Miscellaneous problems

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1 Carbon flux

Question 1

The Mauna Loa observatory has been measuring the atmospheric makeup of Earth since 1958. As part of ongoing research projects, the Mauna Loa observatory collects monthly averaged data on the concentration of CO_2 in the atmosphere near its observation point. The data is shown below for the year 2022.

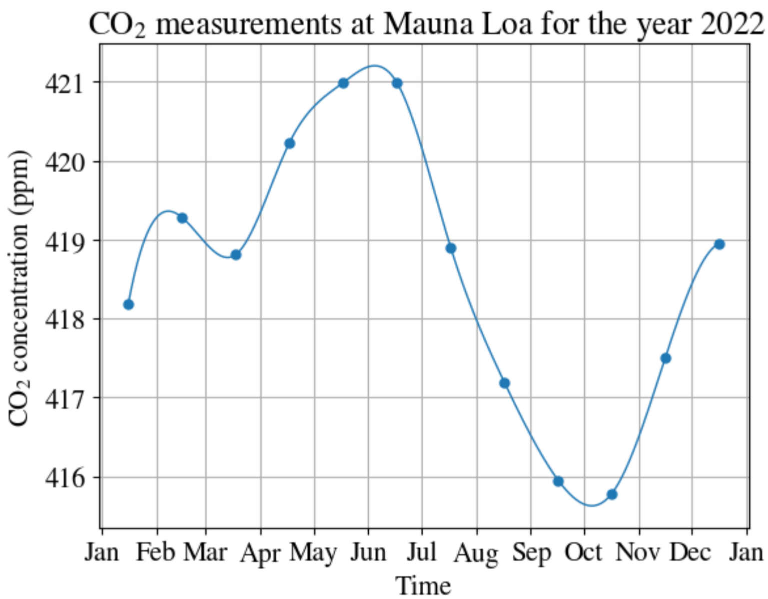


Figure 1. Graph showing the concentration of CO_2 for different months in 2022. Markers represent data. A curve was fitted between datapoints to show continuity. Data source is https://gml.noaa.gov/ccgg/trends/data.html.

1. Let $C(t)$ be the concentration of CO_2. The net flux is defined as the time-derivative of the CO_2 concentration, $\frac{dC}{dt}$. **Which** of the graphs in Figure 2. accurately represents the net flux in the CO_2 concentration with respect to time?

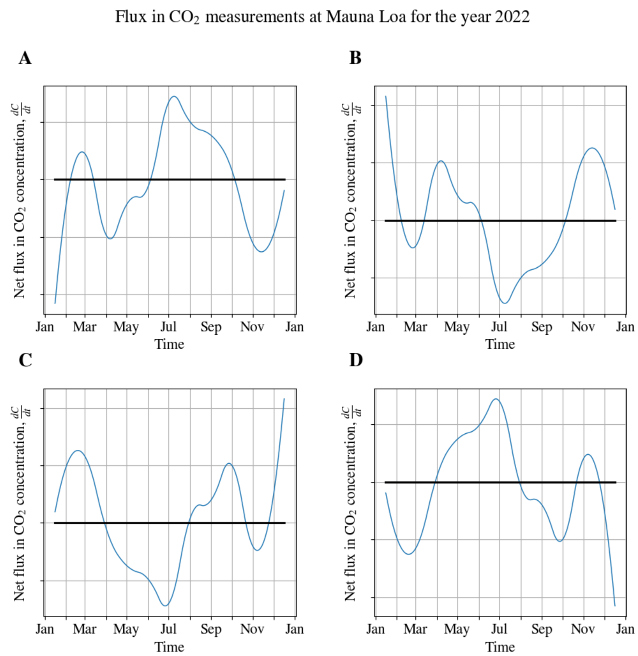


Figure 2. Potential graphs of the net flux in CO_2 concentration in 2022.

2. In practice, the net flux is a combination of various effects, some contributing to an increase in the CO_2 concentration called the influx, others contributing to a decrease in the CO_2 concentration called the outflux. **During which** months is there a point in time at which carbon influx matches outflux?

