$$\mathrm{Stab}^{\circ}\,\mathcal{D}_{fd}(\mathbb{S}_\Delta) \quad \& \quad \pi_1 = 0$$

$$\cong\; T$$

$$\mathrm{FQuad}(\mathbb{S}_\Delta)$$

Cluster Exchange Groupoids & Framed Quadratic Differentials

with Alastair King

§ Motivation:

$$\mathrm{DFuk}(\Sigma_{\mathbf{S}}) \cong D^b(\mathrm{Coh}\,\Sigma^\vee)$$

H.M.S.

$$\frac{\mathrm{Stab}^\circ D_3(\mathbf{S})}{\mathrm{Aut}\, D_3(\mathbf{S})} \cong \mathrm{Quad}_3(\mathbf{S}) \quad \text{[Bridgeland–Smith]}$$

Aim: Stab° is the universal cover of Quad.

Key ideas:

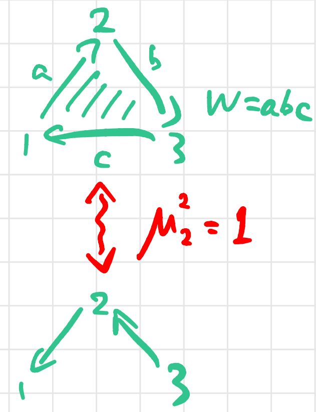
1° Topological model for $D_3(\mathbf{S})$

2° Cluster theory (exchange graphs of triangulations)

§ Cluster theory.

- (Q,W): a quiver with potential (QP)
- Mutation of QP (Fomin-Zelevinsky, DWZ, ...)
- $\boxed{\mathcal{C}(\boxed{\mp}) := \text{cluster category}}$

  of mutation equiv. class $\boxed{\mp}$ of (Q,W).



Lem: is 2-Calabi-Yau.

$\Gamma(Q,W)$: Ginzburg dga

$\left.\begin{array}{l}\mathrm{per}\\ D_{fd}\end{array}\right\}$ derived cat.

e.g. $\mathcal{C}(Q) := D^b(kQ) / \tau^{-1}\circ[1]$

$\mathcal{C}(Q,W) := \mathrm{per}\,\Gamma(Q,W) / D_{fd}(\Gamma(Q,W))$

Def A **cluster** $\mathscr{C}$ in $\mathscr{C}(\Pi)$ is a max. collection of indecomposable objects with Ext^1-vanishing property. $Q_0 = \{1, \cdots, n\}$

N.B. $|\mathscr{C}| = n$ e.g. $\mathscr{C} = \{M_1, \cdots, M_n\}$.

$\exists$ associated quiver $Q_{\mathscr{C}}$ = Gabriel quiver of $\mathrm{End}(\oplus M_i)$

loop free & 2-cycle free

Def. **Mutation of $\mathscr{C}$ at M_i** (or $i \in Q_{\mathscr{C}}$): (is an involution)

$$(\mathscr{C}, Q_{\mathscr{C}}) \overset{\mu_i}{\longleftrightarrow} (\mathscr{C}', Q_{\mathscr{C}'})$$

$Q_{\mathscr{C}'} = \mu_i(Q_{\mathscr{C}})$

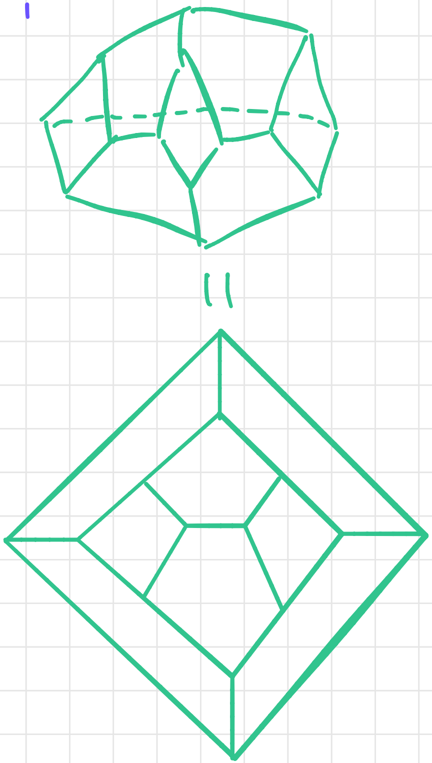
$\mathscr{C}' = \mathscr{C} \setminus \{M_i\} \cup \{M_i'\}$

