Module 2

Matias

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0.1 Finite Sample Space: special case

In many cases, the sample space contains finite number of possible outcomes.

Moreover, it is justifiable to assign equal probability to every possible outcome

0.2 Example

- Experiment: roll a fair die;
- Sample space: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Typically assume (a) and (b):

$$
\begin{aligned}
(a) \quad & \mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) && (1) \\
(b) \quad & \mathbb{P}(\Omega) = 1 && (2) \\
& \Longrightarrow \mathbb{P}(\omega) = 1/6 \text{ for any } \omega = \{1\}, \dots, \{6\}. && (3)
\end{aligned}
$$

0.3 Essence of this chapter

When equal likely is declared/assumed,

- subsequent probability calculation is conceptually easy,
- can be challenging technically.

This chapter goes over some general techniques for probability calculation of this nature.

1 Counting equally likely outcomes

1.1 Equally likely outcomes

Let A be a subset (event) of a sample space Ω.

$$\mathbb{P}(A) = \frac{\text{number of elements in } A}{\text{number of elements in } \Omega}$$

All that matters in this case is the numbers of elements is different sets.

1.2 Basic principle

Suppose that a random experiment has **2 steps**.

- Step 1 has n_1 possible outcomes, and
- Step 2 has n_2 possible outcomes.

Then,

$$\text{total number of possible outcomes} \;=\; n_1 \times n_2$$

1.3 Unstated Assumption:

the outcome of the first step does not affect the outcomes of the second step.

1.4 Extension to "k step" experiments

If a random experiment has **k steps**.

- Step 1 has n_1 possible outcomes,
- Step 2 has n_2 possible outcomes,

  ⋮

- Step k has n_k possible outcomes.

Then,

$$\text{total number of outcomes} \;=\; n_1 \times n_2 \times n_3 \times \cdots \times n_k$$

> **Note**
>
> the outcomes of these steps do not affect each other.

1.5 License plates

Example

How many different license plates with 7 characters are possible if the first 3 places are letters and the last 4 are numbers?

- 26 choices for the 1st, 2nd and 3rd entry, and
- 10 choices for each of the last 4 entries.

$$\begin{aligned} \#\text{of plates} &= 26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 \\ &= 26^3 \times 10^4 \\ &= 175,760,000 \end{aligned}$$

The assumption holds that the outcomes of these steps do not affect each other.

1.6 Special license plates

Example

What happens if letters and numbers cannot be repeated?

- 26 choices for the 1st entry,
- 25 choices for the 2nd entry,
- 24 choices for the 3rd entry,
- 10 choices for the 4th entry,
- 9 choices for the 5th entry,
- 8 choices for the 6th entry, and
- 7 choices for the 7th entry.

$$\begin{aligned} \#\text{of plates w/o rep} &= 26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 \\ &= 78,624,000 \end{aligned}$$

1.7 Shuffling license plates

Example

What is the probability that a randomly chosen plate has no repetitions?

We haven't specified that some plates are more or less likely to be chosen.

So we assume that each is equally likely.

Be aware of the background information we have just presented.

1.8 Shuffling license plates (solution)

- The event in question is

$$A = \{\text{license plate without repetition}\}.$$

- The sample space contains all plates whose first 3 are letters and the last 4 are numbers.

Being explicit with the definitions will help you avoid mistakes.

The probability that a randomly chosen plate has no repetitions is given by

$$\begin{aligned}\mathbb{P}(A) &= \frac{\#\text{ of plates w/o rep}}{\#\text{ of plates}}\\ &= \frac{78,624,000}{175,760,000}\\ &= 0.44734\\ &\approx 0.45\end{aligned}$$

2 When order matters

2.1 Permutation

Definition

A permutation of a set is an arrangement of its elements in a specific order.

Example

Consider the set $A = \{1, 2, 3, 4, 5\}$

- Some permutations of the elements of A are:

$$(3, 5, 2, 4, 1)\,,\ (2, 1, 3, 4, 5)\,,\ (5, 4, 2, 1, 3)\,, \quad \text{etc.}$$

2.2 Counting permutations

How many permutations of the set $\{1, 2, \dots, n\}$ are there?

There are

- n choices for the 1st entry,
- $(n-1)$ choices for the 2nd entry,
- $(n-2)$ choices for the 3rd entry,

  $\vdots$

- 2 choices for the $(n-1)^{\text{th}}$ entry, and
- 1 choice for the last (n^{th}) entry.

