Module 1

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1 Welcome & "House rules"

2 Syllabus - Expectations - Grades

3 Basics of probability

3.1 Experiments

3.1.1 Definition

An action undertaken to make a discovery, test a hypothesis, or confirm a known fact.

3.1.2 Example

Release your pen from 4.9 meters above the ground.

3.1.3 Predicted outcomes

- The pen will fall to the ground.
- It will take about 1 sec to reach the ground.

3.2 Actual observations

- The pen did touch the ground
- Less sure if it took exactly 1 second to do so

3.3 Uncertainty

The outcome of some experiments cannot be determined beforehand.

Example

- Roll a die: which side will show?
- Draw a card from a well-shuffled deck: which one you will get?
- How many students will be in the classroom today?

3.4 Probability theory

- Even though die rolls are random, patterns emerge when we repeat the experiment many times.
- Probability Theory describes such patterns via mathematical models.
- It is a branch of Mathematics, and is based on a set of Axioms.
- Axioms: Statements or propositions accepted to hold true
- Theorems: Propositions which are established to hold true using sound logical reasoning.

3.5 Sample Space

Definition

Sample space is the set of all possible outcomes of a random experiment.

We denote it by Ω, and a generic outcome, also called sample point, by ω (i.e. $\omega \in \Omega$).

Example

- Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\} \subset \mathbb{N}$.
- Draw a card from a poker deck: $\Omega = \{2\spadesuit, 2\diamondsuit, \dots, A\clubsuit, A\heartsuit\}$
- Wind speed at YVR (km/h): $\Omega = [0, \infty) \subset \mathbb{R}$.
- Wait time for R4 at UBC (min): $\Omega = [0, 720) \subset \mathbb{R}$

3.6 Events

Definition

An event is a subset of the sample space Ω.

Notation: We commonly use upper case letters (A, B, C, ...) for events.

Events are sets:

- $\omega \in A$ means "ω is an element of A".
- $C \subset D$ means "C is a subset of D".

3.7 Examples

Events are often formed by outcomes sharing some property. It's a good idea to practice listing explicitly the sample points of events described with words.

Example

- Roll a dice:
  - $A =$ "roll an even number" $= \{2, 4, 6\}$
  - $B =$ "roll a 3 or less" $= \{1, 2, 3\}$
  - $C =$ "roll an even number no higher than 3" $= \{2\}$
- Bus wait time: $C =$ "wait is less than half an hour" $= [15, 30]$
- Max-wind-speed: $A =$ "wind is over 80 km/hour" $= (80, \infty)$

3.8 Set Operations

Suppose A, B are events (subsets of Ω).

- **Union:** $A \cup B$

$$\omega \in A \cup B \Leftrightarrow \omega \in A \text{ or } \omega \in B$$

- **Intersection:** $A \cap B$

$$\omega \in A \cap B \Leftrightarrow \omega \in A \text{ and } \omega \in B$$

- **Complement:** A^c

$$\omega \in A^c \Leftrightarrow \omega \notin A$$

- **Symmetric difference:** $A \triangle B$

$$A \triangle B \;=\; (A \cap B^c) \,\cup\, (A^c \cap B)$$

3.9 Properties of set operations

- Commutative:
  - $A \cup B \;=\; B \cup A$
  - $A \cap B \;=\; B \cap A$
- Associative:
  - $A \cup B \cup C \;=\; (A \cup B) \cup C = A \cup (B \cup C)$
  - $A \cap B \cap C \;=\; (A \cap B) \cap C = A \cap (B \cap C)$
- Distributive:
  - $(A \cup B) \cap C \;=\; (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C \;=\; (A \cup C) \cap (B \cup C)$
- $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

3.10 De Morgan's Laws

Theorem

For any two events (sets) A and B, we have

$$(A \cup B)^c \;=\; A^c \cap B^c.$$

To prove two events are equal as above, the general approach is to show both

$$(A \cup B)^c \subseteq A^c \cap B^c$$

and

$$A^c \cap B^c \subseteq (A \cup B)^c.$$

We give part of a proof on the next slide.

