MATH 257/316 Class Project

A model to investigate the parameters that determine the possible extinction of a fish population

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Many species are constrained within a habitat of a given size. The size of the habitat may be limited due to the environment (e.g. there are only certain temperatures a species can tolerate, competition with other species, prey availability, or human induced mortality). A classic problem in mathematical ecology is to determine the minimum habitat size required for a species with a given growth rate to survive (Kot 2001).
The Fisher equation (1) is one model for the growth and dispersal of a species within it's habitat. Here let us consider a fish population $P(X,T)$ within a marine reserve of length L. Intense fishing occurs outside of the reserve, so the fish population can only survive in the habitat $X \in (0,L)$. The boundary conditions $P(0,T) = 0$ and $P(L,T) = 0$ represent the mortality due to fishing outside of the reserve. Here we are assuming the fishing fleet is very efficient and removes all fish that reach $X = 0$ and $X = L$.

$$\begin{aligned}
\frac{\partial P}{\partial T} &= D\frac{\partial^2 P}{\partial X^2} + \gamma P(1 - \frac{P}{P_c}), \quad 0 < X < L, \quad T > 0 \qquad (1)\\
BC &: \; P(0,T) = 0, \quad P(L,T) = 0\\
IC &: \; P(X,0) = F(X)
\end{aligned}$$

Here D represents the rate at which the fish disperse within the marine reserve, γ represents the birth rate of the fish at low population densities, P_c represents the carrying capacity of the environment, and $F(X)$ represents the fish population sampled at some initial time.
(a) By introducing the scaled variables $x = X/L$, $t = T/T_0$, and $u = P/P_c$, reduce the boundary value problem (1) to the following dimensionless form:

$$\begin{aligned}
u_t &= \alpha^2 u_{xx} + u(1-u), \quad 0 < x < 1, \quad t > 0 \qquad (2)\\
BC &: \; u(0,t) = 0, \quad u(1,t) = 0\\
IC &: \; u(x,0) = f(x),
\end{aligned}$$

where the dimensionless diffusion coefficient α^2 controls the evolution of the fishery.
(b) Now explore the possibility of extinction of the fish population $u \equiv 0$ by considering whether a small perturbation $\widetilde{u}$ to the zero solution, i.e., $u = 0 + \widetilde{u}$, will grow or decay. By substituting this perturbation into (2) and retaining only first order terms derive the linearized Fisher equation:

$$\begin{aligned}
\tilde{u}_t &= \alpha^2 \tilde{u}_{xx} + \tilde{u}, \quad 0 < x < 1, \quad t > 0 \qquad (3)\\
BC &: \; \tilde{u}_x(0,t) = 0, \quad \tilde{u}(1,t) = 0\\
IC &: \; \tilde{u}(x,0) = \widetilde{f}(x)
\end{aligned}$$

(c) Use the method of separation of variables to solve the above boundary value problem (3) for $\tilde{u}(x,t)$.

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(d) Identify a condition on the dimensionless parameter α^2 that characterizes the boundary α_c^2 between extinction and persistence of the fish population. Interpret your results to determine the minimal length of the marine reserve that will ensure persistence of the fish population as a function of the other parameters in the form $L = g(D, \gamma)$.
(e) Modify the MATLAB code provided in lecture 8 to solve the fully nonlinear Fisher equation (2) using finite differences. Now explore the sharpness of the extinction/persistence boundary by observing the solution for $\alpha^2 = 0.4$ and $\alpha^2 = 0.41$ larger than and just smaller than α_c^2. Use an initial perturbation

$$u(x,0) = 0.1e^{(-64(x-\frac{1}{2})^2)}$$

Integrate the solution till $t = 800$, determine $u(0,800)$, and plot $u(x,t=800)$ in both cases $\alpha^2 = 0.4$ and $\alpha^2 = 0.41$. For stability adjust the parameter Nt to $Nt = 2e6;$.

Interesting related references:

H. Kierstead and L.B. Slobodkin. The size of water masses containing plankton blooms. *Journalof Marine Research*,12(1):141-147,1953.

John G. Skellam. Random dispersal in theoretical populations. *Biometrika*,38(1/2):196-218,1951.

Mark Kot. Elements of Mathematical Ecology, Chapters 15-17, Cambridge University Press, 2001 (Free online through UBC library)

Michael G. Neubert. Marine reserves and optimal harvesting, *Ecology Letters*, 6:843-849, 2003.