Thus, there are

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1 \;=\; n!$$

possible permutations of $\{1, 2, \dots, n\}$

2.3 Counting permutations

! Key assumption in counting permutations this way:

all n objects are distinct.

- When some of them are not distinct, the number of possible distinct outcomes by re-arrangement ("permutation") is not given by this formula.
- We need a different formula.

2.4 Example

In how many different ways can the letters of "pepper" be arranged? Answer:

- If the letters were all different, we would have $6! = 720$ arrangements.
- But each arrangement does not change if:
  - we permute the 3 p's (3! ways of doing this)
  - we permute the 2 e's (2! ways of doing this)
- Thus, in the list of 6! arrangements, each "p/e" pattern appears $3! \times 2!$ times:

$$\text{number of arrangements} = \frac{6!}{3!2!} = 60.$$

2.5 International chess

A chess tournament with 10 players:

# of players	Country of origin
4	Russia
3	USA
2	Argentina
1	Brazil

Example: Counting outcomes

If we see only nationalities in the final rank, how many outcomes are possible?

Example: Go Argentina

If all players have an equal chance to win, what is the probability that Argentina wins the tournament?

2.6 Counting outcomes

If we see only nationalities in the final rank, how many outcomes are possible?

- There are 10! permutations for the players.
- Each country-ranking pattern appears $4! \times 3! \times 2!$ times (Russian players can be permuted in 4! ways, USA players in 3! ways, etc.).

- If we see only nationalities, there are

$$\frac{10!}{4! \times 3! \times 2!} = \frac{3628800}{288} = 12600$$

  possible outcomes.

2.7 Go Argentina

Count favourable outcomes: an Argentinian player is ranked first.

- There are 2 choices for the top ranked player,
- There are 9! ways of arranging (ranking) the other 9 players.
- Thus, we have

$$2 \times 9! = 725760$$

  rankings where an Argentinian player is first.

But if we see only nationalities, these outcomes are not distinct!

2.8 Go Argentina

Each country pattern appears $4! \times 3! \times 2!$ times.

The number of country rankings where Argentina appears 1st is

$$\frac{725760}{4! \times 3! \times 2!} = 2520.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left(\{\text{Argentina wins}\}\right) &= \frac{\#\text{of favorable outcomes}}{\#\text{of possible outcomes}} \\ &= \frac{2520}{12600} \\ &= 0.20. \end{aligned}$$

For assignments, keep only 3 significant digits in general if no neat numerical outcome.

2.9 Simpler approach

What is the probability the Argentina player wins the tournament?

- If all players have equal chance to win.
- There are 10 players, and two of them are Argentina.
- The probability is therefore 2/10.

3 When order doesn't matter

3.1 Combinations

Definition

A combination of size m is a subset of m items from a set of size n with $m \leq n$.

- Consider the set

$$S = \{1, 2, 3, 4, 5\}$$

- The following are all the sets of size 3 from S:

$\{1, 2, 3\}$	$\{1, 4, 5\}$
$\{1, 2, 4\}$	$\{2, 3, 4\}$
$\{1, 2, 5\}$	$\{2, 3, 5\}$
$\{1, 3, 4\}$	$\{2, 4, 5\}$
$\{1, 3, 5\}$	$\{3, 4, 5\}$

- We call these sets "combinations" when we only care which elements are in the set.

3.2 Combinations

- Note that the order of the elements does not matter (these are sets, not sequences).

$$\Big\{1, 2, 4\Big\} = \Big\{4, 1, 2\Big\} = \Big\{2, 4, 1\Big\} = \cdots$$

- Given a set of n distinct items

$$S = \{s_1,\ s_2,\ \ldots, s_n\}$$

how many different combinations of size $m \leq n$ can be formed?

3.3 Number of combinations

This concept, the number of combinations of size m out of set of size $n \geq m$ has various notations

$$\binom{n}{m} = {}_nC_m = C_m^n$$

- n: size of the set from which combinations are drawn
- m: size of the combinations

> **Tip**
>
> In this course, we use $\binom{n}{m}$.