3.11 Proof of De Morgan's Laws

Proof

Let us show $(A \cup B)^c \subseteq A^c \cap B^c$:

- take any $\omega \in (A \cup B)^c$, we must have $\omega \notin A \cup B$;
- This implies $\omega \notin A$ and $\omega \notin B$ (because if either $\omega \in A$ or $\omega \in B$ then we'd have $\omega \in A \cup B$).
- Hence $\omega \in A^c$ and $\omega \in B^c$,
- This is the same as $\omega \in A^c \cap B^c$.
- That is, $\omega \in (A \cup B)^c$ implies $\omega \in A^c \cap B^c$ which is

$$(A \cup B)^c \subseteq A^c \cap B^c.$$

Try it yourself to prove $A^c \cap B^c \subseteq (A \cup B)^c$.

3.12 Useful Identities

- $A = (A \cap B) \cup (A \cap B^c)$
  - First: $A \subseteq (A \cap B) \cup (A \cap B^c)$: Take $\omega \in A$, then either $\omega \in B$ or $\omega \notin B$. In the first case: $\omega \in A \cap B$, in the second case: $\omega \in A \cap B^c$. Thus $\omega \in (A \cap B) \cup (A \cap B^c)$.
  - Also: $(A \cap B) \cup (A \cap B^c) \subseteq A$: Take $\omega \in (A \cap B) \cup (A \cap B^c)$. If $\omega \in (A \cap B)$ then $\omega \in A$. If $\omega \in (A \cap B^c)$ then $\omega \in A$. Thus, we always have $\omega \in A$.
- $A \cup B = A \cup (B \cap A^c)$
  - Prove it!

3.13 Power set

The power set of Ω (denoted 2^Ω) is the set of all possible subsets of Ω. **Example:** Suppose $\Omega = \{1, 2, 3\}$. Then

$$2^\Omega = \Big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\Big\}$$

- The symbol $\emptyset$ denotes the empty set.

- If Ω has n elements, then 2^Ω has 2^n elements. In symbols:

$$\#(2^\Omega) \;=\; 2^{\#\Omega}.$$

3.14 Reasoning for the size of power set

List the n elements of Ω as

$$\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$$

Any event $A \subseteq \Omega$ is uniquely obtained by including or excluding each of $\omega_1, \omega_2, \ldots, \omega_n$.

Hence, we have 2 options for each of the n sample points, leading to 2^n distinct outcomes.

4 Probability

4.1 Probability of an event, an possible outcome

Even though we cannot predict the outcome of an experiment well enough in many cases, we have an idea about the chance of various outcomes:

- If you toss a coin, the chance of observing a head is formidable.
- If you buy a lottery ticket, the chance of winning the grand price is negligible.

In probability theory, we quantify the chance for "every subset" of the sample space in a self-consistent way.

Such a system was first proposed by Kolmogorov.

4.2 Probability Axioms

Let Ω be a sample space and B be the collection of "all subsets" of Ω.

Definition

A probability function is a function $\mathbb{P}$ with domain B so that

1. **Axiom 1:** $\mathbb{P}(\Omega) = 1$.
2. **Axiom 2:** For all $A \in B$, we have $\mathbb{P}(A) \geq 0$.

3. **Axiom 3:** If $\{A_n\}_{n\geq 1}$ are a sequence of **disjoint** events, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

4.3 Probability

- The value of $\mathbb{P}(A)$ is called the probability of event A.
- Apparently, $0 \leq \mathbb{P}(A) \leq 1$: trivial but useful. If your calculation gives an event negative probability value, re-do the calculation.
- The key phrase **a sequence of** is important: this rule does not go beyond.
- Being **disjoint** means for all $i \neq j$,

$$A_i \cap A_j = \emptyset.$$

- As a mathematical theory, probability definition does not rely on an "uncertainty experiment".

4.4 Properties of the probability function

Unless otherwise specified, A, B and so on are events, and Ω is the sample space.

- Probability of the complement:

$$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

- Monotonicity:

$$A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$$

- Probability of the union:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

- Boole's inequality:

$$\mathbb{P}(\bigcup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} \mathbb{P}(A_i)$$

4.5 Proof of formula for complement

Recall **Mathematical Theorem:** A general proposition not self-evident but proved by a chain of reasoning; a truth established by means of accepted truths.

