

$\text{Stab}^{\circ} D_{\text{fd}}(S_\Delta)$
||> $\pi_i = 0$

$D_{\text{Quad}}(S_\Delta)$

Cluster Exchange Groupoids & Framed Quadratic Differentials

with Alastair King

§ Motivation:

$$DF_{\text{Fuk}}(\Sigma_S) \cong D^b(\text{Coh } \Sigma^\vee)$$

↑
H.M.S.

$$\frac{\text{Stab}^0 D_3(S)}{\text{Aut } D_3(S)} \cong \text{Quad}_3(S) \quad [\text{Bridge (and -Smith)}]$$

Aim: Stab^0 is the universal cover of Quad.

Key ideas:

1° Topological model for $D_3(S)$

2° Cluster theory (exchange graphs of triangulations)

§ Cluster theory.

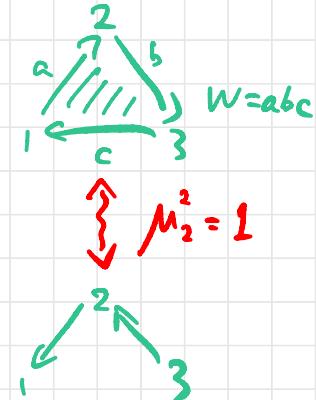
- (Q, W) : a quiver with potential (Q, P)
- Mutation of QP (Fomin-Zelevinsky, DWZ, ...)
- $\boxed{C(\square) := \text{cluster category}}$

of mutation equiv. class \square of (Q, W) .

Lem: is 2-Calabi-Yau.

e.g. $C(Q) := D^b(kQ) / \mathcal{T}^{-1}[1]$

$C(Q, W) := \text{per } I(Q, W) / D_{\text{fd}}(I(Q, W))$



$I(Q, W)$: Ginzburg dga

per } derived cat.
D_{fd}

Def A cluster \mathcal{C} in $\mathcal{C}(\mathbb{D})$ is a max. collection of indecomposable objects with Ext'-vanishing property. $Q_0 = \{1, \dots, n\}$

N.B. $|\mathcal{C}| = n$ e.g. $\mathcal{C} = \{M_1, \dots, M_n\}$.

\exists associated quiver $Q_{\mathcal{C}}$ = Gabriel quiver of $\text{End}(\oplus M_i)$

Def. Mutation of \mathcal{C} : (is an involution)

at M_i ($\forall i \in Q_0$)

\parallel
 $M_i(Q_{\mathcal{C}})$

loop free
&
2-cycle free

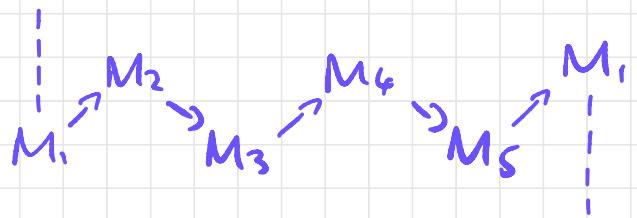
$(\mathcal{C}, Q_{\mathcal{C}}) \xleftrightarrow{M_i} (\mathcal{C}', Q_{\mathcal{C}'})$

