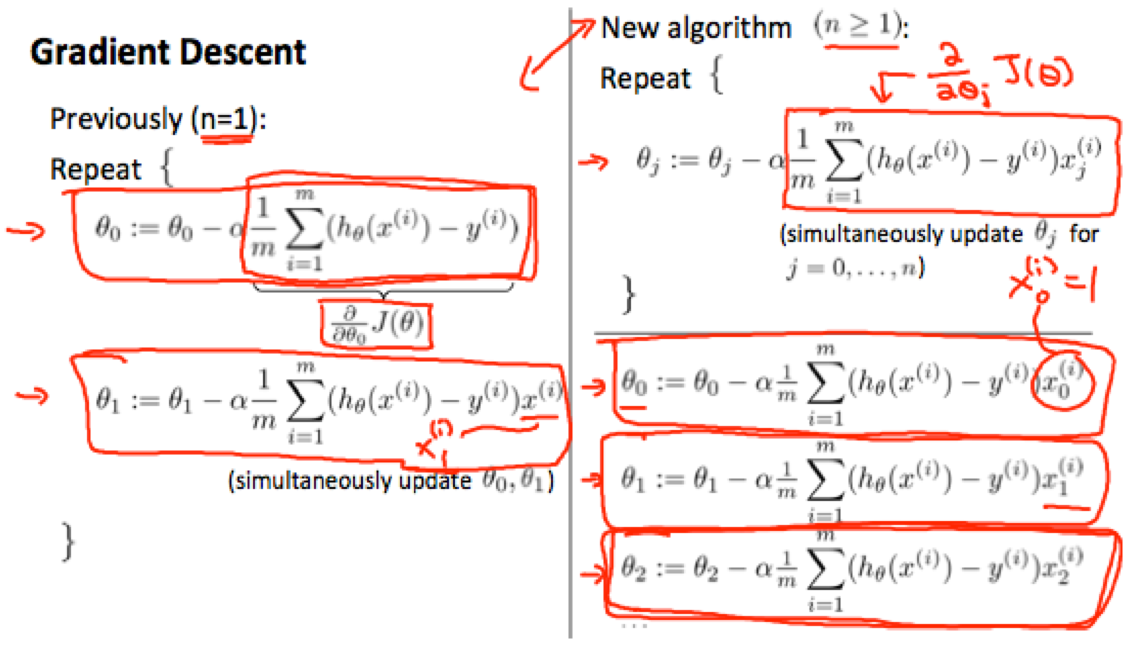
Multivariate linear regression

Previously n was 1 i.e. only one feature, like based on area we predict the housing price, now multiple n, i.e. multiple features like area of house, age of house, no of floors etc.



Gradient Descent

Previously (n=1):

Repeat {

$$\theta_0 := \theta_0 - \alpha \underbrace{\frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})}_{\frac{\partial}{\partial\theta_0}J(\theta)}$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x^{(i)}$$

$x^{(i)}_1$

(simultaneously update θ_0, θ_1)

}

New algorithm $(n \geq 1)$:

Repeat {

$\frac{\partial}{\partial\theta_j}J(\theta)$

$$\theta_j := \theta_j - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_j^{(i)}$$

(simultaneously update θ_j for $j = 0, \ldots, n$)

}

$x_0^{(i)} = 1$

$$\theta_0 := \theta_0 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_0^{(i)}$$

$$\theta_1 := \theta_1 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_1^{(i)}$$

$$\theta_2 := \theta_2 - \alpha \frac{1}{m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})x_2^{(i)}$$

...

GD Feature scaling

We can speed up gradient descent by having each of our input values in roughly the same range. This is because θ will descend quickly on small ranges and slowly on large ranges, and so will oscillate inefficiently down to the optimum when the variables are very uneven.

The way to prevent this is to modify the ranges of our input variables so that they are all roughly the same. Ideally:

−1 ≤ x(i)x_{(i)}x(i) ≤ 1

or

−0.5 ≤ x(i)x_{(i)}x(i) ≤ 0.5

These aren't exact requirements; we are only trying to speed things up. The goal is to get all input variables into roughly one of these ranges, give or take a few.