By "proof" regarding probability formulas, we use a chain of reasoning to show they are implied by means of the accepted three Axioms.

4.6 Proof of formula for complement

Proof: $\mathbb{P}(A^c) \; = \; 1 - \mathbb{P}(A)$

- Note that (1) $\Omega = A \cup A^c$ and (2) A and A^c are disjoint.
- By Axioms 1 and 3, we have

  $$1 = \mathbb{P}(\Omega) = \mathbb{P}(A) + \mathbb{P}(A^c)$$

  which implies $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$.

4.7 Monotonicity

Proof: $A \subset B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

- We note that $B = (B \cap A) \cup (B \cap A^c)$;
- Since $A \subset B$ is given, we have $B \cap A = A$. Hence, $B = A \cup (B \cap A^c)$
- In addition, A and $B \cap A^c$ are disjoint. By Axiom 3 , we get $\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$.
- By Axiom 2, $\mathbb{P}(B \cap A^c) \geq 0$. Thus,

  $$\mathbb{P}(B) = \mathbb{P}(A) + P(B \cap A^c) \geq \mathbb{P}(A).$$

4.8 Probability of a union

Proof: $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

- First recall that $A \cup B = A \cup (B \cap A^c)$.
- Note that A and $B \cap A^c$ are disjoint events. Hence by Axiom 3:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$$

- Splitting B into union of two disjoint events: $B = (B \cap A) \cup (B \cap A^c)$ and applying Axiom 3, we get

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$$

- The formula is obtained by combining the above two conclusions.

4.9 Proof of Boole's inequality

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathbb{P}(A_i)$$

Proof

- We will prove it by induction.
- For $n = 2$:

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$$

$$\leq \mathbb{P}(A_1) + \mathbb{P}(A_2)$$

because $\mathbb{P}(A_1 \cap A_2) \geq 0$

4.10 Boole's inequality (continued)

Proof

- Assume now that it holds for n. We then have

$$\begin{aligned}
\mathbb{P}\Big(\bigcup_{i=1}^{n+1} A_i\Big) &= \mathbb{P}\Big[\Big(\bigcup_{i=1}^{n} A_i\Big) \bigcup A_{n+1}\Big] && \text{associative prop.} \\
&\leq \mathbb{P}\Big[\bigcup_{i=1}^{n} A_i\Big] + \mathbb{P}\left(A_{n+1}\right) && \text{case n=2} \\
&\leq \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) + \mathbb{P}\left(A_{n+1}\right) && \text{induction} \\
&= \sum_{i=1}^{n+1} \mathbb{P}\left(A_i\right).
\end{aligned}$$

4.11 Example of applying these formulas

John borrows 2 books.

Suppose there is a 0.5 probability he likes the first book, 0.4 that he likes the second book, and 0.3 that he likes both.

What is the probability that he will **NOT** like either of the 2 books?

4.12 Solution

Introduce two notations for events:

$$A = \{\text{ John likes book 1 }\}, B = \{\text{ John likes book 2 }\}.$$

We are told that $\mathbb{P}(A) = 0.5\,, \mathbb{P}(B) = 0.4\,, \mathbb{P}(A \cap B) = 0.3$.

We are asked to calculate $\mathbb{P}\left(A^c \cap B^c\right)$.

Note that $A^c \cap B^c = (A \cup B)^c$. Hence,

$$\mathbb{P}\left(A^c \cap B^c\right) = \mathbb{P}\left[(A \cup B)^c\right] = 1 - \mathbb{P}\left(A \cup B\right).$$

Making use of another formula:

$$\mathbb{P}\left(A \cup B\right) = \mathbb{P}\left(A\right) + \mathbb{P}\left(B\right) - \mathbb{P}\left(A \cap B\right) = 0.6.$$

We get $\mathbb{P}(A^c \cap B^c) = 1 - 0.6 = 0.4$.

General line of approach:

Connect $A^c \cap B^c$ with events A, B and $A \cap B$, because the probabilities of latter are provided.

