Calculus aspects of the Greenhouse Effect

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Introduction – Blackbody radiation

Visible light is only a part of the spectrum of *electromagnetic radiation* (EMR). EMR travels in waves and their frequency determines the energy they carry. The frequency ν and wavelength λ are related by the equation

$$\nu\lambda = c$$

where c is the speed of light and it is a universal constant of nature. Within the visible spectrum, the wavelength of light is associated with a specific colour, as shown in Figure 1.
The unit of energy is the Joule J, the unit of frequency is s^{-1} (also called the Herz Hz) and the unit of wavelength is m (in particular, $\nu\lambda$ indeed has units of a velocity).

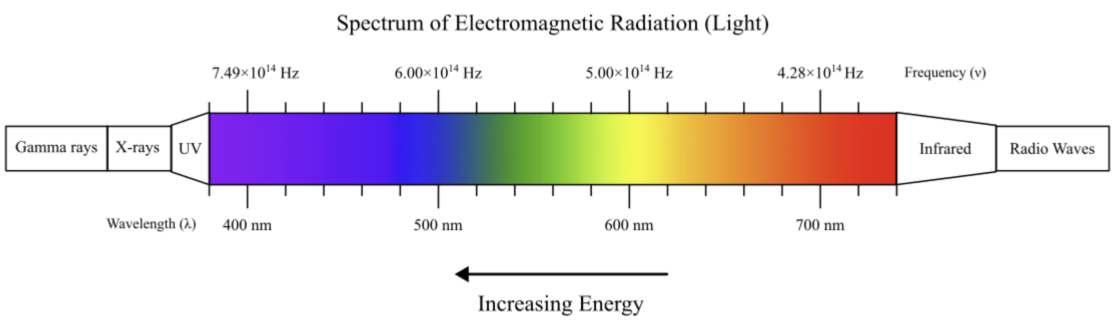

Figure 1. The electromagnetic spectrum showing visible light as well as the classifications of light beyond the visible spectrum. Diagram not drawn to scale.

Matter in thermal equilibrium constantly absorbs and emits EMR. The physical processes underpinning these interactions depend on the temperature of the object and on the wavelength of the EMR. The intensity of the radiation at various wavelengths is given by the following *Planck's law* of blackbody radiation[1]:

$$B_T(\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}.$$

Specifically, the *spectral radiance* $B_T(\lambda)$ is the rate of emitted radiation with wavelength λ per unit surface area.

[1] What this exactly means and a derivation of the formula would require much more physics, it is sufficient here to know that this blackbody is an idealization which is sufficiently good for our purpose.

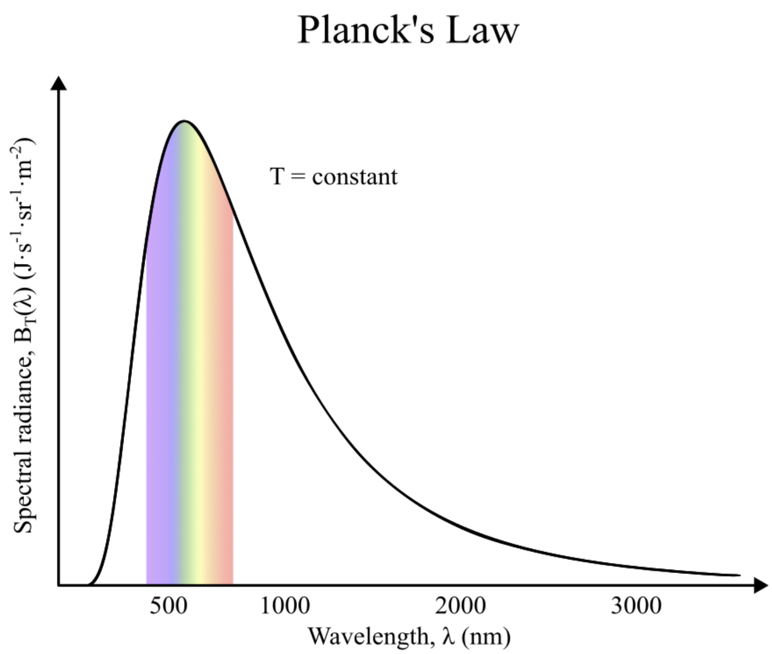

Figure 2. The graph of $B_T(\lambda)$ for a fixed value of temperature $T = 5000K$.

