SCIENCE ONE, MATHEMATICS - HOMEWORK #3

In this assignment you will use your calculus knowledge to verify a law of Physics you learned in your Physics classes a couple of weeks ago. Specifically, in Part 1 you will derive Wien's displacement law and in Part 2 you will use it to explore the greenhouse effect, that is the process through which heat is trapped near the Earth's surface.

We begin by recalling that visible light is only a part of the spectrum of *electromagnetic radiation* (EMR). Within the visible spectrum, the wavelength of light is associated with a specific colour, as shown in Figure 1.

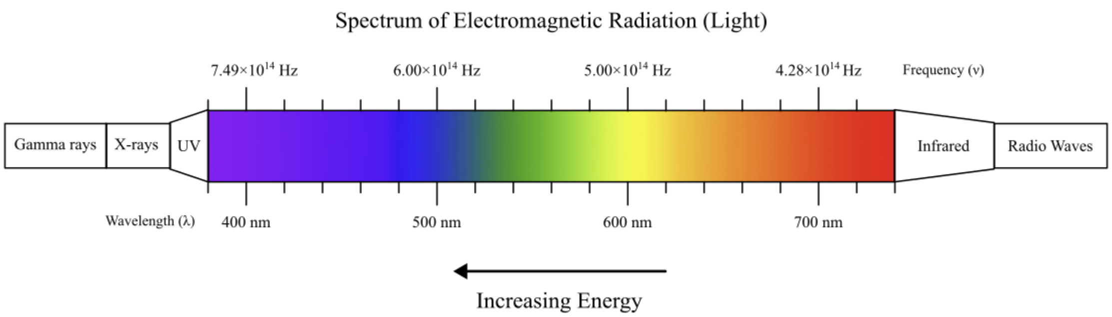

Figure 1. The electromagnetic spectrum showing visible light as well as the classifications of light beyond the visible spectrum. Diagram not drawn to scale.

Matter in thermal equilibrium constantly absorbs and emits EMR. The physical processes underpinning this interaction depend on the temperature of the object and on the wavelength of the EMR. For an ideal object, the intensity of the radiation at various wavelengths is given by the following *Planck's law* of blackbody radiation:

$$B_T(\lambda) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k_B T}} - 1}, \ 0 < \lambda < \infty$$

Specifically, the function $B_T(\lambda)$ is the rate of emitted radiation with wavelength λ per unit surface area at a given temperature T, while c, h, k_B are constants defined in the table below.

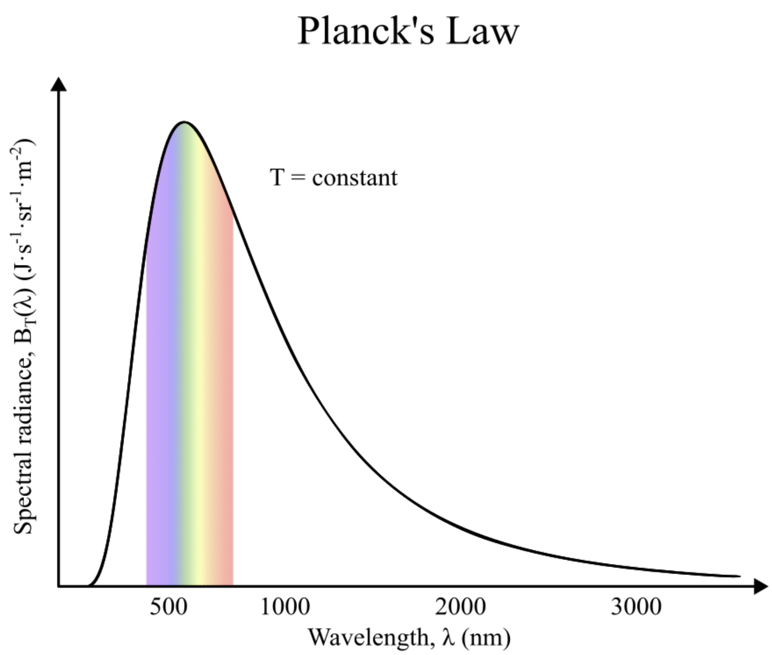

Figure 2. The graph of $B_T(\lambda)$ for a fixed value of temperature $T = 5000K$.

Consider the table of variables and units below.

Symbol	Definition	(Value) Units
B_T	Spectral blackbody radiance	$J \cdot s^{-1} \cdot sr^{-1} \cdot m^{-2}$
λ	Wavelength	m
h	Planck's constant	$(6.63 \times 10^{-34}) J \cdot Hz^{-1}$
c	Speed of light	$(3.00 \times 10^{8}) m \cdot s^{-1}$
k_B	Boltzmann constant	$(1.38 \times 10^{-23}) J \cdot K^{-1}$

Part 1: The math behind the greenhouse effect

Wien's law of displacement states that the black-body radiation curve for different temperatures will peak at different wavelengths, and the wavelength at the peak is inversely proportional to the temperature. More precisely. using the notation introduced above, **Wien's law** asserts that $B_T(\lambda)$ has a unique critical point $\lambda_{\max} = \frac{W}{T}$ for some universal constant W, and $B_T(\lambda)$ has a global maximum at $\lambda_{\max}$ (consistent with the shape of Figure 2). We will use basic calculus to verify this law.

