partial w_e}{\partial z} = Bu\alpha^2\Pi$$

So our boundary condition in terms of w_e is

$$w_e = \frac{|N^2|Z}{g\alpha^2 Bu}\frac{\partial w_e}{\partial z} \text{ at } z = 0$$

"External Mode"

For this mode $\alpha^2 \ll 1$ and the differential equation can be approximated:

$$\frac{\partial^2 w_e}{\partial z^2} = 0$$

which has a solution $w_e = W_o(z + \tilde{H})$ where the boundary condition $w_e = 0$ at $z = -\tilde{H}$ has already been applied.
Applying the surface boundary condition:

$$W_o \tilde{H} = \frac{|N^2| Z}{g \alpha^2 Bu} W_o$$

which expanding Bu gives

$$\frac{\alpha^2}{X^2} = \frac{f_o^2}{gH} \approx \frac{1}{(2000\,\mathrm{km})^2}$$

or a $c_e = 200\mathrm{m\ s}^{-1}$ where A is the scale for α
(Exercise: show that for this mode Π is approximately constant with depth as are u_g and v_g)

Quasi-geostrophic Waves

Writing $\alpha^2 = 1/R_i^2$ where i is the lengthscale for the ith mode, our horizontal equation is

$$\left[Ro_t \frac{\partial}{\partial t} + Ro\left(\frac{\partial \bar{p}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \bar{p}}{\partial y}\frac{\partial}{\partial x}\right)\right]\left[\nabla_h^2 \bar{p} - \frac{\bar{p}}{R_i^2}\right] + Beta\frac{\partial \bar{p}}{\partial x} = 0$$

We will look for linear waves and thus formally assume that $Ro \ll \epsilon$.

$$Ro_t \frac{\partial}{\partial t}\left[\nabla_h^2 \bar{p} - \frac{\bar{p}}{R_i^2}\right] + Beta\frac{\partial \bar{p}}{\partial x} = 0$$

(Exercise: Show that this equation has solutions of the form $\exp[i(kx + \ell y - \omega t)]$ and find the relation between the frequency ω and the wave-numbers k, ℓ)

Rossby Waves

These waves occur in both the atmosphere and ocean and explain phenomena such as waves on the Jet Stream, the Gulf Stream and the Agulhas Current.
(Exercise: In what directions can these waves propagate?)
https://www.youtube.com/watch?v=ELDkYJWHNiU

Potential Vorticity Form

Our horizontal equation

$$\left[Ro_t \frac{\partial}{\partial t} + Ro\left(\frac{\partial \bar{p}}{\partial x}\frac{\partial}{\partial y} - \frac{\partial \bar{p}}{\partial y}\frac{\partial}{\partial x}\right)\right]\left[\nabla_h^2 \bar{p} - \frac{\bar{p}}{R_i^2}\right] + Beta\frac{\partial \bar{p}}{\partial x} = 0$$

can also be written

$$Ro_t \frac{\partial q}{\partial t} + \mathcal{J}(Ro\bar{p}, q) = 0$$

where $q = \nabla_h^2 \bar{p} - \bar{p}/R_i^2 + (Beta/Ro)\, y$ and $\mathcal{J}$ is the Jacobian.
(Exercise: Show these relations are equivalent)

Compact, Steadily Translating Eddy

We will assume that $\bar{p} = \bar{p}(x - ct, y)$ and that $\bar{p}$ is compact in form. As x and t only occur combined, we can include the time-derivative in the Jacobian and assume $Ro == Ro_t$

$$\mathcal{J}(\bar{p} + cy, q) = 0$$

which has a solution $q = \mathcal{F}(\bar{p} + cy)$ for some function $\mathcal{F}$.
We can assume two types of contours of $\bar{p}$.
1) Those that are closed near the center of the eddy $r < a$ which we will ignore for now.
2) Those that extend to great distances. A long way from the eddy q should approach $(Beta/Ro)\, y$ as the eddy has compact form and $\bar{p}$, the perturbation pressure, should vanish. Thus along these contours, $\mathcal{F}$ should be linear, that is $\mathcal{F}(\gamma) = (Beta/Ro)\gamma/c$.

Eddy Solution

So in the outer region we have an equation

$$\nabla_h^2 \bar{p} - \frac{1}{R_i^2}\bar{p} = \frac{Beta}{Ro\, c}\bar{p}$$

which boundary condition that $\bar{p} \to 0$ at large distance from the origin and constant at $r = a$.

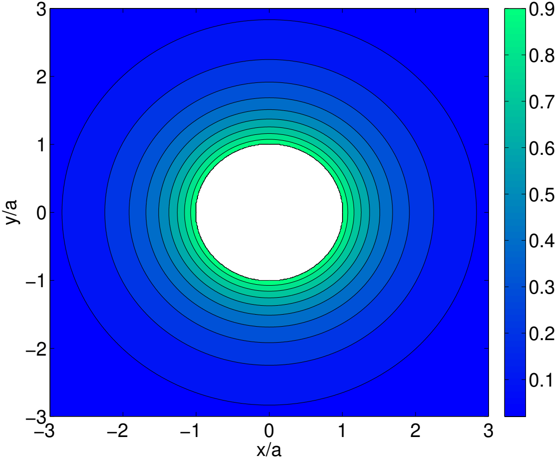
Exercise: Show that

$$\bar{p} = P_o K_o(mr)$$

is a solution, where K_o is the gravest modified Bessel function and where

$$m^2 = \frac{1}{R_i^2} + \frac{Beta}{Ro\, c}$$

"Ocean Eddy"



SEA

(Exercise: Given that m is real, what real values can c take?)
(Exercise: what direction to eddies large compared to R_i move? Small?)

Theme Movie and Image Credits

Movie

https://www.youtube.com/watch?v=CCmTY0PKGDs

Image Credits

- Theme Movie & Still: NASA: https://svs.gsfc.nasa.gov/10841/
- Sea Surface Height, page 16 NOAA: was www.nodc.noaa.gov/sog/Jason2/qa.html No longer available.
- Map: page 20 Google Maps
- Right Contour Plot, page 21 Sverdrup, H.U., M.W. Johnson and R.H. Fleming. The Oceans, their Physics, Chemistry and General Biology. Prentice-Hall, Inc. 1942.