Question 2

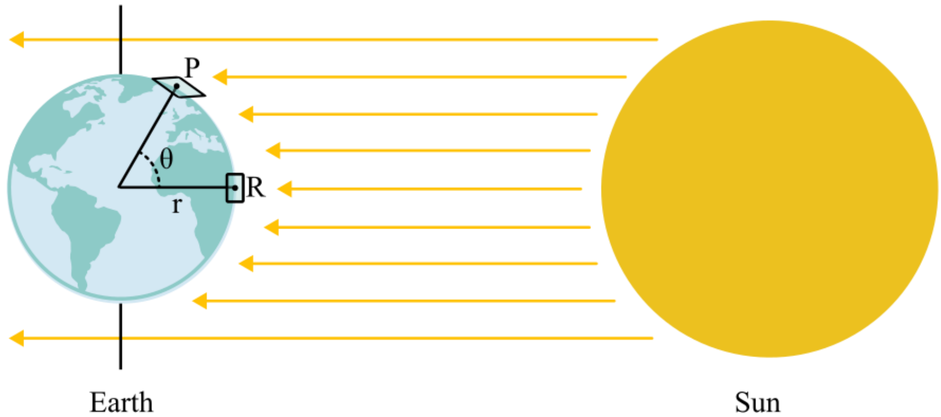
At equilibrium, a fraction of CO_2 is dissolved in water. In order to discuss this effect, we consider three variables: T for the temperature, C for the concentration of CO_2, and S for the solubility. The solubility characterizes the ability of a substance (here CO_2) to form a solution with another substance (here water). All other parameters being constant, increasing the solubility will increase the concentration. Physical processes are so that, all other parameters being constant, solubility decreases as temperature increases.

1. Is $\frac{dS}{dT}$ positive or negative?
2. Is $\frac{dC}{dS}$ positive or negative?
3. Is $\frac{dC}{dT}$ positive or negative?

2 Solar radiation

Question 1

1. Consider the Earth to be a perfect sphere of radius r. The Sun has a diameter of 109 times the Earth, so light rays incident upon the Earth from the Sun are approximately parallel to each other. Assume the Earth has no axial tilt. Consider the diagram shown below.



Consider a stationary Earth with a vertical, non-tilted axis of rotation. At a point R on the equator, the solar rays are perpendicular to the Earth's surface, so the energy received is equal to the solar constant, $Q = 1.361 kWm^{-2}$. Consider a point P at the same longitude as R, but at an angle θ away from R, for $\theta \in [-\pi, \pi]$. Let $E(\theta)$ be the amount of solar energy that reaches $1m^2$ of on Earth at an angle θ away from the point R.

(a) **Determine** $E(\theta)$ ensuring that your function is accurate across the entire domain $\theta \in [-\pi, \pi]$.

(b) The total amount of solar energy arriving at the Earth is $Q\pi r^2 \times$ (1.36 Wm^{-2}), where r is the radius of the Earth. Use this information to **determine** the average amount of solar radiation reaching each meter squared on the surface of the Earth.

(c) For some angle θ_c, the amount of solar energy received per m^2 at point P when $\theta = \theta_c$ is exactly equal to the average amount of solar energy per squared meter across the entire Earth. Determine θ_c.

Extensions to question 1

(a) We will consider the rotation of the Earth by considering a second dimension, and utilizing the time t in days. Consider the graphic below.

In this case, the amount of Energy reaching the Earth at point P' is a function of the angle

$\phi \in [0, 2\pi]$ relative to point R. Since the Earth makes a full rotation in 1 day, we will use the function $\phi = 2\pi t$ to represent the relationship between these two variables. Now, the term $F(t)$ is defined to be the fraction of the energy at the same latitude $E(\theta)$ that is absorbed by each square meter of the Earth as a function of the time of day t. **Construct** an expression for $F(t)$, ensuring that it is valid for the entire domain $t \in [0, 1]$.