4.13 Example of applying formulas

Jane must take two tests, call them T_1 and T_2.

Suppose the probability she passes test T_1 is **0.8**, test T_2 is **0.7** and both tests is **0.6**.

Calculate the probability that:

(a) She passes at least one test.

(b) She passes at most one test.

(c) She fails both tests.

(d) She passes only one test.

4.14 Solutions

Notation:

$$A = \{\text{Jane passes test } T_1\}$$
$$B = \{\text{Jane passes test } T_2\}$$

We are told that

$$\mathbb{P}(A) = 0.80, \mathbb{P}(B) = 0.70, \quad \text{and} \quad \mathbb{P}(A \cap B) = 0.6.$$

Our task is to use these information to figure out the probabilities of various events.

4.15 Solution to (a) Jane passes at least one test

We note that

$$\{\text{Passes at least one test}\} = A \cup B.$$

Hence,

$$\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
&= 0.80 + 0.70 - 0.60 \\
&= 0.90.
\end{aligned}$$

The probability that Jane passes at least 1 test is 0.9.

4.16 Solution to (b) Jane passes at most one test

We notice that

$$\begin{aligned}\{\text{Passes at most one test}\} &= \{\text{Fails at least one test}\} \\ &= A^c \cup B^c.\end{aligned}$$

And we notice

$$A^c \cup B^c = (A \cap B)^c \quad \text{De Morgan rule}$$

which gives one way of computing its probability.

$$\begin{aligned}\mathbb{P}(\{\text{Passes at most one test}\}) &= \mathbb{P}\left((A \cap B)^c\right) \\ &= 1 - \mathbb{P}(A \cap B) \\ &= 1 - 0.60 = 0.40.\end{aligned}$$

The probability that Jane passes at most 1 test is 0.4.

4.17 Solution to (c) She fails both tests

Similarly, we observe by De Morgan rule,

$$\begin{aligned}\{\text{Fails both tests}\} &= A^c \cap B^c \\ &= (A \cup B)^c\end{aligned}$$

which gives one way of computing its probability.

$$\begin{aligned}\mathbb{P}(\{\text{Fails both tests}\}) &= \mathbb{P}\left((A \cup B)^c\right) \\ &= 1 - \mathbb{P}(A \cup B) \\ &= 1 - 0.90 \\ &= 0.10 \text{ from Part (a).}\end{aligned}$$

Therefore, the probability that Jane fails both tests is 0.1.

4.18 Solution to (d) She passes only one test

We first decompose the event into the union of two disjoint events:

$$\{\text{Passes only one test}\} = (A \cap B^c) \cup (A^c \cap B)$$

We hence have

$$\begin{aligned}
\mathbb{P}(\{\text{Passes only one test}\}) &= \mathbb{P}(A \cap B^c) + \mathbb{P}(A^c \cap B) \\
&= [\mathbb{P}(A) - \mathbb{P}(A \cap B)] + [\mathbb{P}(B) - \mathbb{P}(A \cap B)] \\
&= \mathbb{P}(A) + \mathbb{P}(B) - 2 \times \mathbb{P}(A \cap B) \\
&= 0.80 + 0.70 - 2 \times 0.60 = 0.30.
\end{aligned}$$

The probability that Jane passes only 1 test is 0.3.

4.19 Remarks

- It is typical in introductory probability theory course to give a story first, followed by specifying events verbally.
- In these cases, your answer should be a sentence, not just with the context included.
- The best mathematical approach is to **define** some events, and express other events of interest using those events.
- Rather than relying on algebra as in our examples, it can be easier to use Venn diagram to have these events connected.
- There can be many ways to connect them, all lead to a viable probability calculation.

