SCIENCE ONE, MATHEMATICS - HOMEWORK #2.3

Due on Thursday, March 27th by 10PM.

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1. Recall that the probability density function for the electron-proton distance r in the hydrogen atom ground state (i.e., the 1s orbital) is given by

$$\begin{aligned} f(r) &= 4\pi r^2 \psi_{100}^2(r) \\ &= Ar^2 e^{-\frac{2r}{a_0}}, \end{aligned}$$

where a_0 is the Bohr radius.

   (a) Find the cumulative distribution function $F(r)$.

   (b) Determine the value of A (i.e., "normalize" the distribution).

   (c) Determine the probability that the electron will be found within $2a_0$ of the proton. Does this seem reasonable?

   (d) Determine the "most probable" and the "expected" radius for the hydrogen atom in the ground state.

2. Recall the Greenhouse Effect problem from Written Homework 1.3, and that for an ideal object, the intensity of the radiation at various wavelengths is given by Planck's Law of Blackbody Radiation:

$$B(\lambda) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1},$$

where h is Plank's constant, c is the speed of light, λ is the wavelength, T is the temperature, and k_B is the Boltzmann constant.

We can define the total emitted energy per unit time and per unit surface area $I(T)$ by

$$I(T) = \int_0^\infty B_T(\lambda) d\lambda.$$

   (a) Show that $I(T) = \int_0^\infty \hat{B}_T(\nu) d\nu$, where $\hat{B}_T(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$. Hint: use the fact that $\lambda\nu = c$.

   (b) Show that $\int_0^\infty \hat{B}_T(\nu) d\nu = A \int_0^\infty \frac{u^3}{e^u - 1} du$. Be sure to explicitly define the constant A and the variable u.

   (c) Determine whether the integral $I(T) = A \int_0^\infty \frac{u^3}{e^u - 1} du$ is convergent or divergent.

3. *Newton's universal law of gravitation* states that the force of attraction between two point masses m and M has magnitude

$$F = \frac{GmM}{r^2},$$

where r is the distance between the masses and $G = 6.67 \times 10^{-11}$ N·m^2/kg^2 is a constant.

   (a) If M represents the mass of the centre of the earth and we regard it as a point mass concentrated at its centre, show that Newton's universal law of gravitation at the earth's surface reduces to $F = mg$, where $g = 9.82$ m/s^2. Assume for the calculation of M that the earth is a sphere with radius 6370 km and mean density 5.52×10^3 kg/m^3.

   (b) Use the original $F = \frac{GmM}{r^2}$ with the earth regarded as a point mass to calculate the work required to lift a mass of 10kg from the earth's surface to a height of 10 km.

   (c) Calculate the work in part (b) using the constant gravitational force $F = mg$ in part (a). Is there a significant difference?

4. In the next problem you will be working with an example of a fractal. A fractal is a mathematical set that displays a self-similarity property, that is, it exhibits a repeating pattern that displays at every scale.

   (a) In this problem, you will construct Helga von Koch's snowflake curve. Start with an equilateral triangle with sides of length 1. Step 1 in the construction is to divide each side into three equal parts, construct an equilateral triangle on the middle part, and then remove the middle part. Step 2 is to repeat Step 1 for each side of the resulting polygon. This process is repeated at each succeeding step. The snowflake curve is the curve that results from repeating this process indefinitely. Animated visualizations of Helga von Koch's snowflake curve (and many variants of the same curve) are available on Wikipedia (`https://en.wikipedia.org/wiki/Koch_snowflake`).

   Here you will show that the snowflake curve is a curve of infinite length that encloses a region of finite area.

      i. Sketch the three curves that are obtained after the first two steps in the construction process described above.

      ii. Let p_n be the perimeter of the polygon obtained after step n of the construction. Find a formula for p_n. *Hint: First find a formula for the number of sides of the polygon, then find the length of each side.*

      iii. Show that $p_n \to \infty$ as $n \to \infty$.

      iv. Sum an infinite series to find the area enclosed by the snowflake curve. *Hint: First compute the area of the original triangle, then add the area of each smaller triangle.*