

q -deformations
of

Stability Conditions
&
Quadratic Differentials

with A.Ikeda

$$Q_{\text{stab}} = Q_{\text{Dress}}$$

2019

q -Deformations of categories, Stab & Quad

with Akishi Ikeda.

§1. Motivations

- Homological Mirror Symmetry (Kontsevich)

$$DF_{\text{Fuk}}(X) \cong D^b(\text{Coh } X^\vee)$$

classically
Calabi-Yau-3

- A conj. geometric picture:

Bridgeland space
of stability conditions

$$\text{Stab } \mathcal{D} \cong \mathcal{M}_{\text{cpt}}(X^\vee)$$

Moduli space
of complex str.

Remark

In general: it is even hard to construct geo. Stab.

- Known results:

Thm(Bridgeland-Smith)

$$\text{Stab}^{\circ} \underline{D_3(S)} \cong \text{FQuad}_3(S) \quad \xrightarrow{\quad DF_{\text{Fuk}}(X_S) \quad}$$

King-Q's
version

Thm(Haiden-Kazarkov-Kontsevich)

$$\text{Stab}^{\circ} \underline{D_{\infty}(S)} \cong \text{FQuad}_{\infty}(S)$$

$\coloneqq \text{TFuk}(S)$

Aim: use HKK to obtain a β -deformation of BS.

& with application to representation theory & cluster theory.

(i.e. relate $D_3(S)$ & $D_{\infty}(S)$ categorically)

§2. Categories' q -deformation.

- $Q = A_n$ quiver (ADE, acyclic, QP from surfaces & general)

$$D_{\infty}(Q) := D^b(kQ) \quad (\cong \text{TFnk}(\mathcal{S}) \text{ in HKK})$$

$$kD_{\infty}(Q) \cong \mathbb{Z}^n, \text{ where } n = \#Q_0.$$

$\mathbb{Z} \oplus \mathbb{Z} X$ grading

Construction (IQ, after Ginzburg & Keller)

$$\bar{Q} = \text{double of } Q = (\underbrace{Q_0, Q_1, Q_i^*, Q_o^*}_{\sim})$$

\mathbb{Z}^2 -grading on \bar{Q} : $(0,0) \quad (2,-1) \quad (1,-1)$

diff. d with $\deg = (1,0)$

$$e_i^* : G_1 \xrightarrow{a} G_2$$

Consider dga $P_x Q := (k\bar{Q}, d)$

$$d \sum_{e \in Q_0} e^* = \sum_{a \in Q_1} [a, a^*]$$

if $\exists w, \deg = (3, -1)$

$D_x(Q) := D_{fd}(L_x Q)$ which is Calabi-Yau- X $\boxed{[X]}$
 for $X = (0, 1)$ -grading shift (Adams grading)

Key observation:

$$KD_x(Q) = R^{\oplus n}$$

$$R = \mathbb{Z}[\varrho^{\pm 1}]$$

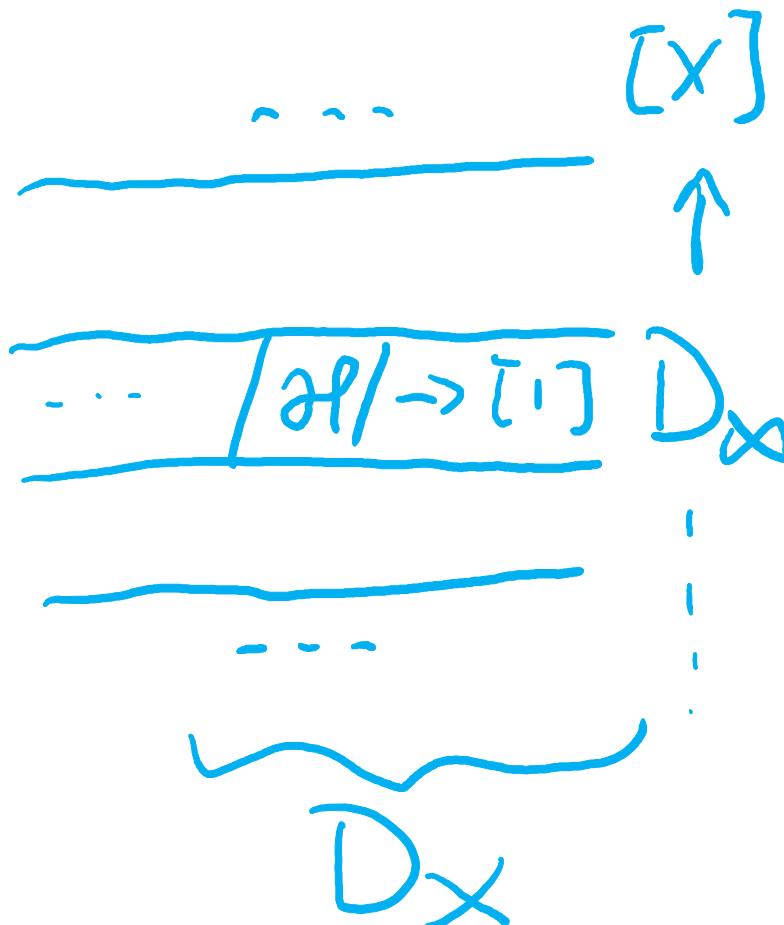
Lem \exists f.f. embedding
 $D_\infty(Q) \hookrightarrow D_x(Q)$ as on X -baric heart.