\parallel

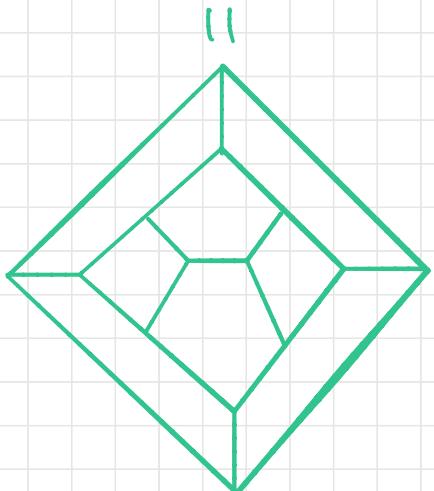
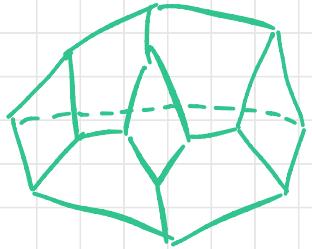
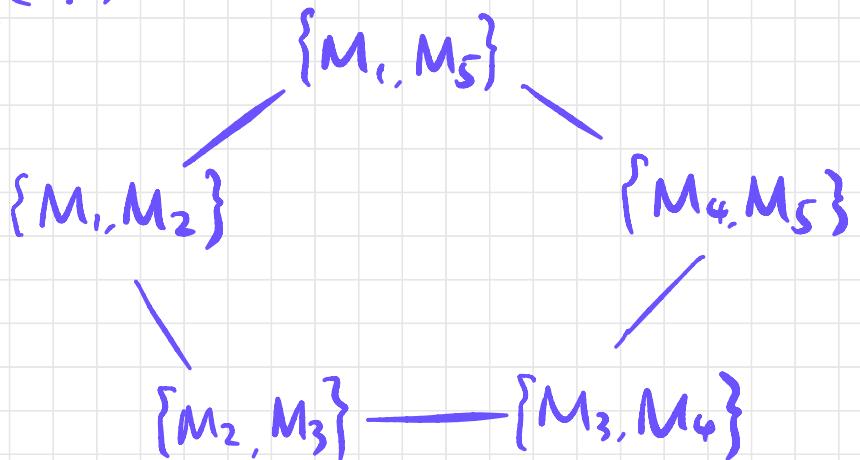
$\mathcal{C} \setminus \{M_i\} \cup \{M'_i\}$

Def. CEG = {vertices = clusters
edges = mutations} unoriented
cluster exchange graph.

e.g. $C(A_2)$:
AR-quiver:

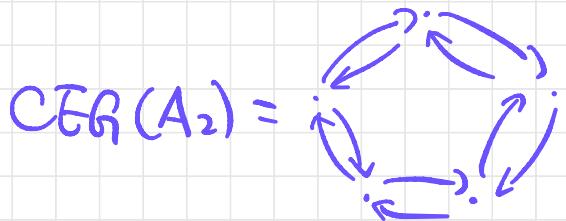


$CEG(A_2)$:



e.g. $CEG(A_n)$ = Associahedron of A_n

• $\text{CEG}(\Pi)$ = Oriented version of $\text{CEG}(\Pi)$

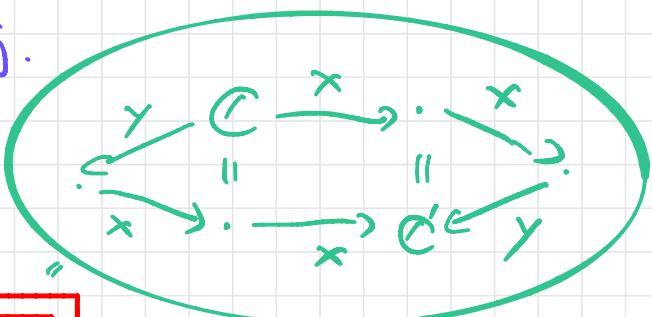


$$C - C' \xrightarrow{\mu_j} C \xleftarrow{j} \overset{x}{\overbrace{C' \xrightarrow{y}}} C'$$

- \exists local twists (e.g. $t_j = xy$ at C , $t'_j = yx$ at C')

Def. Let $i, j \in Q_C$ s.t. $\# \text{arrows } i \rightarrow j$.

$$(*) \quad \begin{array}{c} \text{graph} \\ \text{with nodes } i, j \\ \text{and arrows } i \xrightarrow{x} j, j \xrightarrow{x} i, i \xrightarrow{y} i, j \xrightarrow{y} j \end{array}$$