Two techniques to help with this are **feature scaling** and **mean normalization**. Feature scaling involves dividing the input values by the range (i.e. the maximum value minus the minimum value) of the input variable, resulting in a new range of just 1. Mean normalization involves subtracting the average value for an input variable from the values for that input variable resulting in a new average value for the input variable of just zero. To implement both of these techniques, adjust your input values as shown in this formula:

$$x_i := \frac{x_i - \mu_i}{s_i}$$

Where μ_i is the **average** of all the values for feature (i) and s_i is the range of values (max - min), or s_i is the standard deviation.

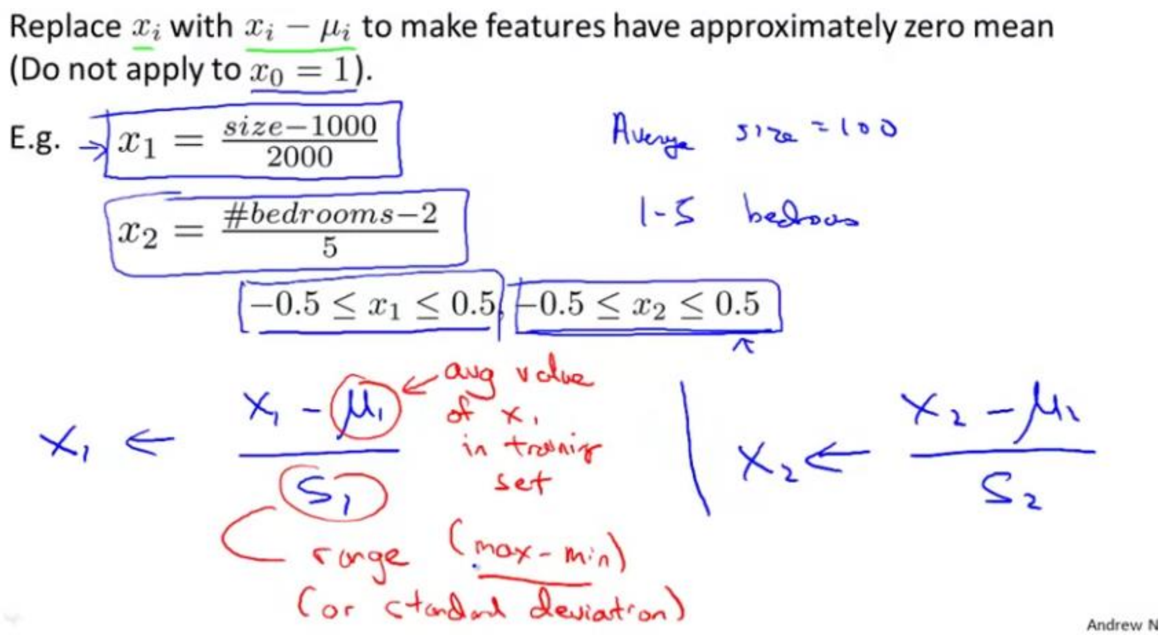
Note that dividing by the range, or dividing by the standard deviation, give different results. The quizzes in this course use range - the programming exercises use standard deviation.

For example, if x_i represents housing prices with a range of 100 to 2000 and a mean value of 1000, then, $x_i := \frac{price - 1000}{1900}$.

Example to feature scaling and mean normalization

Say, you have two features i.e. size of bedroom and no of bedrooms. Size ranges from 0 to 2000m2 and no ranges from 1-5, so this makes the gradient descent algorithm more difficult to reach the local minimum, as the concentric ovals look tall and slim. To make it more oval or circle like the values have to scaled in a way that the values lie between -1 to 1, like divide the area of rooms/2000 and the no of bedrooms by 5.

Scale the features in a way that the mean value of them is zero



Replace x_i with $x_i - \mu_i$ to make features have approximately zero mean (Do not apply to $x_0 = 1$).