Recall that:

$$\text{mod } kQ =: \mathcal{H}_Q \hookrightarrow D_\infty(Q)$$

$$\& \quad \langle \mathcal{H}_Q[\mathbb{Z}] \rangle = D_\infty(Q)$$

Analogously: $\langle D_\infty(Q)[\mathbb{Z}X] \rangle = D_x(Q)$



§3. Cluster-X categories

Classically (BMRRRT, Keller)

$$C_2(Q) := D_{\infty}(Q) / \tau_0[-1]$$

Amiot, Keller, Guo:

$$\Gamma_X Q \rightarrow \Gamma_3 Q$$

$$(0,1) = (3,0)$$



$$[X] = [3]$$

$$0 \rightarrow \text{per } \Gamma_3 Q \rightarrow D_{\text{fd}} \Gamma_3 Q \rightarrow C_2(Q) \rightarrow 0$$

IQ:

$$0 \rightarrow \text{per } \Gamma_X Q \rightarrow D_X(Q) \rightarrow C_X(Q) \rightarrow 0$$

$\uparrow \text{}/[x-3] \quad \uparrow \text{}/[x-3] \quad \uparrow \text{}/[x-3] = " \tau_0[-1]"$

$$\text{D}_{\infty}(X)$$

(this picture in full generality is in working progress)

§4. Stability condition's ϱ -deformation

Def.: \mathcal{D} is a triangulated cat.

A stzb. cond. $\sigma = (Z, P)$ consists of

a central charge $Z: K\mathcal{D} \rightarrow \mathbb{C}$.

in slicing $P = \{P(\phi) \mid \phi \in R\}$

= a R -collection of t -structures.

& they are compatible.

Thm (B): Stab \mathcal{D} forms a \mathbb{C} -mfld of $\dim = n$

for $n = \text{rank } K\mathcal{D} < \infty$

- \exists 2 natural actions:

$$s \in \mathbb{C} \setminus \text{Stab } D / \text{Aut}_{\mathfrak{D}} \Phi$$

\Downarrow

$$\sigma = (z, p)$$

$s \cdot (z, p)$	$\Phi(z, p)$
$(e^{-i\pi s} \cdot z, P_{\text{Re}(s)})$	$(z \circ \Phi^{-1}, \Phi \circ p)$

- \exists local coordinate:

$$Z: \boxed{\text{Stab } D \rightarrow \text{Hom}_k(kD, \mathbb{C})}$$

\Downarrow

$$\sigma = (z, p) \mapsto Z$$

(+ tech. condition)

IQ's g -deformation

$$q = e^{i\pi s}$$

$D = D_\infty \rightsquigarrow D_X$ with $X \in \text{Aut } S$.

$KD \cong \mathbb{Z}^n$ $KD \cong R^{\oplus n}$, $R = \mathbb{Z}[\rho^{\pm 1}]$.

Def. A g -stab. cond. is a pair (σ, s) s.t.

0°) σ is a Bridgeland stab. cond., $s \in \mathbb{C}$

1°) $s \cdot \sigma = X(\sigma)$

\Leftrightarrow 1. 1° $z \in \text{Hom}_R(KD_X, \mathbb{C}_s)$

1. 2° $P(\phi)[x] = P(\phi + 2\pi i s)$

$\mathbb{C}_s = \mathbb{C}$ is
a R -mod

$$q \cdot z = e^{i\pi s} \cdot z$$

Thm (rQ) $\mathbb{Q} \text{Stab}_s D_x$ is a \mathbb{C} -mfld with $\dim = n$.