Consider the table of variables and units below.

Symbol	Definition	(Value) Units
B_T	Spectral blackbody radiance	$J \cdot s^{-1} \cdot sr^{-1} \cdot m^{-2}$
λ	Wavelength	m
h	Planck's constant	$(6.63 \times 10^{-34}) J \cdot Hz^{-1}$
c	Speed of light	$(3.00 \times 10^{8}) m \cdot s^{-1}$
k_B	Boltzmann constant	$(1.38 \times 10^{-23}) J \cdot K^{-1}$

The units for B_T are $Js^{-1}sr^{-1}m^{-2}$. The first part, Js^{-1}, denotes a rate of energy transfer (Joules per second). The second part, $sr^{-1}m^{-2}$, are (inverse) units of surface area. The symbol sr stands for steradians which are a way to measure angles in three-dimensions. Just as 2π radians make up a circle, 4π steradians make up a sphere.

Part 1 – Wien's displacement law (mainly Math 100 content)

As can be observed in Figure 2, Planck's law has one local maximum which is also its global maximum. Let $\lambda_{\max}$ be the critical point. In this question, we derive *Wien's law*, which is the functional dependence of $\lambda_{\max}$ on the temperature T.

1. Let the temperature T be a fixed constant. Compute $\frac{d}{d\lambda}B_T(\lambda)$.
2. Consider the variable
$$t = \frac{hc}{\lambda k_B T}.$$
Determine the equation characterizing the critical points of $B_T(\lambda)$ and write it in terms of the variable t. Simplify the equation as much as possible but do not solve this equation.
3. Whether this is appropriate depends on how much of that is taught. Show that there exists a solution $t_0 \in (0, \infty)$ to the equation obtained in 2.
4. Same remark as above. Show that there is exactly one solution $t_0 \in (0, \infty)$ to the equation obtained in 2.
5. Use the above to show that $B_T(\lambda)$ has a unique critical point $\lambda_{\max}$ and express it in terms of t_0.
6. Conclude that Wien's displacement law holds:
$$\lambda_{\max} = \frac{b}{T}$$
for some constant b that is independent of the temperature. Express b in terms of t_0 and the constants appearing the Planck's law.
7. Show that $\lambda_{\max}$ is a global maximum of $B_T(\lambda)$ on the interval $(0, \infty)$.
8. Using the information gathered so far, briefly explain why a flame changes colour from red to yellow to blue as it gets hotter.

Part 2 - Stefan-Boltzmann's law (mainly Math 101 content)

The total emitted energy per unit time and per unit surface area $I(T)$ is given by integrating over all wavelengths, namely
$$I(T) = \int_0^\infty B_T(\lambda) d\lambda.$$
Stefan-Boltzmann's Law, which we shall derive here, is the fact that $I(T)$ is proportional to the fourth power of temperature. This steep dependence plays an important role in the earth's energy balance.

1. Recall that $\lambda\nu = c$. Show that
$$I(T) = \int_0^\infty \tilde{B}_T(\nu) d\nu, \qquad \text{where} \qquad \tilde{B}_T(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}.$$

2. Determine

$$\lim_{\nu \to 0+} \tilde{B}_T(\nu).$$

3. Determine

$$\lim_{\nu \to \infty} \tilde{B}_T(\nu).$$

We now wish to understand the asymptotic behaviour of $\tilde{B}_T(\nu)$ near the boundaries of the integration domain. For this we use the notation $f(x) \approx g(x)$ as $x \to a$ whenever

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

4. Explain why

$$\tilde{B}_T(\nu) \approx \frac{2\nu^2 k_B T}{c^2} \qquad \text{as} \qquad \nu \to 0^+,$$

using a Taylor series expansion of the exponential function.

5. Show that

$$\tilde{B}_T(\nu) \approx \frac{2h\nu^3}{c^2} e^{-\frac{h\nu}{k_B T}} \qquad \text{as} \qquad \nu \to \infty.$$

6. Use the asymptotics above to argue that $\int_0^\infty \tilde{B}_T(\nu) d\nu$ is a convergent integral.
Hint: You can use that $x^3 e^{-x}$ is smaller that $Ce^{-x/2}$ for a large enough constant C.