(b) The function $E_T = E(\theta)F(t)/Q$ represents the total amount of energy experienced at any latitude of the Earth at any particular time. $E_T(\theta, t)$ is a function of two variables. There is only one critical point of this function for which $E_T \neq 0$. Find this non-zero critical point and classify it as a max, min, or saddle.

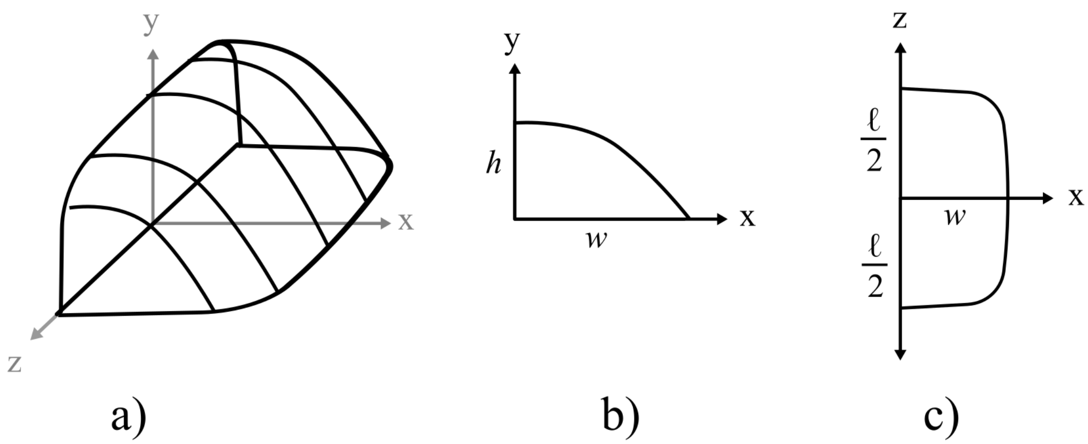
(c)

Consider an altered graphic, which represents the same situation as in a) (before the extension questions), except with the Earth having an axial tilt of 23.4° relative to the vertical. The point where light incident upon the Earth is at a maximum now sits at an angle of $\theta = 23.4°$.

Determine a new expression for $E(\theta)$, representing the amount of solar energy reaching each square meter of the Earth where θ is now given in degress, still ensuring that your function is accurate across the entire domain $\theta \in (-180, 180)$ degrees.

3 Glaciers

Glaciers play a critical role in maintaining the energy balance of the Earth. A climate scientist is studying the effect that solar energy has on the geometry of glacier. Because glaciers are made of water, they are susceptible to melting from solar energy, which can cause viscous flow to alter its shape over time. This scientist has decided to create a 3D model of the edge of a glacier. A graphical representation of the shape of this object is displayed in **Figure 1**.



The shape of this model can be reconstructed by cutting the 3D shape into infinitely many 2D slices of the same shape in the xy plane, and then stacking these slices along the z axis. Consider a 3D coordinate plane with an x, y, and z axis.

- The length of the entire shape (in the z-direction) is given by l.
- The width of the entire shape (in the x-direction) is given by w.
- The height of the entire shape (in the y-direction) is given by h.

Further, consider the following information.

- Each 2D slice has the same proportions, but a different size. The shape of each slice is given by the area under the curve of

$$y = H(z)\cos\left(\frac{\pi x}{2W(z)}\right) \qquad x \in [0, W(z)],$$

  where $H(z)$ and $W(z)$ are the height and width of a specific slice along the z axis.
- When $z = 0$, the shape of the slice is given by the area under the curve of

$$y = h\cos\left(\frac{\pi x}{2w}\right) \qquad x \in [0, w].$$

- Each 2D slice is placed orthogonal to the z-axis. The corner of every slice sits on the z-axis.
- The width of each slice, $W(z)$, depends on the value of z. $W(z)$ and z always obey the following relationship:

$$\left(\frac{W(z)}{w}\right)^2 + \left(\frac{2z}{l}\right)^4 = 1.$$

Part A

1. **Determine** an expression for the area of each 2D slice in terms of h, w, and $W(z)$.
2. **Determine** an expression for the total volume of this shape in terms of h, w, and l.