Specialization to $s=N \in \mathbb{Z}_{\geq 2}$

$$D_N := D_x / [x-N] \quad \text{CY-N}$$

e.g. Puiseux
case

Thm

$$\mathbb{Q} \text{Stab}_N D_x \hookrightarrow \text{Stab } D_N$$

open + closed.

§5. Quadratic differentials' g -deformation

S : a Riemann surface

A meromorphic quad. diff is a section of $\omega_s^{\otimes 2}$

Locally $\phi(z) = g(z) dz^{\otimes 2}$



a foliation = a direction in $T_x S$ for $\forall x \in S$

with singularities = critical points

$$= \text{Zero}(\phi) \cup \text{Pole}(\phi).$$

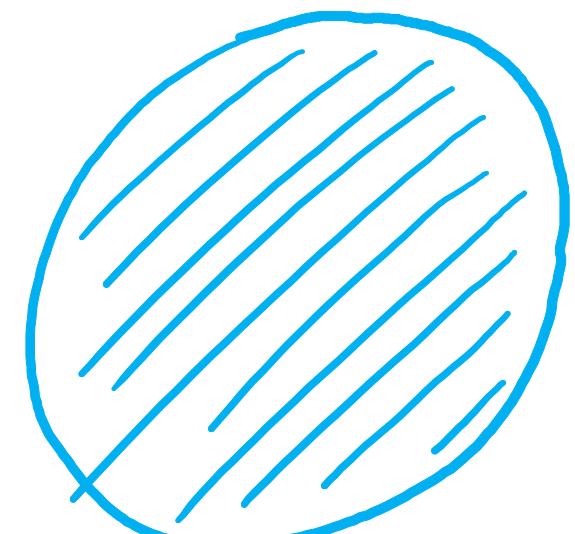
* For CY-3 (BS case)