3.4 Example

For example, when $n = 5$ and $m = 3$ we have

$\{1, 2, 3\}$	$\{1, 4, 5\}$
$\{1, 2, 4\}$	$\{2, 3, 4\}$
$\{1, 2, 5\}$	$\{2, 3, 5\}$
$\{1, 3, 4\}$	$\{2, 4, 5\}$
$\{1, 3, 5\}$	$\{3, 4, 5\}$

and a direct count shows that the number of combinations is

$$\binom{5}{3} = 10.$$

3.5 General definition of combination

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

. . .

These are also called binomial coefficients.

They have many beautiful interpretations.

3.6 How many is $\binom{n}{m}$?

Note that combinations can be constructed using permutations as follows:

- pick m items out of the n items, in order;
- each of these m-long permutations generates a subset;
- but many (how many?) of these permutations correspond to the same subset

3.7 Intuition for the combination formula (first version)

Given a set of n distinct items

$$S = \left\{ s_1, s_2, \dots, s_n \right\}$$

we can form

$$n \times (n-1) \times (n-2) \times \cdots \times (n-m+1)$$

different m-tuples by choosing from elements of S

3.8 Intuition for the combination formula (first version)

- However, any permutation of these m elements corresponds to the same subset
- There are $m!$ ways to reorder these m elements
- In other words, each different subset appears $m!$ times in the list of ordered m-tuples

- Hence, the number of combinations is:

$$\frac{n\,(n-1)\,(n-2)\,\cdots\,(n-m+1)}{m!} = \frac{n\,(n-1)\,(n-2)\,\cdots\,(n-m+1)}{m!}\frac{n!}{m!\,(n-m)!}.$$

3.9 Intuition for the combination formula (second version)

Another way to construct combinations using permutations uses the intuition:

- take a permutation of all n items in the set
- keep the first m terms of the permutation, and discard the remaining $(n-m)$ terms.

3.10 Intuition for the combination formula (second version)

$$\overbrace{(i_1, \dots, i_m, i_{m+1}, \dots, i_n)}^{\text{permutation}} \to \overbrace{i_1, \dots, i_m}^{m}\ \overbrace{i_{m+1}, \dots, i_n}^{n-m} \to \overbrace{\{i_1, \dots, i_m\}}^{\text{combination}}.$$

Concrete example — $m = 3$ and $n = 5$.

$$
\begin{aligned}
\overbrace{(5,1,4,2,3)}^{\text{permutation}} &\longrightarrow \overbrace{5,1,4}\,\overbrace{2,3} &\longrightarrow \overbrace{\{1,4,5\}}^{\text{combination}}\\
\overbrace{(5,1,4,3,2)}^{\text{permutation}} &\longrightarrow \overbrace{5,1,4}\,\overbrace{3,2} &\longrightarrow \overbrace{\{1,4,5\}}^{\text{combination}}\\
\overbrace{(1,5,4,3,2)}^{\text{permutation}} &\longrightarrow \overbrace{1,5,4}\,\overbrace{3,2} &\longrightarrow \overbrace{\{1,4,5\}}^{\text{combination}}
\end{aligned}
$$

> **i** Permutations sharing the same first 3 and last 2 elements give the same combination.

3.11 Intuition for the combination formula (second version)

- Each permutation generates a combination
- But many permutations generate the same combination
- The number of permutation needs to be corrected to remove repetitions
- Two permutations lead to the same combination if and only if their first m elements are identical, and they need not to be in the same order.

3.12 Intuition for the combination formula (second version)

- For a permutation of length n,
- there are $m!$ ways to re-order its first m elements and
- $(n-m)!$ ways to re-order its last $n-m$ elements:
- The total number of repetitions is

$$m! \times (n-m)!$$

- There are $n!$ distinct length-n permutations.
- Therefore, the number of distinct combinations is

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}.$$

3.13 Conventions

We define

$$0! = 1$$

So, for example

$$\binom{5}{5} = \frac{5!}{5!\,(5-5)!} = \frac{5!}{5!\;0!} = 1 \qquad \text{(as you would guess)}$$
$$\binom{5}{0} = \frac{5!}{0!\;5!} = 1 \qquad \text{(convention, but sensical).}$$

3.14 Lotto 6/49

- Player chooses 6 distinct integers between 0 and 49.
- Dealer randomly selects 6 distinct integers between 0 and 49.
- The more of matches, the bigger the prize.
- What is the probability that the player matches is $k \in \{0, \dots, 6\}$ integers?