Def. **CEG** $= \begin{cases} \text{vertices} = \text{clusters} \\ \text{edges} = \text{mutations} \end{cases}$ unoriented cluster exchange graph.

e.g. $\mathcal{C}(A_2)$:

AR-quiver:

$$\begin{tikzcd} & M_2 \arrow[dr] & & M_4 \arrow[dr] & & M_1 \\ M_1 \arrow[ur] & & M_3 \arrow[ur] & & M_5 \arrow[ur] & \end{tikzcd}$$

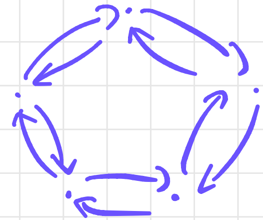


$CEG(A_2)$:

$$\begin{tikzcd} & & \{M_1, M_5\} \arrow[dll, dash] \arrow[drr, dash] & & \\ \{M_1, M_2\} \arrow[dr, dash] & & & & \{M_4, M_5\} \arrow[dl, dash] \\ & \{M_2, M_3\} \arrow[rr, dash] & & \{M_3, M_4\} & \end{tikzcd}$$

e.g. $CEG(A_n)$ = Associahedron of A_n

• $\mathrm{CEG}(\square)$: Oriented version of $\mathrm{CEG}(\square)$

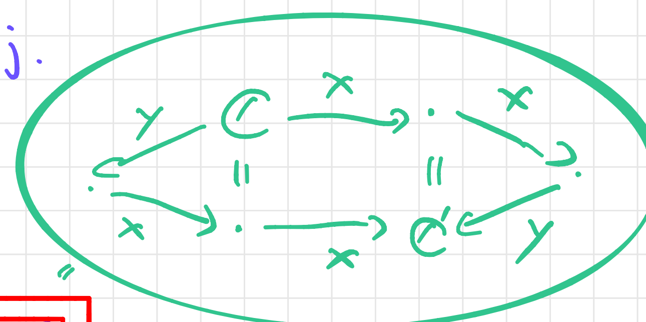
$\mathrm{CEG}(A_2)$ = 

$\mathbb{C} \overset{\mu_j}{\text{———}} \mathbb{C}' \rightsquigarrow \mathbb{C} \underset{y}{\overset{x}{\rightleftarrows}} \mathbb{C}'$ (labelled j)

• $\exists$ local twists (e.g. $t_j = xy$ at $\mathbb{C}$, $t_j' = yx$ at $\mathbb{C}'$)

Def. Let $i, j \in Q_{\mathbb{C}}$ s.t. $\nexists$ arrows $i \not\to j$.

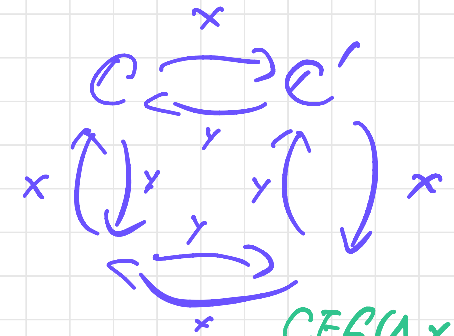
(*) $\cdot \underset{y}{\overset{x}{\rightleftarrows}}_{i} \mathbb{C} \underset{y}{\overset{x}{\rightleftarrows}}_{j} \mathbb{C}' \underset{y}{\overset{x}{\rightleftarrows}}_{i} \cdot$



① Hexagonal relation: $\boxed{\boxed{x^2 y = y x^2}}$ at $\mathbb{C}$ w.r.t. (i,j)

② Square relation: if $\nexists$ arrow $j \to i$

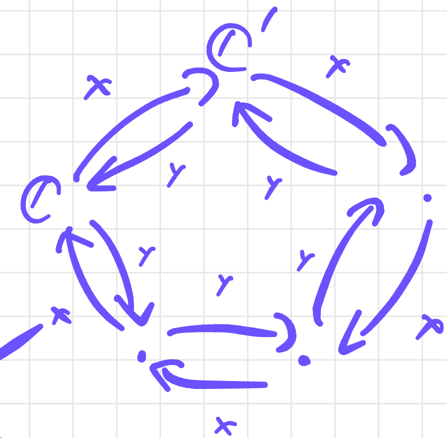
$\boxed{x^2=y^2}$ for (*) is completed as