• simple zeros $z' dz^{\otimes 2}$

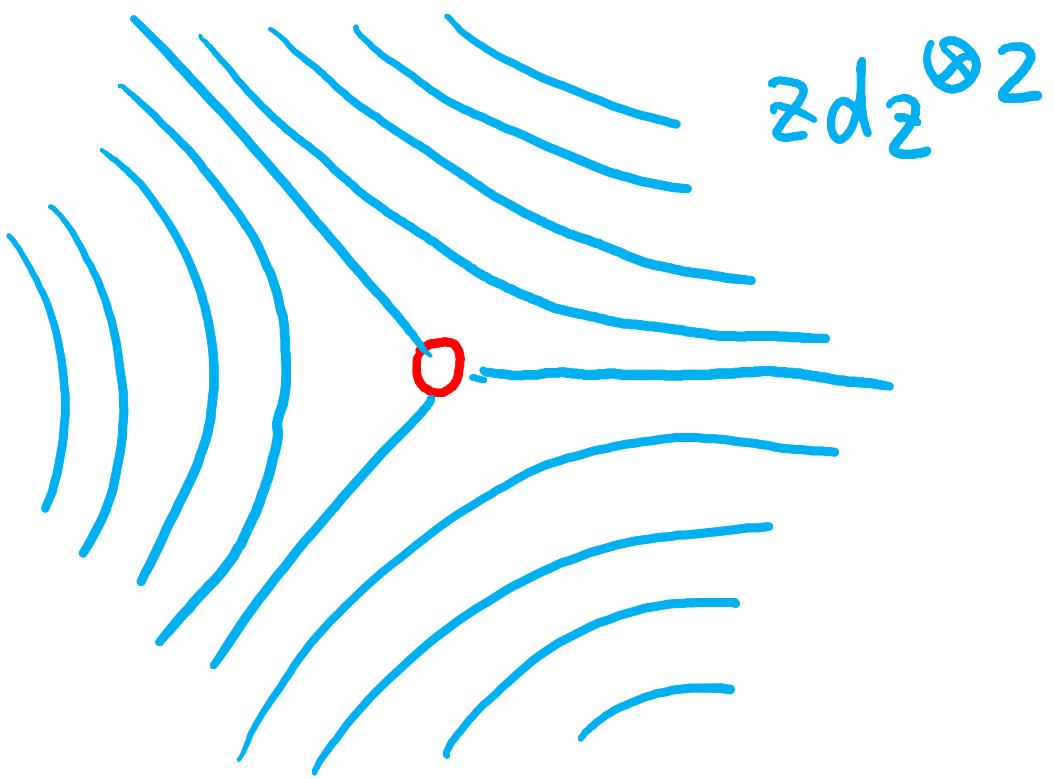
• (higher order) poles

$$z^{-k} dz^{\otimes 2}$$

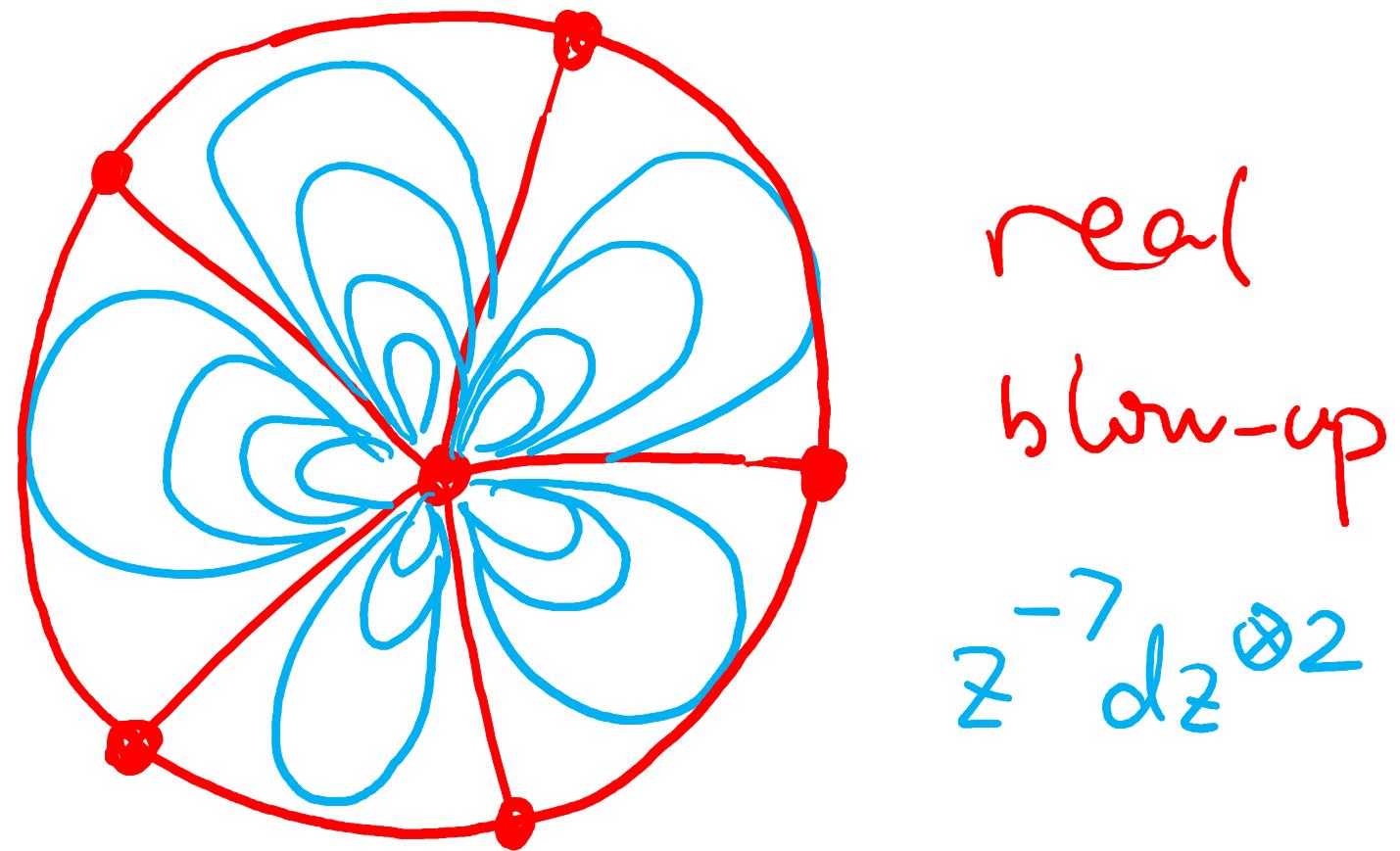
near
smooth point



$\text{Im } z$
"const.
in \mathbb{C} .