3.15 Lotto 6/49 solution

- Simplify everything: an urn contains 50 balls, 6 of which are red (whichever 6 the player picked).
- What is the probability that the dealer selects k red balls?

$$A_k = \text{the number of matches is } k; \qquad D_k = \text{dealer selects are } k \text{ red balls.}$$

Claim: $\mathbb{P}(A_k) = \mathbb{P}(D_k)$.

- The number of the possible outcomes of dealer's draw is $\#\Omega = \binom{50}{6}$.
- Number of favorable dealer's draws is

$$\#D_k = \binom{6}{k} \times \binom{44}{6-k}.$$

- Therefore, the desired probability is given by

$$\mathbb{P}(A_k) = \frac{\binom{6}{k} \times \binom{44}{6-k}}{\binom{50}{6}}.$$

3.16 Lotto 6/49 solution: numerical values

k	$P\,(k$ matches $)$
0	0.444
1	0.410
2	0.128
3	0.017
4	0.001
5	$1.66\times\ 10^{-5}$
6	$6.29\times\ 10^{-8}$

3.17 Summary

- We exclusively considered the experiments with a finite number of possible outcomes.
- We only considered cases where all outcomes are equally likely.
- Either by brute-force or by combinatoric algebra, we enumerate the numbers of “outcomes in favour of an event” and “outcomes in the sample space”.
- The ratio is the answer to “the probability of the event”.

Sometimes, using rules about probability can simplify brute-force calculations.

4 Using symmetry to simplify unions

4.1 General formula with proportional probability of intersection

Theorem

Suppose an intersection of any h subsets has the same probability p_h. Then,

$$\mathbb{P}\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) = \sum_{h=1}^{n} (-1)^{h-1} \binom{n}{h} p_h.$$

Every intersection of h subsets has probability p_h:

$$\begin{aligned}
p_1 &= \mathbb{P}\left(A_1\right) = \mathbb{P}\left(A_2\right) = \cdots = \mathbb{P}\left(A_n\right) \\
p_2 &= \mathbb{P}\left(A_1 \cap A_2\right) = \mathbb{P}\left(A_1 \cap A_3\right) = \cdots = \mathbb{P}\left(A_{n-1} \cap A_n\right) \\
p_3 &= \mathbb{P}\left(A_1 \cap A_2 \cap A_3\right) = \cdots = \mathbb{P}\left(A_{n-2} \cap A_{n-3} \cap A_n\right) \\
&\vdots \\
p_n &= \mathbb{P}\left(A_1 \cap A_2 \cap \cdots \cap A_n\right)
\end{aligned}$$

4.2 Gentle proof

Let's build some intuition first.

4.3 The probability of event A, or B, or C, or ...

$$\mathbb{P}\Big(A_1 \cup A_2 \cup \cdots \cup A_n\Big)$$

Suppose $n = 2$:

$$\begin{aligned}
\mathbb{P}\left(A_1 \cup A_2\right) &= \mathbb{P}\left(A_1\right) + \mathbb{P}\left(A_2\right) - \mathbb{P}\left(A_1 \cap A_2\right) && \text{rule} \\
&= \mathbb{P}\left(A_1\right) + \mathbb{P}\left(A_2\right) && \text{desired events (inclusion)} \\
&\quad - \mathbb{P}\left(A_1 \cap A_2\right) && \text{double counting (exclusion)}
\end{aligned}$$

4.4 What about $n = 3$?

Claim:

$$\begin{aligned}
&\mathbb{P}\left(A_1 \cup A_2 \cup A_3\right) \\
&= \mathbb{P}\left(A_1\right) + \mathbb{P}\left(A_2\right) + \mathbb{P}\left(A_3\right) && \text{desired events (inclusion)} \\
&\quad - \ \mathbb{P}\left(A_1 \cap A_2\right) - \mathbb{P}\left(A_1 \cap A_3\right) - P\left(A_2 \cap A_3\right) && \text{double counted (exclusion)} \\
&\quad + \ \mathbb{P}\left(A_1 \cap A_2 \cap A_3\right) && \text{removed too much (inclusion)}
\end{aligned}$$