E.g. $x_1 = \frac{size - 1000}{2000}$

$x_2 = \frac{\#bedrooms - 2}{5}$

$-0.5 \le x_1 \le 0.5$, $-0.5 \le x_2 \le 0.5$

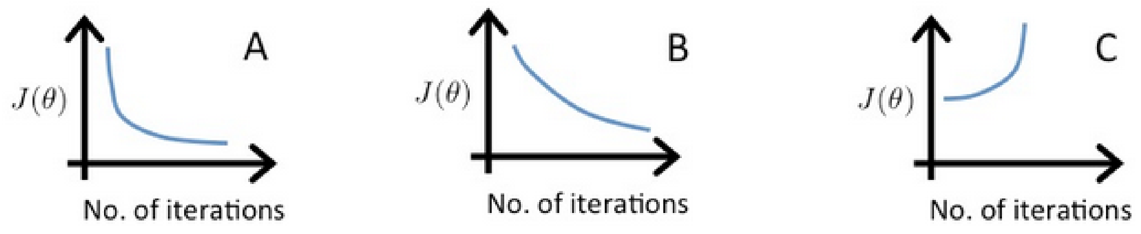
Average size = 100
1-5 bedrooms
$x_1 \leftarrow \frac{x_1 - \mu_1}{s_1}$
avg value of x_1 in training set
range (max - min)
(or standard deviation)
$x_2 \leftarrow \frac{x_2 - \mu_1}{s_2}$
Andrew N

Learning rate

- If α is too small: slow convergence.
- If α is too large: $J(\theta)$ may not decrease on every iteration; may not converge.

● A is $\alpha = 0.1$, B is $\alpha = 0.01$, C is $\alpha = 1$.

Suppose a friend ran gradient descent three times, with $\alpha = 0.01$, $\alpha = 0.1$, and $\alpha = 1$, and got the following three plots (labeled A, B, and C):



To choose α, try

$$\ldots, 0.001, 0.003, 0.01, 0.03, 0.1, 0.3, 1, \ldots$$

3x ≈3x 3x ≈3x

Polynomial regression

If the data points are in curved fashion then linear regression is bad for this data set, instead a polynomial regression is better, for the curves can come down cubic functions can be used.

We can improve our features and the form of our hypothesis function in a couple different ways.

We can **combine** multiple features into one. For example, we can combine x_1 and x_2 into a new feature x_3 by taking $x_1 \cdot x_2$.

Polynomial Regression

Our hypothesis function need not be linear (a straight line) if that does not fit the data well.

We can **change the behavior or curve** of our hypothesis function by making it a quadratic, cubic or square root function (or any other form).

For example, if our hypothesis function is $h_\theta(x) = \theta_0 + \theta_1 x_1$ then we can create additional features based on x_1, to get the quadratic function $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2$ or the cubic function $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_1^2 + \theta_3 x_1^3$

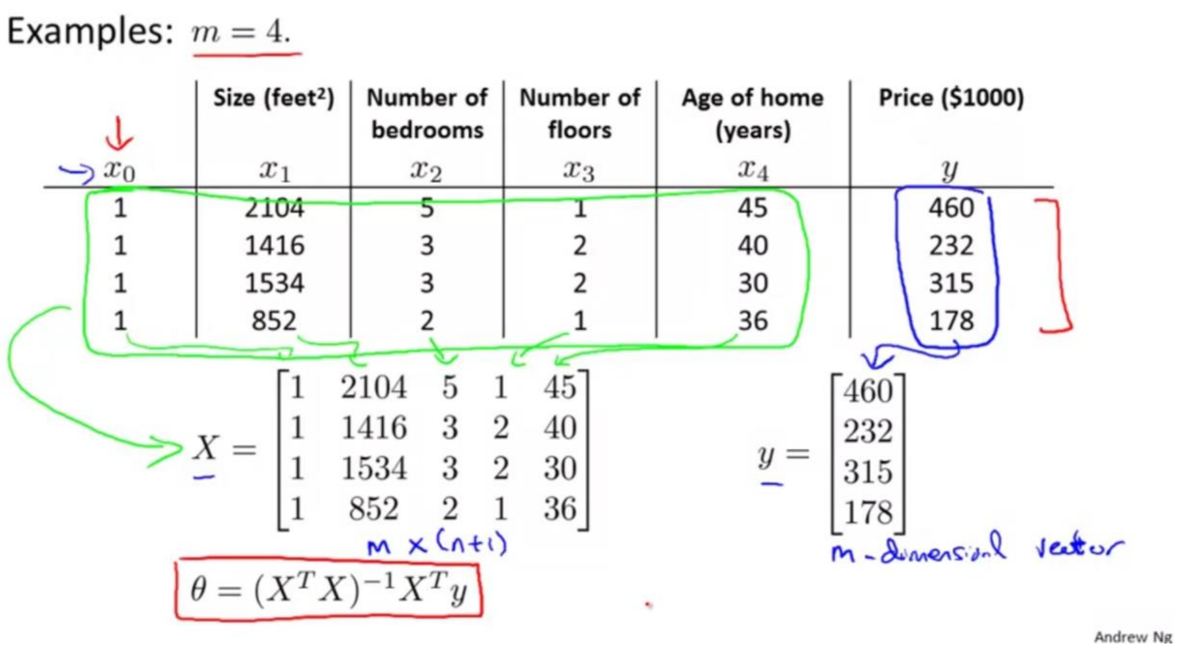
In the cubic version, we have created new features x_2 and x_3 where $x_2 = x_1^2$ and $x_3 = x_1^3$.