Part B

1. Let t be the time in years. A simulation is being performed using the 3D model. Assuming that both the length and volume of the glacier remain constant then if the height of the glacier edge decreases by 10m/yr (i.e. $\frac{dh}{dt} = -0.01$), **determine** the rate the width will change, $\frac{dw}{dt}$, when h is 2.1 km, l is 20.0 km, and w is 9.0 km. (Note for instructors: if Part B is done separately from Part A, then the volume of the glacier should be given).
2. Another simulation is performed with the same initial conditions, but only the volume is constant. If $\frac{dl}{dt} = 0.025$ and $\frac{dw}{dt} = 0.033$, **determine** the rate the height is changing, $\frac{dh}{dt}$, when h is 2.1 km, l is 20.0 km, and w is 9.0 km.
3. Another simulation is performed with the same initial conditions, but only the volume is held constant. If $\frac{dh}{dt} = -0.005$ and $\frac{dw}{dt} = \frac{dl}{dt}$, **determine** the rate the width of the glacier is changing, $\frac{dw}{dt}$, when h is 2.1 km, l is 20.0 km, and w is 9.0 km.
4. Another simulation is performed with the same initial conditions, but no variables are constant. Let the volume of the solid be V. If $\frac{dh}{dt} = -0.005$, $\frac{dw}{dt} = 0.02$, and $\frac{dl}{dt} = 0.034$, **determine** the rate the volume of the glacier is changing $\frac{dV}{dt}$, when h is 2.1 km, l is 20.0 km, and w is 9.0 km.

4 Numerical extension to Math 100 EBM assignment (run on syzygy)

(.ipynb file can be found in folder. If you make changes to ipynb file, you can run 'jupyter nbconvert –to latex Data_Analysis_Questions.ipynb' in the UBC syzygy terminal to export assignment to latex)

The code below imports and creates the functions you need for this assignment. Run the code block below before continuing to the assignment, but do not edit it.

[29]:
```
# DO NOT EDIT

# Imports

import matplotlib
import math
import numpy as np
import matplotlib.pyplot as plt
from scipy import integrate
matplotlib.rcParams['mathtext.fontset'] = 'stix'
matplotlib.rcParams['font.family'] = 'STIXGeneral'
#matplotlib.pyplot.title(r'ABC123 vs $\mathrm{ABC123}^{123}$')
matplotlib.rcParams.update({'font.size': 15})

# Albedo Functions

def albedo1(T):
  if T <= 247:
    return 0.7
  elif T >= 282:
    return 0.3
```

```
  else:
    #return -2/175*T+(1233)/350
    return 3.52296 - 0.011429*T

def albedo2(T):
  return 0.5-0.2 * math.tanh((T-265)/10)

def albedo3(T):
  if T <= 264:
    return 0.7
  else:
    return 0.3

# Constants

C=1e23
r=6.3781e6
Q=1365
sigma=5.6704e-8
epsilon=0.600

# Initial Conditions

T0 = 232.112

# Differential Equation (Energy Balance Model)

def dTdt(t,T):
  Ein = np.pi*r**2.0*Q*(1-albedo(T))
  Eout = 4.0*np.pi*r**2*sigma*epsilon*T**4
  return [Ein/C - Eout/C]

# Solving the Equation

def solveAndGraph():
  xs = np.linspace(220,320,100)
  ys = np.vectorize(albedo)(xs)
  plt.plot(xs,ys)
  plt.title(r"Albedo $\alpha(T)$")
  plt.xlabel("Temperature (K)")
  plt.ylabel("Albedo (unitless)")
  plt.grid()
  plt.show()

  sol = integrate.solve_ivp(dTdt, [0.0,3.156e9], [T0], t_eval=np.linspace(0,3.
↪156e9,100))
  plt.plot(sol.t/3.154e7,sol.y[0])
  plt.title(r"Temperature vs. Time")
  plt.xlabel("Time (years)")
  plt.grid(which='major', linestyle='-')
  plt.grid(which='minor', linestyle='-', alpha=0.2)
  plt.minorticks_on()
  plt.xlim(0,10)
```

```
plt.ylabel("Temperature (Kelvin)")
plt.ticklabel_format(useOffset=False)
plt.show()
```

4.0.1 Background

Instructor note: If used in tandem with the EMB Math 100 problem set, the background information below may be redundant.