4.5 Proof of claim: A union of $n = 3$ sets

Proof

$$
\begin{aligned}
&\mathbb{P}\left(A_1 \cup A_2 \cup A_3\right) \\
&= \mathbb{P}\left[\left(A_1 \cup A_2\right) \cup A_3\right] \\
&= \mathbb{P}\left(A_1 \cup A_2\right) + \mathbb{P}\left(A_3\right) - \mathbb{P}\left[\left(A_1 \cup A_2\right) \cap A_3\right] \\
&= \mathbb{P}\left(A_1\right) + \mathbb{P}\left(A_2\right) - \mathbb{P}\left(A_1 \cap A_2\right) + \mathbb{P}\left(A_3\right) \\
&\quad - \ \mathbb{P}\left[\left(A_1 \cap A_3\right) \cup \left(A_2 \cap A_3\right)\right] \\
&= \mathbb{P}\left(A_1\right) + \mathbb{P}\left(A_2\right) + \mathbb{P}\left(A_3\right) - \mathbb{P}\left(A_1 \cap A_2\right) \\
&\quad - \ \left[\mathbb{P}\left(A_1 \cap A_3\right) + \mathbb{P}\left(A_2 \cap A_3\right) - \mathbb{P}\left(A_1 \cap A_2 \cap A_3\right)\right] \\
&= \mathbb{P}\left(A_1\right) + P\left(A_2\right) + P\left(A_3\right) && \text{inclusion} \\
&\quad - \ \mathbb{P}\left(A_1 \cap A_2\right) - \mathbb{P}\left(A_1 \cap A_3\right) - \mathbb{P}\left(A_2 \cap A_3\right) && \text{exclusion} \\
&\quad + \ \mathbb{P}\left(A_1 \cap A_2 \cap A_3\right) && \text{inclusion}
\end{aligned}
$$

4.6 General formula

$$
\begin{aligned}
\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = & \sum_{1 \le i \le n} \mathbb{P}\left(A_i\right) && \text{inclusion} \\
& - \sum_{i<j} \mathbb{P}\left(A_i \cap A_j\right) && \text{exclusion} \\
& + \sum_{i<j<k} \mathbb{P}\left(A_i \cap A_j \cap A_k\right) && \text{inclusion} \\
& - \sum_{i<j<k<h} \mathbb{P}\left(A_i \cap A_j \cap A_k \cap A_h\right) && \text{exclusion} \\
& \qquad \vdots \\
& + (-1)^{n-1} \mathbb{P}\left(A_1 \cap A_2 \cdots \cap A_n\right) && \text{inclusion}
\end{aligned}
$$

4.7

Always holds.

4.8

Didn't use intersection of any h subsets has the same probability p_h

4.9 Proportional probability of intersection

$$
\begin{aligned}
\sum_{i=1}^{n} \mathbb{P}\left(A_i\right) &= \sum_{i} p_1 = np_1 = \binom{n}{1} p_1 \\
\sum_{iMj} \mathbb{P}\left(A_i \cap A_j\right) &= \sum_{i<j} p_2 = \binom{n}{2} p_2 \\
\sum_{i<j<k} \mathbb{P}\left(A_i \cap A_j \cap A_k\right) &= \sum_{i<j<k} p_3 = \binom{n}{3} p_3 \\
&\vdots
\end{aligned}
$$

. . .

Plug this into the formula and you're done!

$$
\mathbb{P}\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) = \sum_{h=1}^{n} (-1)^{h-1} \binom{n}{h} p_h.
$$

5 More worked problems

5.1 Possibly helpful

- An event of interest can be complex if viewed directly.
- Sometimes, it can be decomposed as the result of set operations of simpler events.
- When the probabilities of these simpler events are manageable, we may make use of probability rules.
- Individual steps might be simple, but the architecture may be confusing.

5.2 Student exams

Suppose that the probability that some students in this class gets at least 80 on the midterm. We'll consider 4 students.

Suppose that

$$
\begin{aligned}
\mathbb{P}(\text{one student gets 80+}) &= 1/2, \text{ for any student} \\
\mathbb{P}(\text{two students get 80+}) &= (1/2)^2, \text{ for any 2 students} \\
\mathbb{P}(\text{three students get 80+}) &= (1/2)^3, \text{ for any 3 students} \\
\mathbb{P}(\text{four students get 80+}) &= (1/2)^4.
\end{aligned}
$$

Calculate the probability that

a. at least one student gets at least 80,
b. no student gets at least 80, and
c. only student number 4 gets at least 80.

5.3 Problem setup

First, we define events.

Let A_i be the event that student i gets at least 80.