$CEG(A_1 \times A_1)$

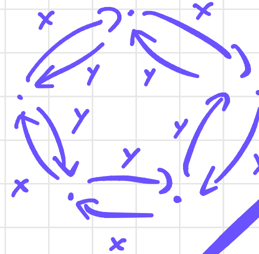
③ Pentagon relation: if $\exists!$ arrow $j \to i$

$\boxed{x^2=y^3}$ for (*) is completed as

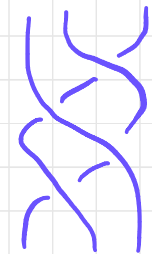


$CEG(A_2)$

<u>Def.</u> $\mathcal{CEG}(\boxplus) = \dfrac{\text{Path Groupoid of } CEG(\boxplus)}{\text{All Hexagon/Square/Pentagon relations}}$

e.g. $\mathcal{CEG}(A_2) :=$ 

$/ x^2 \stackrel{?}{=} y^3$

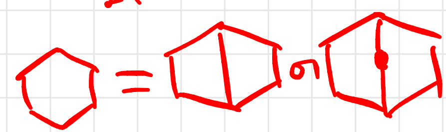


<u>Lem</u>: $\pi_1 \mathcal{CEG}(A_2) = Br(A_2) = Br_3 = \langle a, b \rangle / aba = bab$

<u>Thm</u> [Q-Woolf]. If $\boxminus$ is of Dynkin type, then

$$\pi_1 \mathcal{CEG}(\boxminus) = Br(\boxminus)$$

In this case



<u>Rem</u>: $\pi_1 \underline{CEG}(\boxminus) = \langle \text{squares}, \text{pentagons} \rangle$ [Q]

for $\boxminus$ of Dynkin type

§ Marked Surfaces (Fomin-Shapiro-Thurston)

Def $\mathbb{S}=(S,\mathbb{M})$ consists of data: {marked points}

g = genus, $b = \# \partial S \geq 1$, $\mathbb{M} \subset \partial S$ s.t. $\mathbb{M} \cap \partial_i \neq \emptyset$.

$\partial S = \partial_1 \cup \cdots \cup \partial_b$

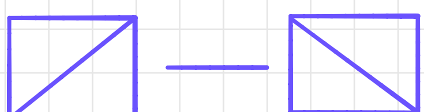
(in general $\exists\, \mathbb{P} \subset S^\circ$)

An open arc γ on S is an isotopy class of curve conn. pts in $\mathbb{M}$.

A triangulation T of S is a max. collection of open arcs with intersection-vanishing.

The flip of T at an arc: $T \rightsquigarrow T' = T \setminus \{\gamma\} \cup \{\gamma'\}$

e.g.



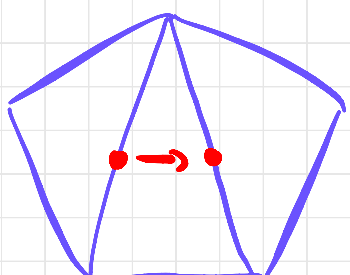
- $\underline{EG}(\mathbb{S}) = \begin{cases} \text{vertices} = \text{triangulations} \\ \text{edges} = \text{flips} \end{cases}$ (exchange graph)

$\underline{\text{Thm}}$ (FST) $\underline{CEG}(\mathbb{S})$ where the class $\mathbb{S}$ is given by:

$\forall$ associated QP (Q_T, W_T) of $\forall$ T of $\mathbb{S}$:

- $(Q_T)_0 = \{\text{open arcs}\}$
- $(Q_T)_1 = \{\text{angles in triangles in } T\}$
- $W_T = \sum$ 3-cycles induced by triangles in T

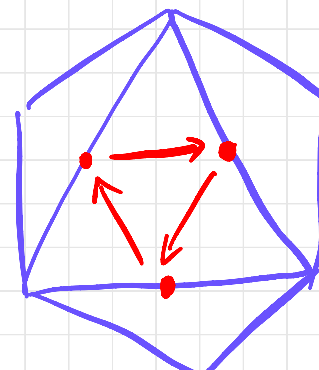
e.g.