4.20 Exercise without a story

Exercise 2

(a) Suppose that $\mathbb{P}(A) = 0.85$ and $\mathbb{P}(B) = 0.75$. Show that

$$\mathbb{P}(A \cap B) \geq 0.60.$$

(b) More generally, prove the Bonferroni inequality:

$$\mathbb{P}\left(\cap_{i=1}^n A_i\right) \geq \sum_{i=1}^n \mathbb{P}(A_i) - (n-1).$$

4.21 Solution to 2 (a)

We know that $\mathbb{P}(A) = 0.85$ and $\mathbb{P}(B) = 0.75$ and wish to show that

$$\mathbb{P}(A \cap B) \geq 0.60.$$

Note that $A \cap B = (A^c \cup B^c)^c$ (De Morgan rule)

Hence:

$$\mathbb{P}(A \cap B) = 1 - \mathbb{P}(A^c \cup B^c)$$

Also

$$\mathbb{P}(A^c \cup B^c) \leq \mathbb{P}(A^c) + \mathbb{P}(B^c)$$

Therefore

$$\begin{aligned}
\mathbb{P}(A \cap B) &= \mathbb{P}\left((A^c \cup B^c)^c\right) \\
&= 1 - \mathbb{P}(A^c \cup B^c) \\
&\geq 1 - [\mathbb{P}(A^c) + \mathbb{P}(B^c)] \\
&= 1 - [1 - \mathbb{P}(A)] - [1 - \mathbb{P}(B)] \\
&= \mathbb{P}(A) + \mathbb{P}(B) - 1 \\
&= 0.85 + 0.75 - 1 = 0.60.
\end{aligned}$$

4.22 Solution to Example 2 (b) Prove the Bonferroni inequality

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - (n-1).$$

First note that

$$\bigcap_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n} A_i^c\right)^c.$$

Therefore

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) &= \mathbb{P}\left(\left(\bigcup_{i=1}^{n} A_i^c\right)^c\right) \\
&= 1 - \mathbb{P}\left(\bigcup_{i=1}^{n} A_i^c\right) \\
&\geq 1 - \sum_{i=1}^{n} \mathbb{P}\left(A_i^c\right) \\
&= 1 - \sum_{i=1}^{n} \left(1 - \mathbb{P}\left(A_i\right)\right) && \text{complement rule} \\
&= 1 - \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) = 1 - n + \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) \\
&= \sum_{i=1}^{n} \mathbb{P}\left(A_i\right) - (n-1)\,.
\end{aligned}$$

4.23 Refresh your memory of the probability axioms

Kolmogorov's axioms tell us that if we have (a) sample space, (b) collection of events, and wish to create a probability function $\mathbb{P}(\cdot)$; what properties this function should have.

The last a few examples show that if such a function $\mathbb{P}(\cdot)$ has been given, how one may derive the value of $\mathbb{P}(A)$ from the probabilities of other events.

We next suggest a way to set up (a) sample space, (b) collection of events, and (c) probability function $\mathbb{P}(\cdot)$ for a special type of experiments.

5 Building probabilitie functions

5.1 Equally likely outcomes

Suppose the sample space Ω is finite.

$$\Omega \;=\; \left\{\omega_1, \omega_2, \dots, \omega_n\right\}$$

In many applications, we trust that these distinct outcomes are equally likely so that we wish to make

$$\mathbb{P}(\{\omega_i\}) = a\,, \quad a \in (0, 1].$$

For such an experiment, we get a natural probability function.

5.2 Equally likely outcomes

Notice that $\Omega = \bigcup_{i=1}^{n} \{\omega_i\}$, and $\{\omega_i\}_{i=1}^{n}$ are disjoint events.

The probability theory Axioms require

$$1 = \mathbb{P}(\Omega) = \mathbb{P}\Big(\bigcup_{i=1}^{n} \{\omega_i\}\Big) = \sum_{i=1}^{n} \mathbb{P}(\{\omega_i\}) = \sum_{i=1}^{n} a \, = \, na.$$

Our only option is to let $a = 1/n$.

5.3 Equally likely outcomes

The axioms further require that for any event $A \subseteq \Omega$:

$$\begin{aligned} \mathbb{P}(A) = \sum_{\omega_i \in A} \mathbb{P}(\{\omega_i\}) = \sum_{\omega_i \in A} \frac{1}{n} = \frac{\#A}{\#\Omega} \\ = \frac{\#\{\text{favorable outcomes}\}}{\#\{\text{possible outcomes}\}} \end{aligned}$$

Being “favourable” means those in the event of interest: A in this example.

