Additional Notes on Complex Arithmetic

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Determinants can be computed using basic arithmetic operations. Also, linear systems can be solved and matrix inverses found by appling basic arithmetic operations in Gaussian Elimination. Having learned the arithmetic of complex numbers, we can apply them to finding determinants and inverses of matrices with complex entries (complex matrices) and solutions of complex linear systems.

1. Determinants of Complex Matrices.

The derminant of a square matrix with complex entries is calculated in the same way as for a real matrix As for the real case, a complex matrix is invertible iff its determinant is nonzero.

Ex 1 Calculate the determinant of $\mathbb{A} = \begin{bmatrix} 1+i & 2 \\ 3 & 1-2i \end{bmatrix}$.

$$\det \mathbb{A} = (1+i)(1-2i) - 6$$
$$= (1+2) + i(1-2) - 6 = -3 - i$$

So $\mathbb{A}$ is invertible. We will find its inverse later in these notes.

Ex 2 Calculate the determinant of

$$\mathbb{A} = \begin{bmatrix} i & 0 & 3 \\ 1 & i & -1 \\ 0 & 1 & 3+i \end{bmatrix}$$

$$\det \mathbb{A} = i\,[\,\underbrace{(i)(3+i)}_{3i-1} + 1\,] + 3(1)$$

$$= 3 - 3 = 0.$$

So in this example, $\mathbb{A}$ is not invertible and so has nontrivial homogeneous solutions. We will find these homogeneous solutions in the next section.

2. Homogeneous Solutions to Complex Linear Systems.

Homogeneous linear systems with complex coefficients have solutions that can be found after Gaussian Elimination as was done in Chapter 3 for real systems.

Ex 3 Find all complex numbers x_1, x_2 and x_3 (if any) that satisfy

$$\left.\begin{aligned} (1+i)x_1 + 2x_2 - 3i\,x_3 &= 0 \\ (1-i)x_1 + 3x_2 + (1+i)x_3 &= 0 \end{aligned}\right\} \text{zero RHS so homogeneous}$$

We can put this system into an augmented matrix

$$\left[\begin{array}{ccc|c} (1+i) & 2 & -3i & 0 \\ (1-i) & 3 & (1+i) & 0 \end{array}\right]$$

or as we did in chapter 3 we can just remember the zero RHS (which will stay zero during the elimination process) and write

$$\begin{bmatrix} (1+i) & 2 & -3i \\ (1-i) & 3 & (1+i) \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i & -\frac{3}{2}-\frac{3i}{2} \\ 0 & 3+2i & 4+i \end{bmatrix} \begin{array}{l} (1)\div(1+i) \\ (2)-(1-i)(1) \end{array}$$

$\sim \cdots$

where in the first row on the right above I used

$$\frac{1}{(1+i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}.$$

and in the second row

$$3-(1-i)^2 = 3-(1-2i-1) = 3+2i$$

$$(1+i) - \left(-\frac{3}{2}-\frac{3i}{2}\right)(1-i) = 1+i + \frac{3}{2}(1+i)(1-i)$$

$$= 1+1+3 = 4+i.$$

Continuing

$$\cdots \sim \begin{bmatrix} 1 & 0 & -\frac{57}{26} - \frac{i}{26}. \\ 0 & 1 & \frac{14}{13} - \frac{5i}{13} \end{bmatrix} \begin{array}{l} (1)-(1-i)(2). \\ (2)\div(3+2i) \end{array} \quad ★$$

where in the second row above I used

$$\frac{1}{3+2i} = \frac{3-2i}{13}$$

and $(4+i)(3-2i)/14 = 14/13 - 5i/13$

and in the first row,

$$-\frac{3}{2} - \frac{3i}{2} - (1-i)\left(\frac{14}{13} - \frac{5i}{13}\right)$$

$$= -\frac{3}{2} - \frac{3i}{2} - \frac{9}{13} + \frac{19i}{13} = \frac{-57}{26} - \frac{i}{26}$$

★ is the reduced row echelon form of $/A$. It can be seen that X_3 is not determined, we let it be the parameter t.