The EBM (Energy Balance Model) is an important model in climate research. It suggests that the rate of change in the Earth's temperature is proportional to the difference in the incoming and outgoing rates of energy transfer due to thermal radiation. Consider the following variable definitions.

Symbol	Definition	Units
C	Heat capacity of the Earth	JK^{-1}
$T > 0$	Temperature of the Earth (in Kelvin)	K
$t > 0$	Time	s

The EBM has the following form:

$$C\frac{dT}{dt} = \underbrace{\pi r^2 Q(1-\alpha(T))}_{P_{\text{in}}} - \underbrace{4\pi r^2 \sigma \epsilon T^4}_{P_{\text{out}}}.$$

The total rate of change of the energy of the Earth is given by $C\frac{dT}{dt}$, the rate of incoming energy being absorbed by the Earth is given by P_{in} and the energy being radiated out of the Earth is given by P_{out}. The variables involved in P_{in} are: Q, which represents the rate of incoming solar energy reaching the Earth per square meter; r, which is the radius of the Earth; and $\alpha \in [0, 1]$, which is the Earth's albedo, or the proportion of light reaching the Earth's surface that gets reflected away, which is a function of temperature. The cross sectional area of the Earth that is exposed to solar radiation is πr^2.

The additional variables involved in P_{out} are: σ, which is the Stefan-Boltzmann constant and ϵ which is the proportion of the Earth's theoretical maximum energy output that is actually radiated away from the surface and into space. In other words, $1 - \epsilon$ is the fraction of outgoing radiation that is re-emitted back down to Earth due to greenhouse gases in the atmosphere. The surface area of the Earth (which is radiating the energy) is $4\pi r^2$.

Estimates for the above parameters are

$$\begin{cases} C & = 1.0 \times 10^{23} JK^{-1} \\ r & = 6.3781 \times 10^{6} m \\ Q & = 1365 Js^{-1}m^{-2} \\ \sigma & = 5.6704 \times 10^{-8} Js^{-1}m^{-2}K^{-4}. \end{cases}$$

In the EBM model the albedo, $\alpha(T)$, which is the proportion of light reaching the Earth's surface that gets reflected away, is a function of temperature. When the Earth's is colder it is covered in more snow and ice, which reflect more light, so albedo is a decreasing function of temperature.

There are three different functions that are commonly used to model the albedo, $\alpha(T)$ and in this assignment we will explore the effect that the different functions have on the temperature of the earth as a function of time, $T(t)$ (the solution to the differential equation above):