To make it a square root function, we could do: $h_\theta(x) = \theta_0 + \theta_1 x_1 + \theta_2 \sqrt{x_1}$

One important thing to keep in mind is, if you choose your features this way then feature scaling becomes very important.

eg. if x_1 has range 1 - 1000 then range of x_1^2 becomes 1 - 1000000 and that of x_1^3 becomes 1 - 1000000000

Solving for Theta without GD using Normal method


Examples: $m = 4.$

x_0	Size (feet²) x_1	Number of bedrooms x_2	Number of floors x_3	Age of home (years) x_4	Price ($1000) y
1	2104	5	1	45	460
1	1416	3	2	40	232
1	1534	3	2	30	315
1	852	2	1	36	178

$$X = \underbrace{\begin{bmatrix} 1 & 2104 & 5 & 1 & 45 \\ 1 & 1416 & 3 & 2 & 40 \\ 1 & 1534 & 3 & 2 & 30 \\ 1 & 852 & 2 & 1 & 36 \end{bmatrix}}_{m \times (n+1)} \qquad y = \underbrace{\begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}}_{m\text{-dimensional vector}}$$

$$\boxed{\theta = (X^TX)^{-1}X^Ty}$$

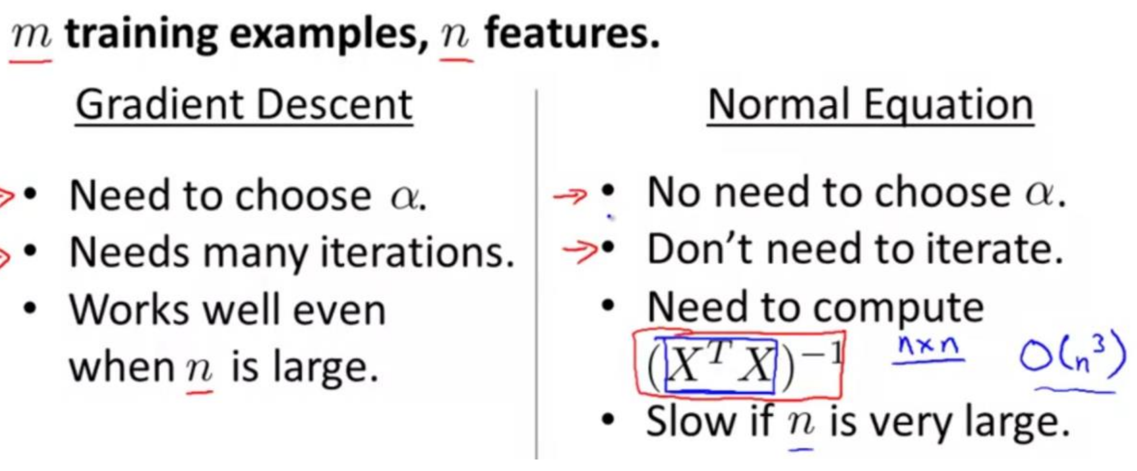
Andrew Ng

Gradient descent is an iterative algorithm, there’s a direct method instead of this

$$\theta = (X^TX)^{-1}X^Ty$$

If number of features ‘n’ is large the normal method will take lots of time in the order of O (n^3), GD is faster here.