5.4 a. at least one student gets at least 80 (solution)

At least one of the A_i's occurs.

Event of interest $A_1 \cup A_2 \cup A_3 \cup A_4$.

$$\begin{aligned}
\mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) &= \binom{4}{1} p_1 - \binom{4}{2} p_2 + \binom{4}{3} p_3 - \binom{4}{4} p_4 \\
&= 4(1/2) - 6(1/2)^2 + 4(1/2)^3 - (1/2)^4 \\
&= \frac{1}{2}\left(4 - \frac{6}{2} + \frac{4}{4} - \frac{1}{8}\right) \\
&= 15/16 = 0.9375.
\end{aligned}$$

5.5 b. no student gets at least 80 (solution)

The event of interest is

$$A_1^c \cap A_2^c \cap A_3^c \cap A_4^c = \left(A_1 \cup A_2 \cup A_3 \cup A_4\right)^c.$$

Therefore

$$\begin{aligned}
\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c\right) &= 1 - \mathbb{P}\left(A_1 \cup A_2 \cup A_3 \cup A_4\right) \\
&= 1 - 15/16 = 1/16 = 0.0625.
\end{aligned}$$

5.6 c. only student number 4 gets at least 80 (solution)

This is the same as "only A_4 occurs".

- The event of interest is
$$A_1^c \cap A_2^c \cap A_3^c \cap A_4$$
.
- Recall that $\mathbb{P}\left(C \cap D\right) = \mathbb{P}\left(C\right) - \mathbb{P}\left(C \cap D^c\right)$
- Take $C = A_1^c \cap A_2^c \cap A_3^c$ and $D = A_4$ to obtain

$$\mathbb{P}(A_1^c \cap A_2^c \cap A_3^c \cap A_4) = \mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) - \mathbb{P}(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c).$$

- We already have the second part, so we need $\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c\right)$

5.7 c. only student number 4 gets at least 80 (solution)

By De Morgan's Law, we have $A_1^c \cap A_2^c \cap A_3^c = (A_1 \cup A_2 \cup A_3)^c$

We also have that $\mathbb{P}((A_1 \cup A_2 \cup A_3)^c) = 1 - \mathbb{P}(A_1 \cup A_2 \cup A_3)$.

We can find,

$$P\left(A_1 \cup A_2 \cup A_3\right) = \binom{3}{1}\frac{1}{2} - \binom{3}{2}\frac{1}{4} + \binom{3}{3}\frac{1}{8} = \frac{1}{2}\left(3 - \frac{3}{2} + \frac{1}{4}\right) = 7/8 = 0.875.$$

And finally,

$$\begin{aligned}\mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4\right) &= \mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c\right) - \mathbb{P}\left(A_1^c \cap A_2^c \cap A_3^c \cap A_4^c\right)\\ &= (1 - 7/8) - 1/16 = 1/16 = 0.0625.\end{aligned}$$

5.8 Couples at a party

- n couples attend a dance party.
- At one moment, each lady randomly picks a gentleman to dance.
- What is the probability that none of them pick their partner?

5.9 Couples at a party (solution)

- Let A_i be the event the ith lady dances with her partner for $i = 1, 2, \ldots$. Clearly, $\mathbb{P}(A_i) = 1/n$.
- The event of interest is $\{A_1 \cup A_2 \cup \cdots \cup A_n\}^c$.
- In general, for $k = 1, 2, \ldots, n$,

$$p_k = \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_k) = \frac{(n-k)!}{n!}.$$

- Therefore,

$$\mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{(n-k)!}{n!} = \sum_{k=1}^{n} (-1)^{k-1} \frac{1}{k!}.$$

- The probability of the event of our interest is therefore

$$1 - \mathbb{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{k=0}^{n} (-1)^k \frac{1}{k!}.$$

What is the limit of this probability when $n \to \infty$?

5.10 The number of failed processors

A cell phone has 4 GPUs to run ChatGPT.

The more working GPUs, the faster you can get answers to your homework.

Let G_i be the event that processor i fails.