5.4 Probability calculation examples

- To calculate probabilities, we count the number of sample points in the event of interest.
- Counting the number of sample points in a set can be mathematically surprisingly complicated.
- **Combinatorial theory** deals with this problem.
- We will learn some basic combinatorial rules and techniques for this purpose.

5.5 Counting Example I

A die is rolled repeatedly until we see an outcome being 6.

(a) Specify/describe the sample space.

(b) Let E_n denote the event that the number of rolls is exactly n ($n = 1, 2, \ldots$). Describe the event E_n.

(c) Describe the event $E_1 \cup E_2$ and $(\bigcup_{n=1}^{\infty} E_n)^c$.

5.6 Solution to (a) Describe the sample space

The sample space consists of all sequences

$$(x_1, x_2, \ldots, x_{n-1}, x_n)$$

with

$$1 \leq x_i \leq 5 \quad \text{for} \quad 1 \leq i \leq n-1, \quad \text{and} \quad x_n = 6.$$

For example:

$$\begin{aligned}
(4,1,5,1,6) &\quad \text{with } n = \ 5 \\
(5,2,6) &\quad \text{with } n = \ 3 \\
(6) &\quad \text{with } n = \ 1 \\
(2,4,5,1,3,4,6) &\quad \text{with } n = \ 7
\end{aligned}$$

5.7 Solution to (b) Describe the event...rolls is n

$$\begin{aligned}
&E_1 = \{(6)\} \\
&E_2 = \{(x_1, 6) : 1 \leq x_1 \leq 5\} \\
&E_3 = \{(x_1, x_2, 6) : 1 \leq x_1, x_2 \leq 5\} \\
&\quad\ \vdots \\
&E_n = \{(x_1, x_2, ..., x_{n-1}, 6) : 1 \leq x_1, x_2, ..., x_{n-1} \leq 5\}
\end{aligned}$$

5.8 Solution (c) $E_1 \cup E_2$ and $(\bigcup_{n=1}^{\infty} E_n)^c$

Verbally interpreting $E_1 \cup E_2$, $\bigcup_{n=1}^{\infty} E_n$, and $(\bigcup_{n=1}^{\infty} E_n)^c$:

Suggestive answers:

$$\begin{aligned}
E_1 \cup E_2 &= \left\{ 6 \text{ appears before the } 3^{rd} \text{ roll} \right\} \\
\bigcup_{n=1}^{\infty} E_n &= \left\{ 6 \ \text{ eventually appears} \right\} \\
\left(\bigcup_{n=1}^{\infty} E_n \right)^c &= \left\{ 6 \text{ never appears} \right\}
\end{aligned}$$

5.9 Counting example II

A system has **5 components**, which can either be **working** or have **failed**.

The experiment consists of observing the current status (W/F) of the 5 components.

(a) Describe the sample space for this experiment.

(b) How much is $\#\Omega$?

(c) Suppose that the system will work if either components (1 and 2), or (3 and 4), or (1, 3 and 5) are working.

List the outcomes in the event $D = \{\text{The system works}\}$?

(d) Let $A = \{\text{components 4 and 5 have failed}\}$. How much is $\#A$?

(e) List the outcomes in $A \cap D$.

5.10 Answer to (a) Describe the sample space

The outcomes of the experiment are sequences

$$(x_1, x_2.x_3, x_4, x_5)$$

where we could have $x_i = W$ or $x_i = F$ depending on when component i is working or has failed.

One typical outcome is (W, W, F, W, F) which indicates that components **1**, **2** and **4** are working and **3** and **5** have failed,

5.11 Answer to (b) How many outcomes are there in Ω?

NOTE: the number of elements of a set A (aka its cardinal number) is denoted by $\#A$. or $|A|$ or (sometimes) $\|A\|$

We have $\#\Omega = 2^5 = 32$.

5.12 Answer to (c) When does it work?

(c) The system works if (1 and 2), or (3 and 4) or (1, 3 and 5) work.

Let $D = \{\text{The system works}\}$. Let us count it.