1. Let $a = 2hc^2$ and $b = \frac{hc}{k_B}$, re-write Planck's law as

$$B_T(\lambda) = \frac{a}{\lambda^5} \frac{1}{e^{\frac{b}{\lambda T}} - 1} \tag{1}$$

Make a change of variable by replacing λ by the variable

$$t = \frac{b}{\lambda T}.$$

Note that since the range of the variable λ is $(0, \infty)$, so is the range of the variable t.

Show that this change of variable yields

$$B_T(t) = \frac{aT^5}{b^5}\frac{t^5}{e^t - 1} \tag{2}$$

2. Now compute $\frac{d}{dt}B_T(t)$. Verify that

$$\frac{d}{dt}B_T(t) = f(t)g(t)$$

where

$$f(t) = \frac{aT^5t^4}{b^5(e^t - 1)}$$

and

$$g(t) = 5 - \frac{t}{1 - e^{-t}}$$

Our goal is to show that $B_T(t)$ has a unique critical number, but first we need to establish some intermediate results.

3. Show that for all $0 < t < \infty, \quad 0 < f(t) < \infty$.

4. Now show that for all $t > 0$, $e^t > 1 + t$.

5. Let $h(t) = \dfrac{t}{1 - e^{-t}}$. Show that there is a unique solution $0 < t = t_0 < 5$ to the equation $h(t) = 5$, and that for all $0 < t < t_0$, $h(t) < 5$ and for all $t > t_0$, $h(t) > 5$ Recall $2 < e < 3$.

6. Using the results you found above, show that $B_T(\lambda)$ has a unique critical point at $\lambda_{\max} := \dfrac{b}{t_0 T}$ and $B_T(\lambda)$ has a global maximum at $\lambda_{\max}$.

7. Let $W = \frac{hc}{t_0 k_B}$. Show that $\lambda_{\max} = \dfrac{W}{T}$, that is, for a given temperature T, $\lambda_{\max}$ is inversely proportional to the temperature. This is **Wien's Law**.

8. Verify that $B_T(\lambda)$ has a horizontal asymptote as $\lambda \to \infty$.

9. Briefly explain why a flame changes colour from red to yellow to blue as it gets hotter.

Part 2: The greenhouse effect

Gas molecules in the atmosphere absorb the energy of EMR that travels through them, but not all wavelengths are absorbed equally. The specific range of wavelengths that a gas tends to absorb is based upon its chemical composition, and can be expressed as an absorbance graph; see Figure 3 below for the absorbance graph of CO_2. For a given gas, a larger value for absorbance at a given wavelength means that the gas absorbs more EMR of that specific wavelength. When a gas absorbs EMR, the incoming energy is transformed into molecular vibrations, resulting in an increase in the temperature of the gas and re-radiation in all directions. Use this information, along with the results from Part 1 and your Physics knowledge to answer the questions below.

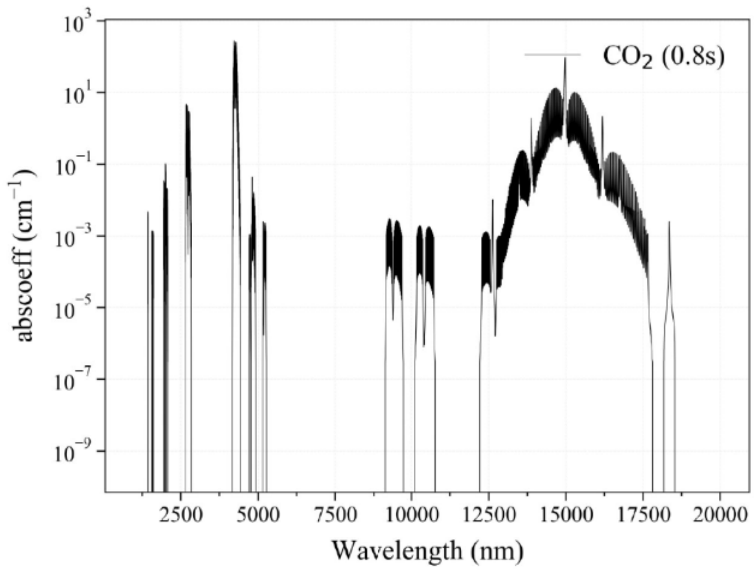

Figure 3. The absorbance graph for CO_2, sourced from HITRAN online. This graph shows that CO_2 absorbs much of the light's energy around 15000 nm, but that it is essentially transparent to light, for example, around 7500 nm. The absorbance is negligible outside of the range of wavelengths displayed in the graph.

1. The sun is approximately a blackbody that emits EMR at its surface temperature of $5778K$. What wavelength of light is radiated the most abundantly by the sun? Recall that $\lambda_{\max} = \frac{W}{T}$. Note that $W = \frac{hc}{t_0 k_B} \approx 2.9 \times 10^{-3} mK$ (the constants h, c, k_B are given in the table above and t_0 is found by approximating the solution to $\frac{t}{1-e^{-t}} = 5$ using an online equation solver).

2. Does CO_2 in the atmosphere absorb much EMR from the Sun?

3. As the radiation that has traversed the atmosphere reaches the earth, a fraction of it is immediately reflected while the rest is absorbed. The earth is in turn also an approximate blackbody and that emits radiation. What wavelength of light is radiated the most abundantly by the earth? You can use the average surface temperature of $283K$.

4. Does CO_2 in the atmosphere absorb much EMR from the Earth?

5. Based on this information, briefly discuss how the thermal emissions and selective absorbtions described above provide a mechanism for the greenhouse effect.