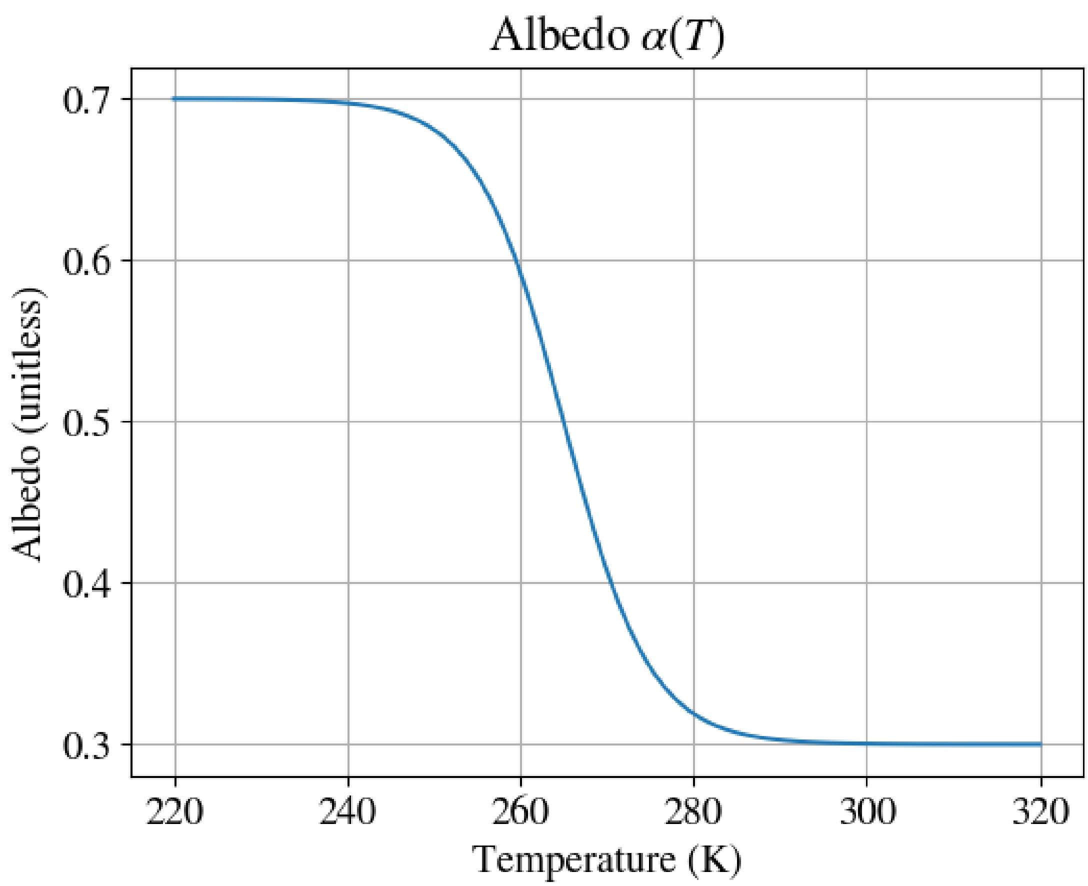
- The first function for albedo is a piecewise function (as utilized in the EBM assignment). Below this function will be referred to as `albedo1`

- The second function for albedo is a hypertangent function, which is a smoothed out approximation of the first expression. Below this function will be referred to as `albedo2`
- The third function for albedo is a discontinuous piecewise function with a single jump discontinuity. Below this function will be referred to as `albedo3`