Here is an exhaustive list of D:

1 and 2 work	3 and 4 work	1, 3 and 5 work
(W, W, W, W, W)	(F, F, W, W, W)	(W, F, W, F, W)
(W, W, F, W, W)	(F, W, W, W, W)	
(W, W, W, F, W)	(W, F, W, W, W)	
(W, W, W, W, F)		
(W, W, W, F, F)	(F, F, W, W, F)	
(W, W, F, W, F)	(F, W, W, W, F)	
(W, W, F, F, W)	(W, F, W, W, F)	
(W, W, F, F, F)		

Apparently, $\#D = 15$.

5.13 Answer to (d) How big is $|A|$?

(d) A$=\{$**4 and 5 have failed**$\}$. How many outcomes are there in A?

The outcomes in A are of the form (x_1, x_2, x_3, F, F) with $x_i = W$ or $x_i = F$ for $i = 1, 2, 3$.

Hence, we have $2^3 = 8$ different outcomes:

$$\#A = 2^3 = 8\,.$$

5.14 Answer to (e) Describe all the outcomes in $A \cap D$

Recall the list of D:

1 and 2 work	3 and 4 work	1, 3 and 5 work
(W, W, W, W, W)	(F, F, W, W, W)	(W, F, W, F, W)
(W, W, F, W, W)	(F, W, W, W, W)	
(W, W, W, F, W)	(W, F, W, W, W)	
(W, W, W, W, F)		
(W, W, W, F, F)	(F, F, W, W, F)	
(W, W, F, W, F)	(F, W, W, W, F)	
(W, W, F, F, W)	(W, F, W, W, F)	
(W, W, F, F, F)		

Clearly, we have

$$W \cap A = \{(W, W, W, F, F), (W, W, F, F, F)\}$$

and $\#(W \cap A) = 2$.

5.15 Example III

Two dice have two sides painted red, two painted black, one painted yellow, and the other painted white.

When this pair of dice is rolled, what is the probability that both dice show the same color facing up?

Remark: If not explicitly declared, this type of problem assumes “equal likely outcomes”.

5.16 Answer to Example III

We answer this question with brute-force counting.

An exhaustive list of sample space is presented as follows:

	R1	R2	B3	B4	Y5	W6
R1	X	X				
R2	X	X				
B3			X	X		
B4			X	X		
Y5					X	
W6						X

The size of sample space is $\#\Omega = 36$.

The number of favourable outcomes is 10.

Hence, under the equal probability model:

$$\mathbb{P}\{\text{Same color}\} = 10/36 = 5/18.$$

5.17 Example IV

A small community consists of 20 families.

Four of them have 1 child, 8 have 2 children, 5 have 3 children, 2 have 4 children, and 1 has 5 children.

(a) What is the probability that a randomly chosen family has i children, for each $1 \leq i \leq 5$?

(b) What is the probability that a randomly chosen child comes from a family with i children, for each $1 \leq i \leq 5$?

Jargon: By "randomly chosen", it says that the **specific unit** is equally likely selected.

Applying the equal probability experiment/model for (a): family; (b): child.

5.18 Answer to Example IV

We organize the information as follows:

i	Families with i children	Children from families w/ i children
1	4	4
2	8	16
3	5	15
4	2	8
5	1	5
Total	**20**	**48**

Catch: There are 20 families, 48 children in this community.

5.19 Answer to (a)

If one family is chosen at random, what is the probability it has i children, $i = 1, 2, 3, 4, 5$?

i	Families with i children	$\mathbb{P}$(family has i children)
1	4	4/20
2	8	8/20
3	5	5/20
4	2	2/20
5	1	1/20
Total	**20**	**1.00**

5.20 Answer to (b)

If one child is randomly chosen, what is the probability that it comes from a family having i children, $i = 1, 2, 3, 4, 5$?

i	Children from families with i children	$\mathbb{P}$(child comes from family with i children)
1	4	4/48
2	16	16/48

i	Children from families with i children	$\mathbb{P}$(child comes from family with i children)
3	15	15/48
4	8	8/48
5	5	5/48
Total	**48**	**1.00**

Remark Part (a) and Part (b) are probability calculations under **Two Different** experiments.