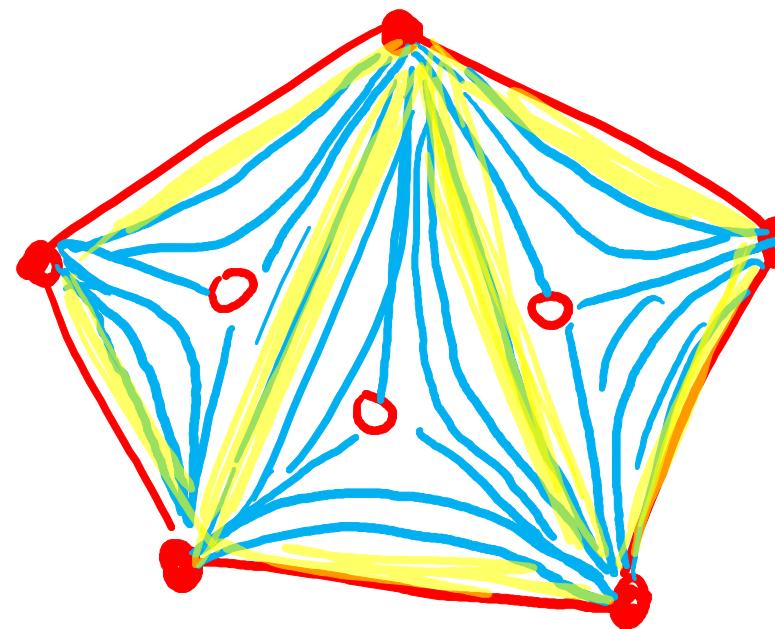
$z \in \text{Zero}$



$p \in \text{Pole} \rightsquigarrow \partial_p$

e.g. On \mathbb{CP}^1 , ϕ of singularities type $(1, 1, 1, -)$

→ after
blow-up



Key:

$$z dz^{\otimes 2}$$

$$z^{-k} dz^{\otimes 2}$$

$$FQuad_3(S) \cong \text{Stab}^{\circ} D_3(S)$$

* For $C^*\alpha$ (HKK case)

Key

$$e^z dz^{\otimes 2}$$

$$e^{z^{-k}} \cdot z^{-l} dz^{\otimes 2}$$

BS.

Ques:
what's
the
relation?

$$FQuad_{\infty}(S) \cong \text{Stab}^{\circ} D_{\infty}(S) \quad \text{HKK}$$

e.g. $D_3(S) = D_3(\mathbb{Q})$, $D_{\infty}(S) = D_{\infty}(\emptyset)$

Construction of the categories:

$$D_3(S) = D_{\text{fd}}(\mathcal{P}(Q_T, W_T))$$

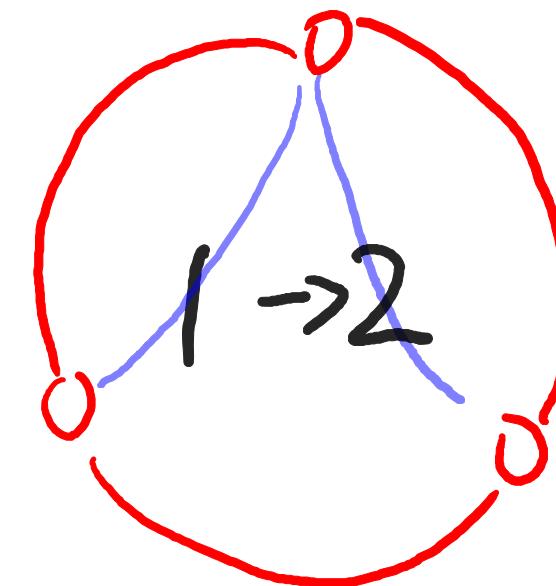
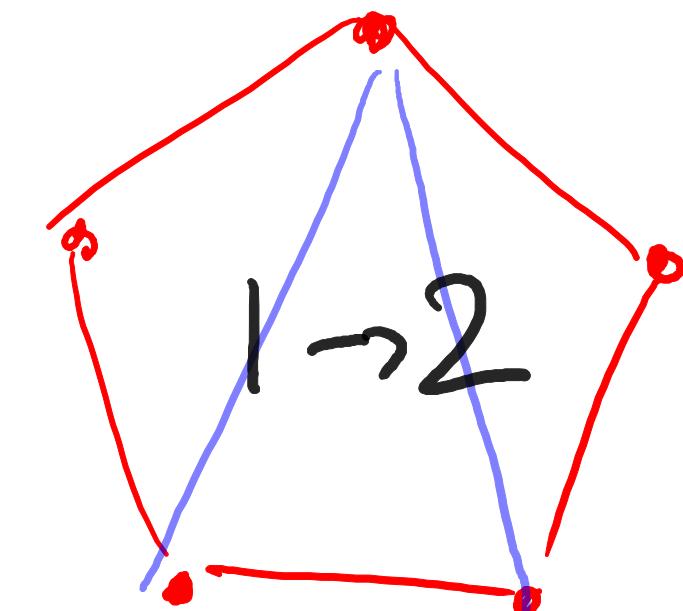
for T : a triangulation

(Q, W) : the ass. OP.

$$D_\infty(S) := \text{TFuk}(S)$$

obj = arcs

morphism = intersections



Answer: ϱ -deformation.