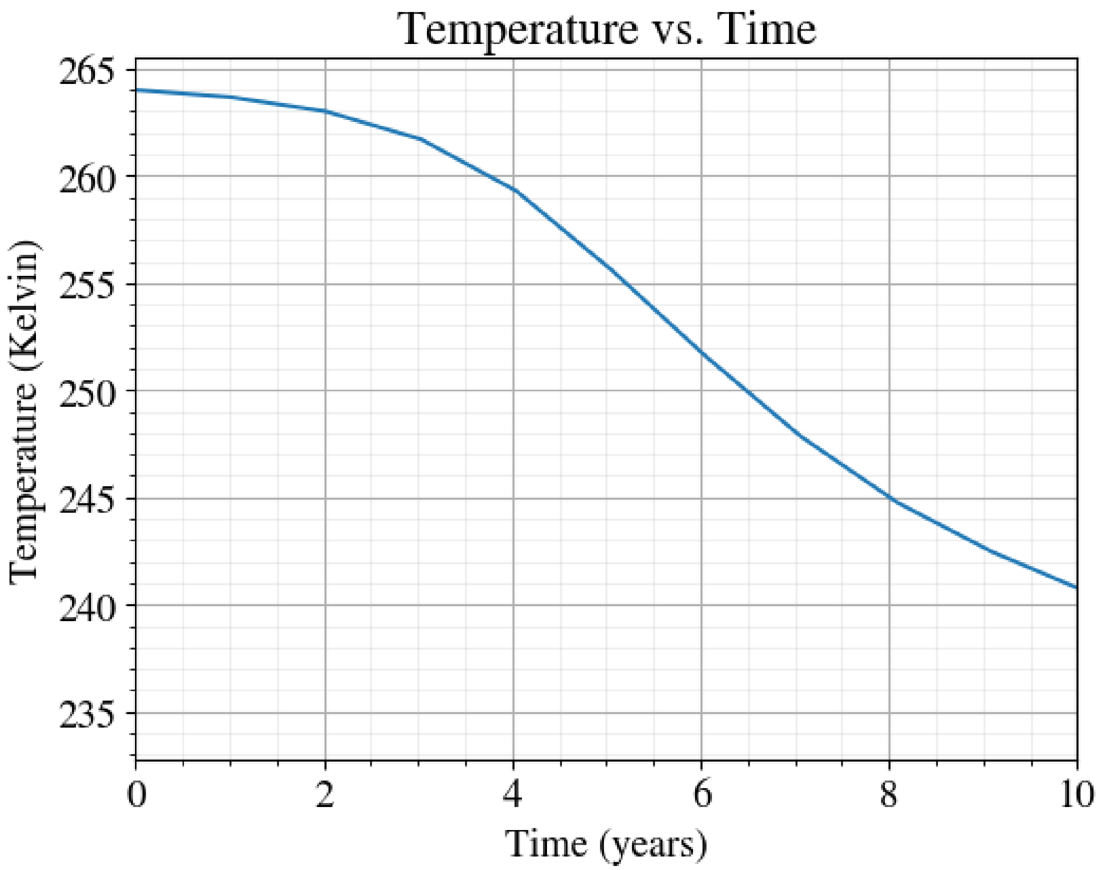
4.0.2 Code

The code below plots the solution to the EBM (the differential equation above) for a given albedo function (`albedo1`, `albedo2`, or `albedo3`) and a given initial condition, T_0. You can alter the albedo function and the initial condition to understand the effect that both have on the temperature of the earth as a function of time, $T(t)$.

[2]:
```
# EDIT BELOW

# Choose which albedo function to use
albedo = albedo2 #albedo1/albedo2/albedo3

# Choose an initial condition
T0 = 264 # [200,300]

# View a solution
solveAndGraph()
```



```
/tmp/ipykernel_691/3697474377.py:27: DeprecationWarning: Conversion of an array
with ndim > 0 to a scalar is deprecated, and will error in future. Ensure you
extract a single element from your array before performing this operation.
(Deprecated NumPy 1.25.)
  return 0.5-0.2 * math.tanh((T-265)/10)
```



4.0.3 Questions

1. Run the simulation using the three different albedo functions. Consider the top graph, which is a graph of each albedo function.

a. For the function `albedo1`, specify which points (if any) are discontinouous. Additionally, specify which points (if any) are non-differentiable.

b. Do the same for the function `albedo2`.

c. Do the same for the function `albedo3`.

2. Now consider the bottom graph, which is the result of the numerical simulation utilizing each albedo function. Run the similation for three different initial conditions: $T_0 = 239$, $T_0 = 267$, and $T_0 = 300$. Compare the plots of the Earth's temperature over time, $T(t)$, when the three different albedo functions (`albedo1`, `albedo2`, and `albedo3`) are used to determine $T(t)$ at each of the different initial conditions. Which initial condition results in the largest qualitative difference in $T(t)$ (Earth's temperature) between the different albedo functions? Why?

3. a. For this question, use the first albedo function (`albedo1`). If there is a uniform probability that the initial temperature, T_0, lies anywhere on the interval $[200, 300]$, determine the probability that $\lim_{t\to\infty} T(t) \geq 250$. Hint: run the simulation above for all integer values of $T_0 \in [260, 265]$ and look at what happens to the graph of $T(t)$.

b. How does your answer change if you use the second albedo function (`albedo2`) instead?