$\mathbb{S} = (n+3)$-gon

$\Updownarrow$

type A_n



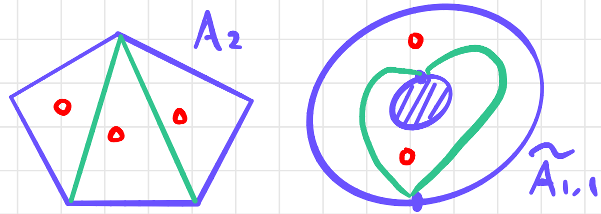
Question: $\pi_1 CEG(S) = \pi_1 EG(S) = ?$

§ Decorated Marked Surfaces (Q)

<u>Def.</u> $S_\Delta = S$ with a set Δ of decorations $\subset S^\circ$

with $|\Delta| = \#$ triangles in $\forall\, T$ of S.

e.g.

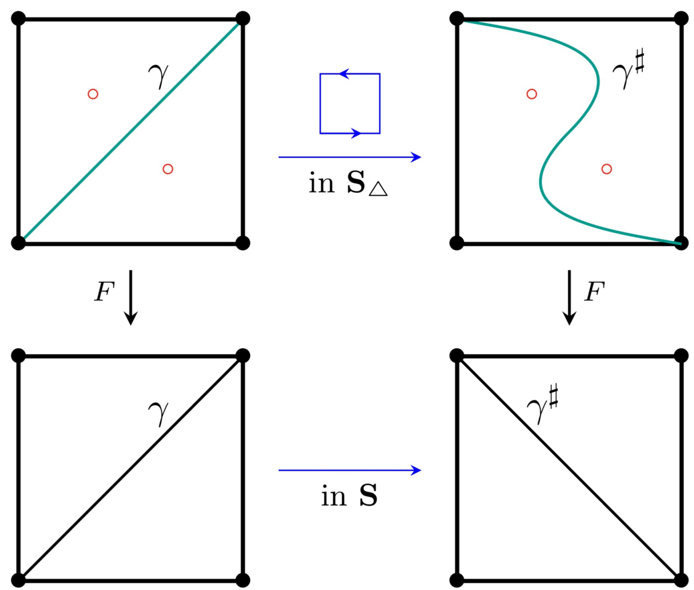


A <u>closed arc</u> η in S_Δ is a curve conn. pts in Δ.

Still have open arcs γ conn. pts in M.

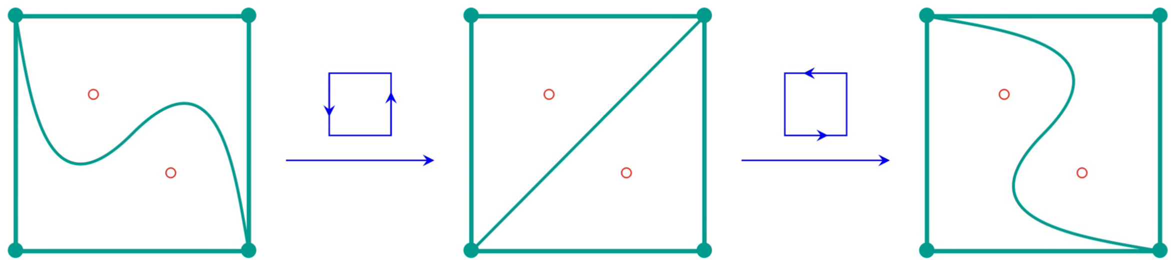
Def. A <u>(decorated) triangulation</u> Π of $S_\circ$ is a max. collection of open arcs that divides $S_\circ$ into once-decorated triangles.

• New flip:



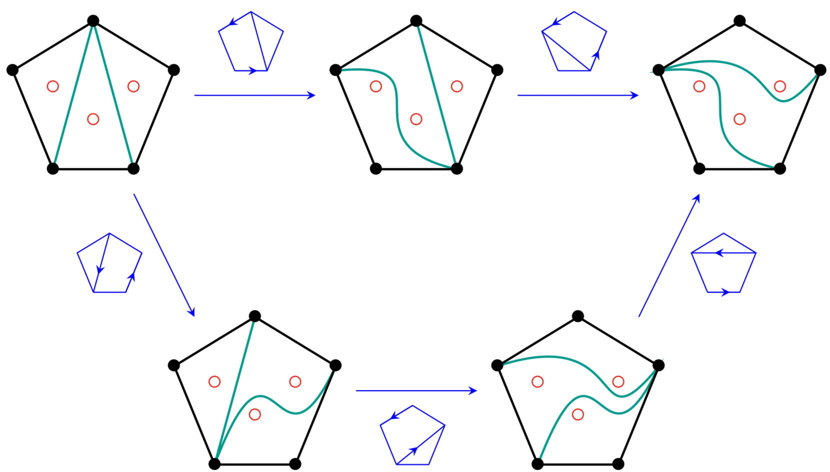
$EG(S_\Delta)$ = exchange graph of tri. of S_Δ

e.g.



braid twist

Some pieces



oriented Pentagon relation

Remark $EG(\mathbb{S}_\Delta)$ is usually not connected.