Normal vs GD


m **training examples,** n **features.**

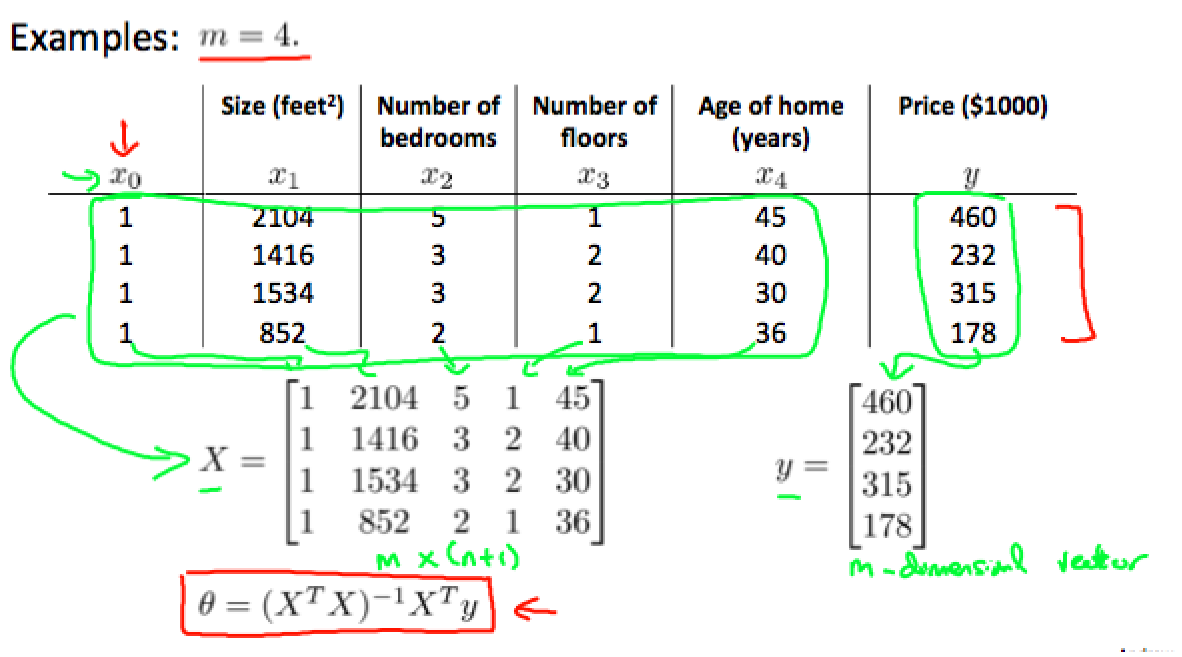
Gradient Descent	Normal Equation
• Need to choose α.	• No need to choose α.
• Needs many iterations.	• Don’t need to iterate.
• Works well even when n is large.	• Need to compute $\boxed{(\underbrace{\boxed{X^TX}}_{n\times n})^{-1}}$ $O(n^3)$
	• Slow if n is very large.

n in the range of 100-10000 is okay, more than that is bad using normal equation.

Note: [8:00 to 8:44 - The design matrix X (in the bottom right side of the slide) given in the example should have elements x with subscript 1 and superscripts varying from 1 to m because for all m training sets there are only 2 features x0x_0x0 and x1x_1x1. 12:56 - The X matrix is m by (n+1) and NOT n by n.]