7. Show that

$$I(T) = \sigma \cdot T^4$$

for some positive constant σ that is independent of the temperature.
Hint: This is an exercise in changing variables in an integral. You do not need to compute any integral to solve this question.

8. In the previous question, you should have encountered the integral

$$\xi = \int_0^\infty \frac{u^3}{e^u - 1} du.$$

Because the integrand decays exponentially fast to zero at ∞ (see Question 3), the numerical value of ξ is quite close to the value of $\int_0^N \frac{u^3}{e^u-1} du$ for $N = 100$.
Numerically estimate the value of ξ using Simpson's method with $N = 100$ and $\Delta u = 0.1$. Retain 5 decimal places and use the value of $\lim_{u \to 0^+} \frac{u^3}{e^u-1}$ in place of $\frac{u^3}{e^u-1}$ evaluated at zero.

9. Compute $\sqrt[4]{15\xi}$ using your numerical approximation of ξ. What number do you recognize?

10. Write a symbolic expression for ξ using what you determined previously and show that $I(T) = \sigma T^4$. *Note: By symbolic, we mean write ξ as a fraction using only integers and numbers like e, π, etc. Do not use any decimals.*

Part 3 – The greenhouse effect (application of Wien's law from Part 1)

We are now ready to understand one of the key processes underlying the role of CO_2 in the *greenhouse effect*. Gas molecules in the atmosphere absorb the energy of EMR that travels through them, but not all wavelengths are absorbed equally. The specific wavelengths that a gas tends to absorb is based upon its chemical composition, and can be expressed as an absorbance graph, see Figure 3. A larger value for absorbance at a given wavelength means that a layer of CO_2 absorbs more EMR of that specific wavelength.

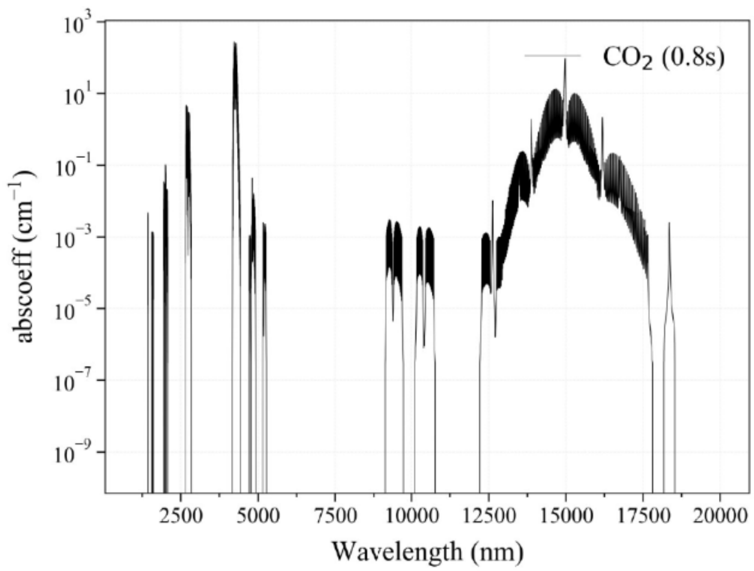

Figure 3. The absorbance graph for CO_2, sourced from HITRAN online. This graph shows for example that CO_2 is essentially transparent to light around $7500nm$, but that it absorbs much of the light's energy around $15000nm$. The absorbance is negligible outside of the range of wavelengths displayed in the graph.

When a gas absorbs EMR, the incoming energy is transformed into molecular vibrations, resulting in an increase in the temperature of the gas.

1. The sun is approximately a blackbody that emits EMR at its surface temperature of $5778K$. What wavelength of light is radiated the most abundantly by the sun? You can use that $b = 2.9 \times 10^{-3} mK$.

2. Does CO_2 in the atmosphere absorb much light from the sun?

3. As the radiation that has traversed the atmosphere reaches the earth, a fraction of it is immediately reflected while the rest is absorbed. The earth is in turn also an approximate blackbody and that emits radiation. What wavelength of light is radiated the most abundantly by the earth? You can use the temperature of the atmosphere at about $10km$, which is $223K$.

4. Does CO_2 absorb much light from the Earth around $\lambda_{\max}$?

5. Based on this information, discuss how the thermal emissions and selective absorbtions described above provide a mechanism for the greenhouse effect.