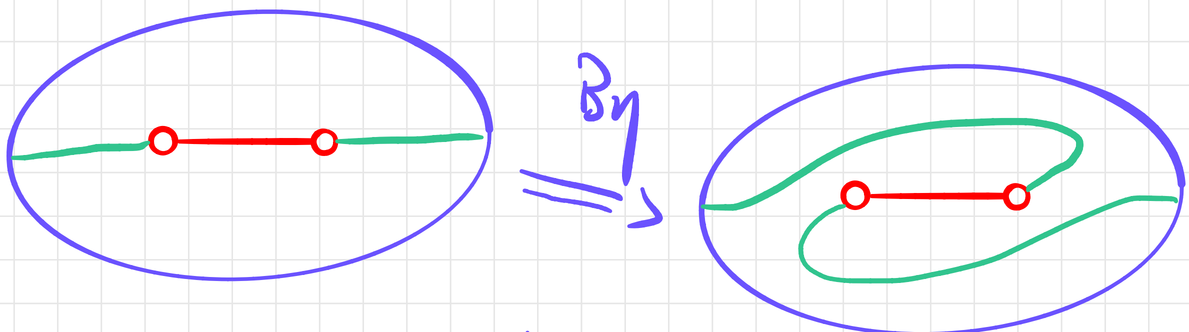
$CA(\mathbb{S}_\Delta) = \{$simple closed arcs with different endpoints$\}$

$\cup\!\!\!|$

$\forall \eta \underset{}{\overset{\text{induces}}{\rightsquigarrow}}$ braid twist $B_\eta \in MCG(\mathbb{S}_\Delta)$

group

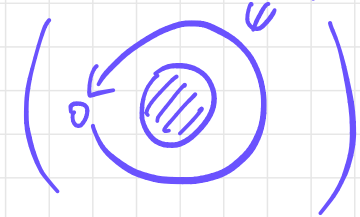
mapping class group of $\mathbb{S}_\Delta$ fixing Δ & M setwise.



$$\boxed{BT(\mathbb{S}_\Delta)} = \langle B_\eta \mid \eta \in CA(\mathbb{S}_\Delta) \rangle \subset MCG(\mathbb{S}_\Delta)$$

Def $SBr(\mathbb{S}_\Delta) = \ker\big(MCG(\mathbb{S}_\Delta) \to MCG(\mathbb{S}_\triangle)\big)$ surface braid group

$$\begin{gathered} SBr(\mathbb{S}_\Delta) \\ \cup \\ BT(\mathbb{S}_\Delta) \end{gathered}$$



<u>Thm</u> (King-Q): $EG(\mathbb{S}_\Delta)$ consists of $SBr(\mathbb{S}_\Delta)/BT(\mathbb{S}_\Delta)$ many connected components, each of which is isomorphic s.t.

$$EG^{\pi}(\mathbb{S}_\Delta)/BT(\mathbb{S}_\Delta) = EG(\mathbb{S})$$

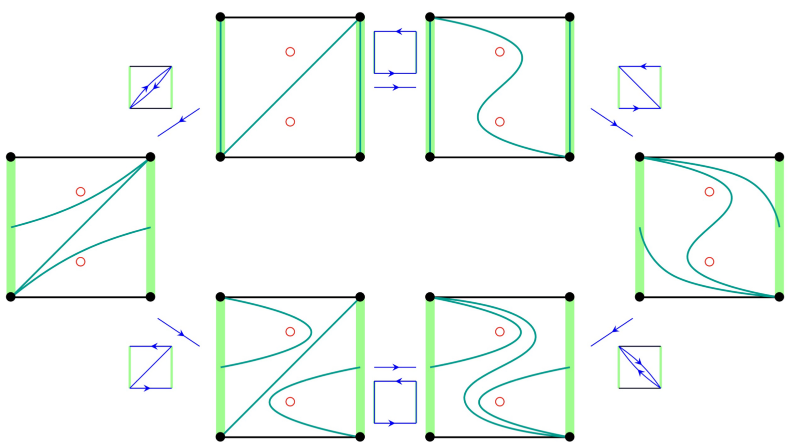
e.g. $BT(\mathbb{S}_\Delta) = SBr(\mathbb{S}_\Delta) = Br_{n+1}$ for $\mathbb{S} = (n+3)$-gon.