Gradient descent gives one way of minimizing J. Let’s discuss a second way of doing so, this time performing the minimization explicitly and without resorting to an iterative algorithm. In the "Normal Equation" method, we will minimize J by explicitly taking its derivatives with respect to the θj ’s, and setting them to zero. This allows us to find the optimum theta without iteration. The normal equation formula is given below:

θ=(XTX)−1XTy\theta = (X^T X)^{-1}X^T yθ=(XTX)−1XTy



Examples: $\underline{m = 4.}$

	Size (feet²)	Number of bedrooms	Number of floors	Age of home (years)	Price ($1000)
x_0	x_1	x_2	x_3	x_4	y
1	2104	5	1	45	460
1	1416	3	2	40	232
1	1534	3	2	30	315
1	852	2	1	36	178

$$\underline{X} = \underbrace{\begin{bmatrix} 1 & 2104 & 5 & 1 & 45 \\ 1 & 1416 & 3 & 2 & 40 \\ 1 & 1534 & 3 & 2 & 30 \\ 1 & 852 & 2 & 1 & 36 \end{bmatrix}}_{m \times (n+1)} \qquad \underline{y} = \underbrace{\begin{bmatrix} 460 \\ 232 \\ 315 \\ 178 \end{bmatrix}}_{m\text{-dimensional vector}}$$

$$\boxed{\theta = (X^T X)^{-1} X^T y} \leftarrow$$

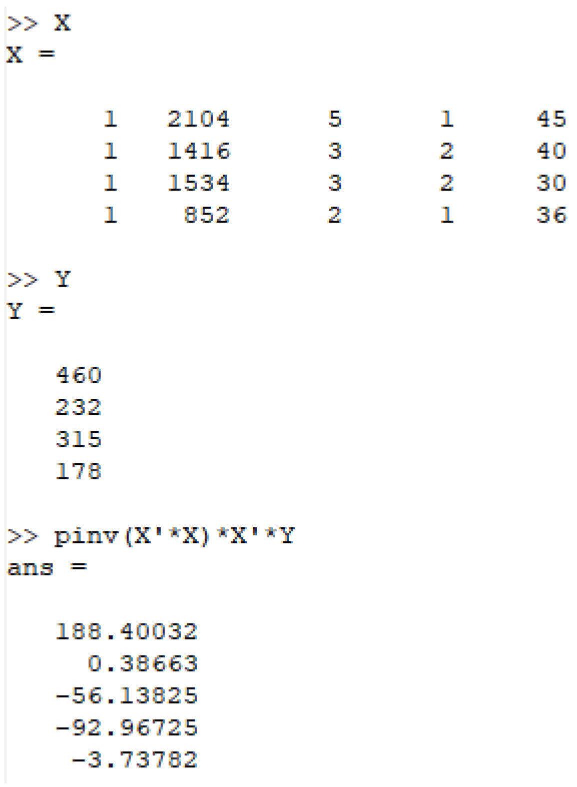
There is **no need** to do feature scaling with the normal equation.

The following is a comparison of gradient descent and the normal equation:

Gradient Descent	Normal Equation
Need to choose alpha	No need to choose alpha
Needs many iterations	No need to iterate
O (kn2kn^2kn2)	O (n^3), need to calculate inverse of X’*X
Works well when n is large	Slow if n is very large

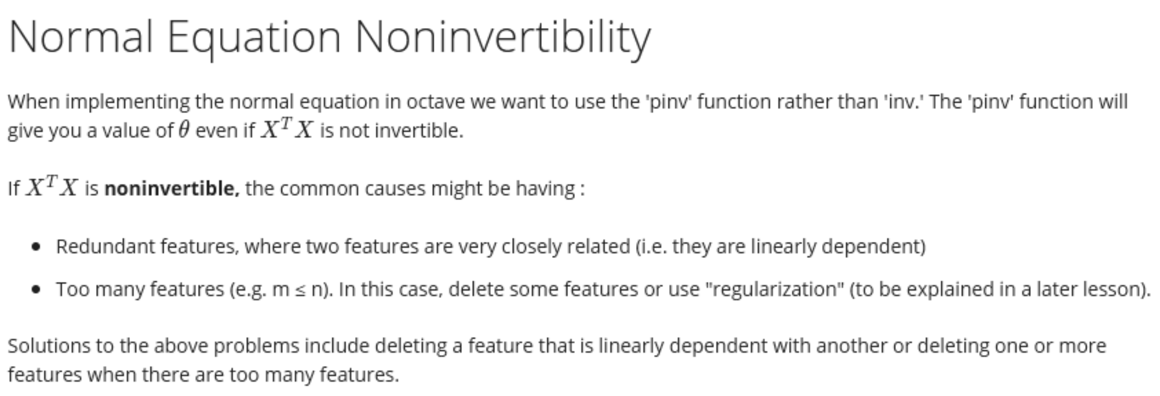
With the normal equation, computing the inversion has complexity, So if we have a very large number of features, the normal equation will be slow. In practice, when n exceeds 10,000 it might be a good time to go from a normal solution to an iterative process.