Categorically $D_\infty \rightsquigarrow D_X \longrightarrow D_3$

CY-X
Completion
 $\cancel{[X-3]}$

$$e^z dz^{\otimes 2} \rightsquigarrow \boxed{z^{s-2} dz^{\otimes 2}} \rightsquigarrow s=3$$

$$e^{z^k} \cdot z^{-l} dz^{\otimes 2} \rightsquigarrow \boxed{z^{-k(s-2)-l} dz^{\otimes 2}} \rightsquigarrow s=3$$

$s \in \mathbb{C}$ in ϱ -stab. cond.

Rem: $l = 2$ - winding number (k =order)

Thm (10)

$$Q\text{Stab}_S^0 D_X(S) \cong Q\text{Quads}_S(S)$$

$$S=3$$

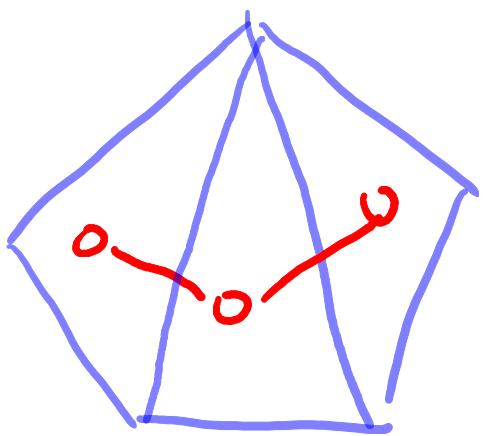
BS result
↓

& here $s \in \mathbb{C}$ with $\operatorname{Re}(s) \gg 1$.

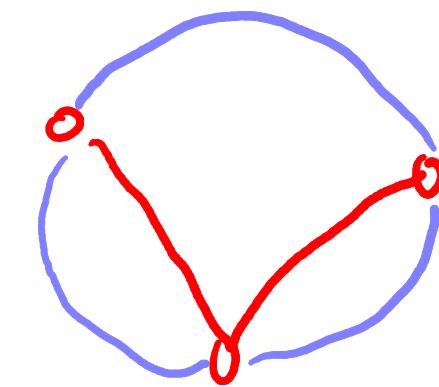
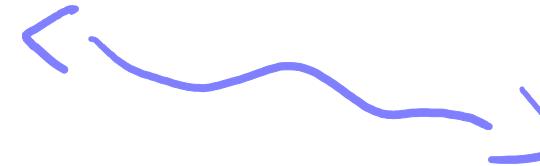
Thank you

Ideas

1° Decorate the marked surface in BS/FST setting



BS
FST



H/KK

dual triangulation
 $\approx \{\text{saddle trajectories}\} = \text{Core}$

2° Construct $Q\text{Stabs } D_x$ from $\text{Stab } D_\infty$

On Grothendieck group: $KD_x \cong KD_\infty \otimes R$

Given $\alpha_\infty = (z_\infty, P_\infty) \in \text{Stab } D_\infty$

$$\begin{array}{ccc} z_\infty: kD_\infty \rightarrow \mathbb{C} & & P(\phi) \\ \downarrow \otimes 1 \quad \downarrow \otimes R \quad \downarrow \otimes R & & \parallel \\ z_{\infty \otimes 1}: kD_x \rightarrow \mathbb{C}[\ell^{\pm 1}] & & \langle P_\infty(\phi - kR e(s)) [kX] \rangle \\ \downarrow \mathcal{Z} & \downarrow q \mapsto e^{is} & \\ \mathbb{C} & & \end{array}$$

So we obtain $(\mathcal{Z}, P) =: \sigma$

Thm (IQ) If $\boxed{\operatorname{Re}(s) \geq \operatorname{gldim} D_\infty + 1}$, then $\sigma \in Q\text{Stab}_s D_x$.

$$\operatorname{gldim} P_\infty := \sup \left\{ \phi_2 - \phi_1 \mid \operatorname{Hom}_{D_\infty}(P(\phi_1), P(\phi_2)) \neq 0 \right\}.$$