The probability of failures is known:

$$\begin{aligned}
&\mathbb{P}(G_i) = 0.1 \quad \forall i \\
&\mathbb{P}(G_i \cap G_j) = 0.01 \quad \forall i \neq j \\
&\mathbb{P}(G_i \cap G_j \cap G_h) = 0.001 \quad \forall i \neq j \neq h \\
&\mathbb{P}(G_1 \cap G_2 \cap G_3 \cap G_4) = 0.0001
\end{aligned}$$

What is the probability that

a. The phone has at least one failed GPU?
b. The phone has all GPUs working?
c. The phone has exactly one failed GPU?
d. Say, the phone can solve your homework if at most one GPU fails. What is the probability that your homework is solved?

5.11 a. The phone has at least one failed GPU? (solution)

Event of interest $A = G_1 \cup G_2 \cup G_3 \cup G_4$

$$
\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(G_1 \cup G_2 \cup G_3 \cup G_4) \\
&= \sum_i \mathbb{P}(G_i) && \text{(inclusion)} \\
&\quad - \sum_{i<j} \mathbb{P}(G_i \cap G_j) && \text{(exclusion)} \\
&\quad + \sum_{i<j<h} \mathbb{P}(G_i \cap G_j \cap G_h) && \text{(inclusion)} \\
&\quad - \mathbb{P}(G_1 \cap G_2 \cap G_3 \cap G_4) && \text{(exclusion)} \\
&= \binom{4}{1} \times 0.1 - \binom{4}{2} \times 0.01 + \binom{4}{3} \times 0.001 + \binom{4}{4} \times 0.0001 \\
&= 0.3439
\end{aligned}
$$

5.12 b. The phone has all GPUs working?

Event of interest: $B_0 = \{\text{Zero failed components}\} = G_1^c \cap G_2^c \cap G_3^c \cap G_4^c$

$$
\begin{aligned}
\mathbb{P}(B_0) &= \mathbb{P}(G_1^c \cap G_2^c \cap G_3^c \cap G_4^c) \\
&= \mathbb{P}((G_1 \cup G_2 \cup G_3 \cup G_4)^c) && \text{De Morgan} \\
&= 1 - \mathbb{P}(G_1 \cup G_2 \cup G_3 \cup G_4) \\
&= 1 - 0.3439 && \text{(from Part (a))} \\
&= 0.6561
\end{aligned}
$$

5.13 c. The phone has exactly one failed GPU? (solution)

Event of interest:

$$
\begin{aligned}
B_1 &= G_i \cap G_j^c \cap G_k^c \cap G_h^c \text{ for } i \neq j \neq k \neq h \\
&= (G_1 \cap G_2^c \cap G_3^c \cap G_4^c) \bigcup (G_1^c \cap G_2 \cap G_3^c \cap G_4^c) \bigcup (G_1^c \cap G_2^c \cap G_3 \cap G_4^c) \\
&\quad \bigcup (G_1^c \cap G_2^c \cap G_3^c \cap G_4) \\
\Rightarrow \mathbb{P}(B_1) &= 4\mathbb{P}(G_1 \cap G_2^c \cap G_3^c \cap G_4^c) \qquad \text{disjoint union, symmetry} \\
&= 4\left(\mathbb{P}(G_2^c \cap G_3^c \cap G_4^c) - \mathbb{P}(G_1^c \cap G_2^c \cap G_3^c \cap G_4^c)\right) \\
&= 4\left(\mathbb{P}((G_2 \cup G_3 \cup G_4)^c) - \mathbb{P}(B_0)\right) \qquad \text{De Morgan} \\
&= 4\left((1 - \mathbb{P}(G_2 \cup G_3 \cup G_4)) - \mathbb{P}(B_0)\right) \\
&= 4\left\{\left(1 - \left[\binom{3}{1} \times 0.1 - \binom{3}{2} \times 0.01 + \binom{3}{3} \times 0.001\right]\right) - \mathbb{P}(B_0)\right\} \\
&= 4(0.7290 - 0.6561) = 0.2916.
\end{aligned}
$$

5.14 d. Homework solved. (solution)

$$
\begin{aligned}
B &= \{\text{Item works}\} \\
&= \{\text{Zero failed components}\} \bigcup \{\text{One failed component}\} \\
&= B_0 \cup B_1 \\
\Rightarrow \mathbb{P}(B) &= \mathbb{P}(B_0) + \mathbb{P}(B_1) \qquad \text{(disjoint)} \\
&= 0.6561 + 0.2916 = 0.9477
\end{aligned}
$$

Note

For further practice, try to verify that

1. the probability of two failed GPUs is 0.0486;
2. and the probability of three failed GPUs is 0.0036.