Thm (King–Q) upgrade to groupoid version: $\mathcal{CEG}(\mathbb{S})$

$\mathcal{EG}^{\mathrm{T}}(\mathbb{S}_\Delta)$ is the unicover of $\mathcal{EG}(\mathbb{S})$

with covering group $BT(\mathbb{S}_\Delta)$

($\exists$ ①, ②, ③ relations)



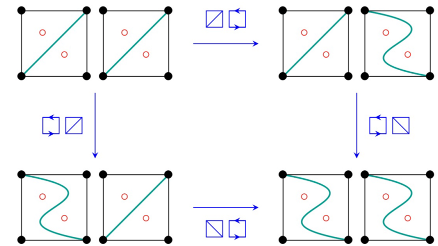


FIGURE 3.9. The square relation for $\mathcal{EG}(\mathbf{S}_\triangle)$

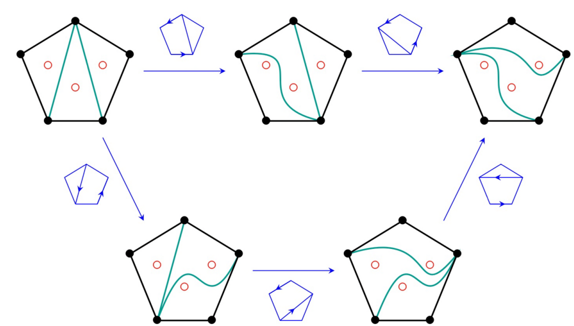


FIGURE 3.10. The pentagon relation for $\mathcal{EG}(\mathbf{S}_\triangle)$

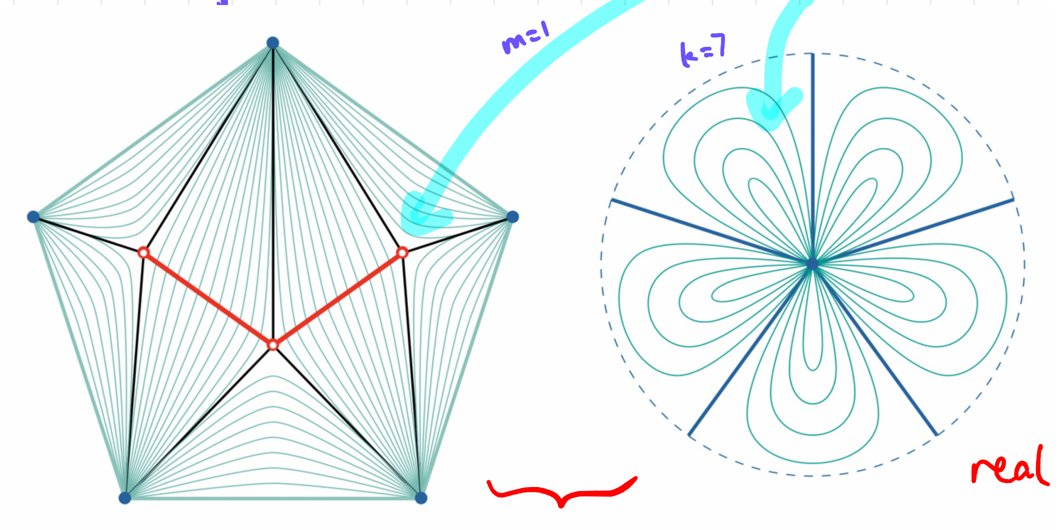
§ Framed Quadratic Differentials

<u>Recall</u> quad. diff $\phi(z) = g(z)\,dz^{\otimes 2}$

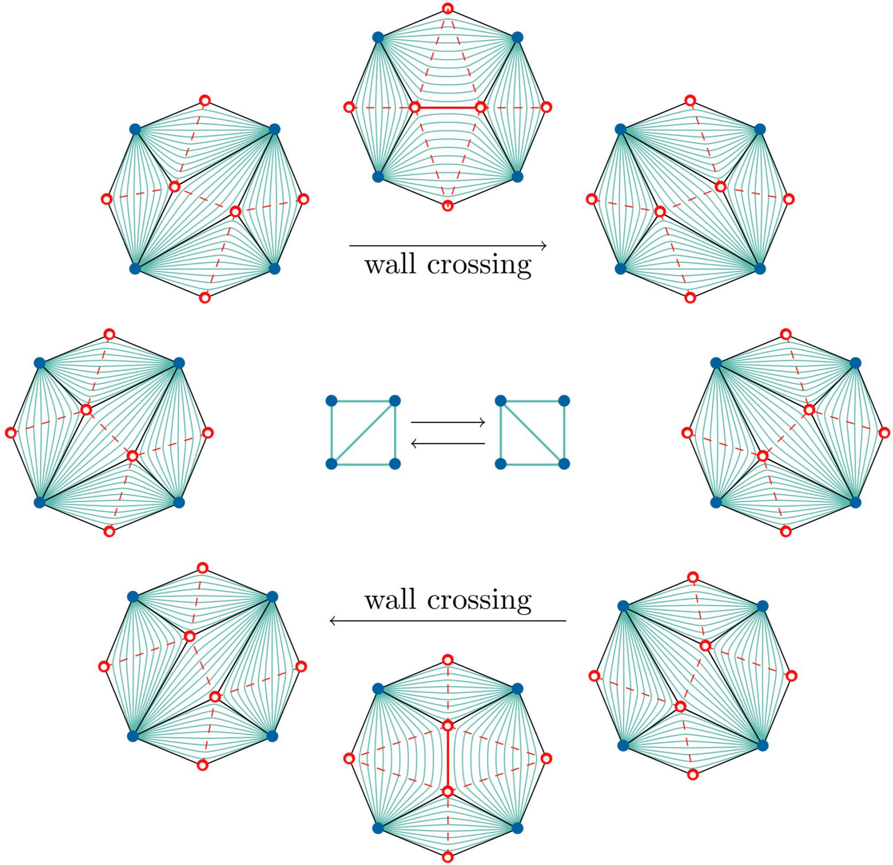
$\hookrightarrow \exists$ zero $\quad g(z) = z^{m} \qquad m>0$

$\qquad\quad$ pole $\quad g(z) = z^{-k} \qquad k>0$

& local foliations:



$\mathbb{CP}^1(1,1,1,-7)$ $\qquad \underbrace{\qquad}_{\mathbb{CP}^1}$ $\qquad$ real blow-up $\quad p \rightsquigarrow \partial_p$


wall crossing
wall crossing

$\mathrm{Quad}(\mathbb{S}) = \{(S, \phi, \xi) : \xi : S^{\phi} \to \mathbb{S}\} / \sim$

S^{ϕ} : real blow-up of S w.r.t. ϕ

- ϕ: a GMN diff on S for a Riemann surface S

GMN diff: a mero quad. diff s.t.
$\begin{cases} \text{zeros with order } 1 \\ \text{poles with order} \geq 3 \end{cases}$

$(S, \phi, \xi) \sim (S', \phi', \xi')$
if $\exists f : S \to S'$.
$\phi' = f^*(\phi)$
& $\xi' \circ f \circ \xi^{-1} \in \dfrac{\mathrm{Diff}(\mathbb{S})}{\mathrm{Diff}_0(\mathbb{S})} = \mathrm{MCG}(\mathbb{S})$

$\mathrm{FQuad}(\mathbb{S}) = \cdots\cdots$

$\Rightarrow \mathrm{Quad}(\mathbb{S}) = \mathrm{FQuad}(\mathbb{S}) / \mathrm{MCG}(\mathbb{S})$