Then from the second row of ⊛

$$x_2 = -\left(\frac{14}{13} - \frac{5i}{13}\right)t$$

and from the first row of ⊛

$$x_1 = -\left(-\frac{57}{26} - \frac{i}{26}\right)t$$

or summarizing

$$\underline{x} = \left(\frac{57}{26} + \frac{i}{26},\ -\frac{14}{3} + \frac{5i}{13},\ 1\right)t \quad , \quad t \text{ any complex number}$$

$t \in \mathbb{C}$

are all the homogeneous solutions to the system.

Ex 4 Find a nontrivial solution to the homogeneous system with the coefficient matrix

$$A = \begin{bmatrix} i & 0 & 3 \\ 1 & i & -1 \\ 0 & 1 & 3+i \end{bmatrix}$$

from Ex 2.

$$\begin{bmatrix} i & 0 & 3 \\ 1 & i & -1 \\ 0 & 1 & 3+i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3i \\ 0 & i & -1+3i \\ 0 & 1 & 3+i \end{bmatrix} \begin{array}{l} \times \frac{1}{i} = -i \\ (2)-(1) \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & -3i \\ 0 & 1 & 3+i \\ 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \times \frac{1}{i} = -i \\ (3)-(2) \end{array}$$

All solutions $\underline{x} = (3i, -3-i, 1)\,t$

We can take $t = 1$ (or any $t \neq 0$) to get a nontrivial solution

$t = 1 \Rightarrow \underline{x} = (3i, -3-i, 1)$.

Note that in Chapter 6 when we are finding eigenvectors corresponding to complex eigenvalues, this is equivalent to finding non trivial solutions to homogeneous systems with square, complex coefficient matrices with determinant zero as we did in Ex 4 above.

3. Nonhomogeneous systems to complex linear systems.

This is done with Gaussian elimination as in the real case.

Ex 5 Find all solutions x_1 and x_2 of the linear system

$$\begin{aligned} (1+i)x_1 + 2x_2 &= 1 \\ 3x_1 + (1-2i)x_2 &= i. \end{aligned}$$

Note: the coefficient matrix of the system is the same as in Ex 1. It has a nonzero determinant so the system above has a unique solution.

Write the system in an augmented matrix and perform elimination

$$\left[\begin{array}{cc:c} 1+i & 2 & 1 \\ 3 & 1-2i & i \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 1-i & \frac{1}{2}-\frac{i}{2} \\ 0 & -2+i & -\frac{3}{2}+\frac{5i}{2} \end{array}\right] \begin{array}{l} \div(1+i) \\ (2)-3(1) \end{array}$$

$$\sim \left[\begin{array}{cc:c} 1 & 0 & \frac{1}{10}+\frac{13}{10}i \\ 0 & 1 & \frac{11}{10}-\frac{7}{10}i \end{array}\right] \begin{array}{l} (1)-(1-i)(2). \\ \times\frac{1}{-2+i} = \frac{-2}{5}-\frac{i}{5} \end{array}$$

4. Finding inverses of Complex Matrices.

This is also done with Gaussian elimination as in the real case.

Ex 6 Find the inverse of the matrix in Ex 1,

$$A = \begin{bmatrix} 1+i & 2 \\ 3 & 1-2i \end{bmatrix}.$$

As usual, form the augmented matrix

$$\left[\begin{array}{cc|cc} 1+i & 2 & 1 & 0 \\ 3 & 1-2i & 0 & 1 \end{array}\right] \sim \cdots$$

Then put the left hand side into reduced row echelon form. We can re-use some of the work from Ex 5 which has the same coefficient matrix A.

$$\sim \cdots \left[\begin{array}{cc|cc} 1 & 1-i & \frac{1}{2} - i/2 & 0 \\ 0 & -2+i & -\frac{3}{2} + \frac{3i}{2} & 1 \end{array}\right].$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{10} + \frac{7}{10} i & \frac{3}{5} - \frac{i}{5} \\ 0 & 1 & \frac{9}{10} - \frac{3}{10} i & -\frac{2}{5} - \frac{i}{5} \end{array}\right] (1)-(1-i)(2)$$

$$\text{So } A^{-1} = \begin{bmatrix} -\frac{1}{10} + \frac{7}{10} i & \frac{3}{5} - \frac{i}{5} \\ \frac{9}{10} - \frac{3}{10} i & -\frac{2}{5} - \frac{i}{5} \end{bmatrix}.$$

5. MATLAB commands.

The MATLAB commands det, rref, inv introduced for real matrices also apply to complex matrices.