Octave implementation



```
>> X
X =

      1   2104      5      1     45
      1   1416      3      2     40
      1   1534      3      2     30
      1    852      2      1     36

>> Y
Y =

   460
   232
   315
   178

>> pinv(X'*X)*X'*Y
ans =

   188.40032
     0.38663
   -56.13825
   -92.96725
    -3.73782
```

Normal equation: non-invertibility



Normal Equation Noninvertibility

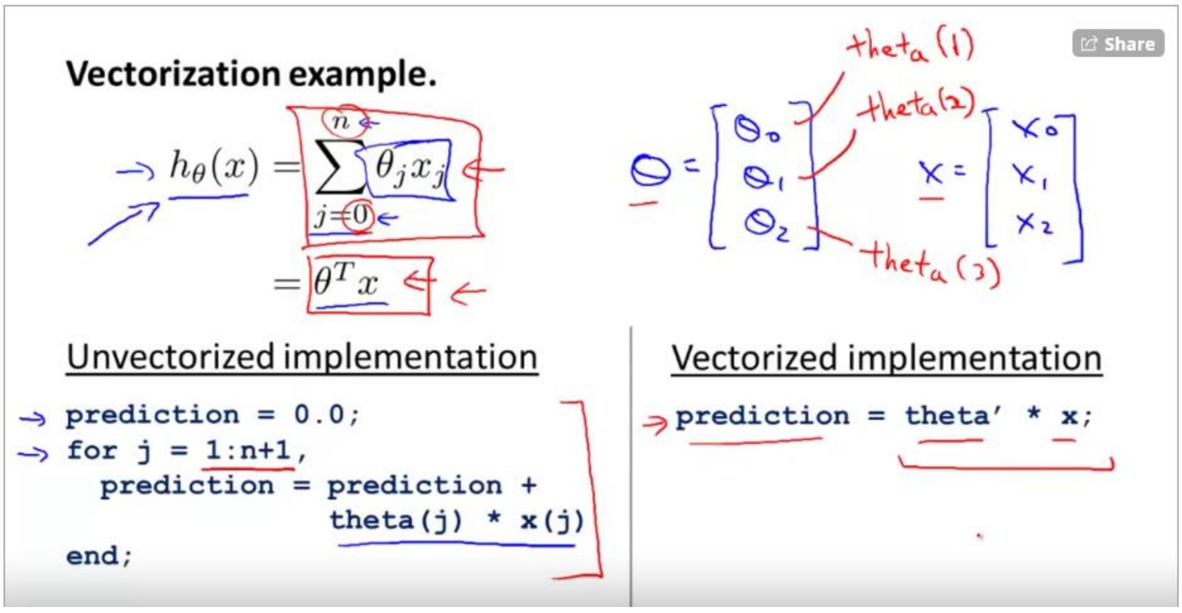
When implementing the normal equation in octave we want to use the 'pinv' function rather than 'inv.' The 'pinv' function will give you a value of θ even if $X^T X$ is not invertible.

If $X^T X$ is **noninvertible,** the common causes might be having :

- Redundant features, where two features are very closely related (i.e. they are linearly dependent)
- Too many features (e.g. m ≤ n). In this case, delete some features or use "regularization" (to be explained in a later lesson).

Solutions to the above problems include deleting a feature that is linearly dependent with another or deleting one or more features when there are too many features.

Vectorization – faster than for loop method



Vectorization example.

$$h_\theta(x) = \boxed{\sum_{j=0}^{n} \theta_j x_j}$$
$$= \boxed{\theta^T x}$$

$$\theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \end{bmatrix} \qquad x = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix}$$

Unvectorized implementation

```
prediction = 0.0;
for j = 1:n+1,
  prediction = prediction +
               theta(j) * x(j)
end;
```

Vectorized implementation

```
prediction = theta' * x;
```