Decorated / Framed version :

$$\begin{tikzcd}
 & \mathrm{FQuad}^{\mathbb{T}}(\mathbf{S}_\triangle) \arrow[dl, hook] \arrow[ddr] \arrow[rrr, "{[\text{King-Q}]}"] & & & \mathrm{Stab}^\circ \mathcal{D}_3(\mathbb{S}) \arrow[ddl] \arrow[dddd, bend left] & \\
 & & \mathrm{BT}(\mathbf{S}_\triangle) \arrow[r, "{[\mathrm{Q}]}"] & \mathrm{ST}\,\mathcal{D}_3(\mathbb{S}) & & \mathrm{ST}\,\mathcal{D}_3(\mathbb{S}) \arrow[d, hook] \\
\mathrm{FQuad}(\mathbf{S}_\triangle) \arrow[rr, "\mathrm{SBr}(\mathbf{S}_\triangle)"] \arrow[ddr, "\mathrm{MCG}(\mathbf{S}_\triangle)"'] & & \mathrm{FQuad}(\mathbf{S}) \arrow[r] \arrow[ddl] & \mathrm{Stab}^\circ \mathcal{D}_3(\mathbb{S}) / \mathrm{ST} \arrow[ddr] & & \mathrm{Aut}^\circ \mathcal{D}_3(\mathbb{S}) \arrow[d, two heads] \\
 & & \mathrm{MCG}(\mathbf{S}) \arrow[r, "{[\mathrm{BS}]}"'] & \mathrm{Aut}^\circ / \mathrm{ST} & & \mathrm{MCG}(\mathbb{S}) \\
 & \mathrm{Quad}(\mathbf{S}) \arrow[rrr, "{[\mathrm{BS}]}"'] & & & \mathrm{Stab}^\circ \mathcal{D}_3(\mathbb{S}) / \mathrm{Aut}^\circ &
\end{tikzcd}$$

FACT: $\mathcal{CEG}(\mathbf{S})$ & $\mathcal{EG}^{T}(\mathbf{S}_\Delta)$

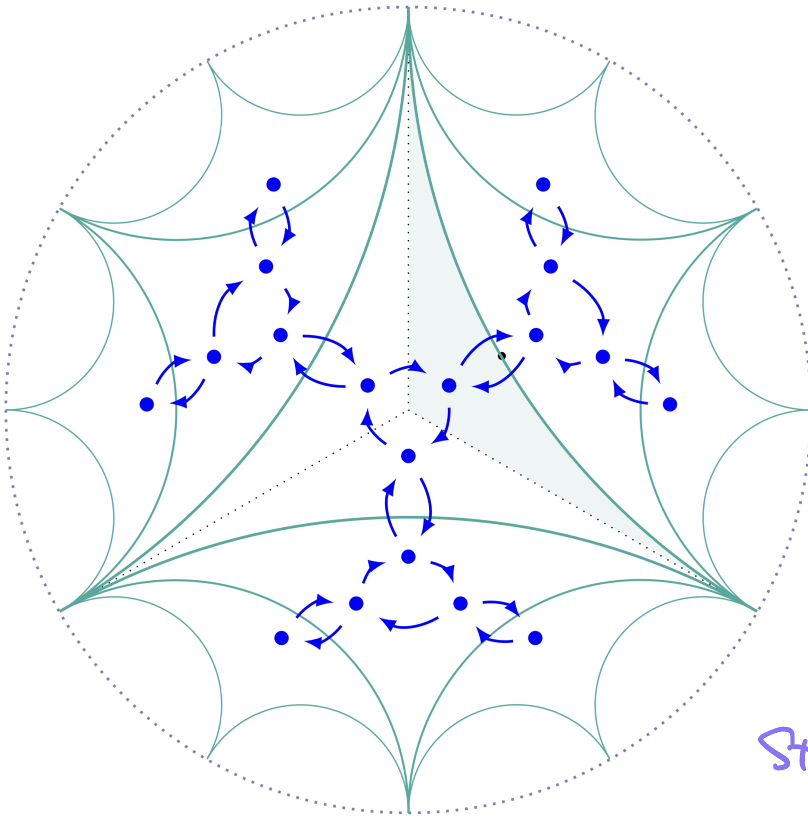
are skeletons of

$\mathrm{Quad}(\mathbf{S})$ & $\mathrm{FQuad}^{T}(\mathbf{S}_\Delta)$

<u>Thm</u> (King–Q) $\mathrm{Stab}^\circ \mathcal{D}_3(\mathbf{S}) \cong \mathrm{FQuad}^{T}(\mathbf{S}_\Delta)$ is simply connected.

<u>Ref</u>. 1805.00030, 1407.0806, 1407.5986, 1111.1010

cf. 1712.09585



Thank you

$$\mathrm{Stab}^{\circ}\mathcal{D}_3(A_2)/\mathbb{C} \cong \overline{\mathbb{CP}^1(1,1,1,-7)}/\mathbb{C}$$