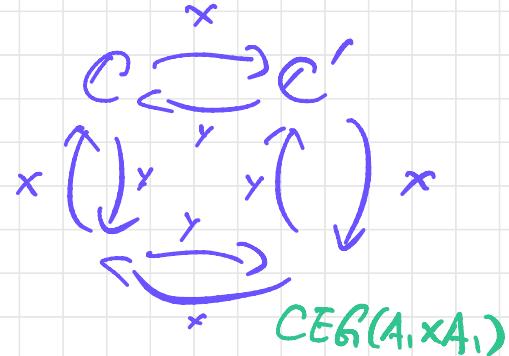
Hexagonal relation:

$$x^2 y = y x^2$$

at C w.r.t. (i, j)

2 Square relation: if \nexists arrow $j \rightarrow i$

$$x^2 = y^2 \quad \text{for } (*) \text{ is completed as}$$



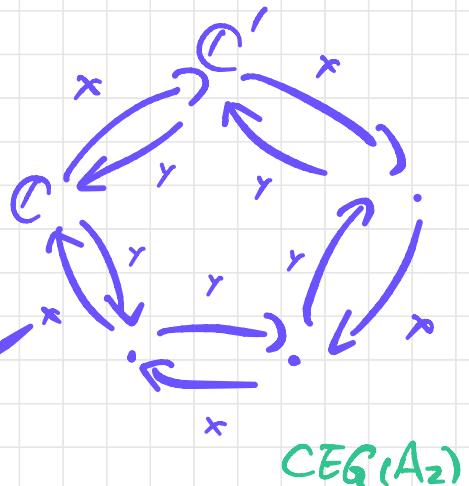
3 Pentagon relation: if $\exists !$ arrow $j \rightarrow i$

$$x^2 = y^3$$

for $(*)$ is completed as

Def. $CEG(\mathbb{D}) = \text{Path Groupoid of } CEG(\mathbb{D})$

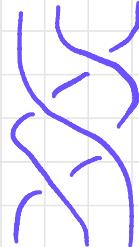
All Hexagon/Square/Pentagon relations



e.g. $\text{CEG}(A_2) :=$

$$x^2 = y^3$$

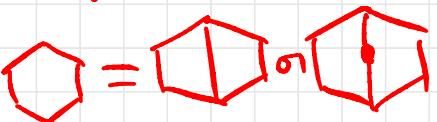
Lem: $\pi_1 \text{CEG}(A_2) = \text{Br}(A_2) = \text{Br}_3 = \langle a, b \rangle / aba = bab$



Thm [Q-Woff]. If \mathbb{D} is of Dynkin type, then

$\pi_1 \text{CEG}(\mathbb{D}) = \text{Br}(\mathbb{D})$

In this case



Rem: $\pi_1 \text{CEG}(\mathbb{D}) = \langle \text{squares, pentagons} \rangle$
for \mathbb{D} of Dynkin type

[Q]

§ Marked Surfaces (Fomin-Sheapiro-Thurston)

Def $S = (S, M)$ consists of data: $\{$ marked points $\}$

$g = \text{genus}, b = \#\partial S > 1, M \subset S$ s.t. $M \cap \partial_i \neq \emptyset$.

$$\partial_1 \cup \dots \cup \partial_b$$

(in general $\exists P \subset S^\circ$)

An open arc γ on S is an isotopy class of curve conn. pts in lM .

A triangulation T of S is a max. collection of open arcs
with intersection-vanishing.

The flip of T at an arc: $T \leadsto T' = T \setminus \{\gamma\} \cup \{\gamma'\}$

e.g.



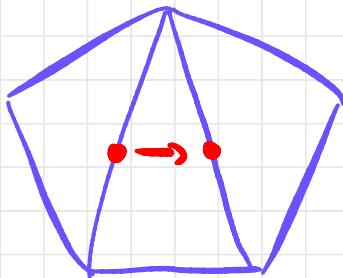
• $\underline{EG}(\mathcal{S}) = \begin{cases} \text{vertices} = \text{triangulations} \\ \text{edges} = \text{flips} \end{cases}$ (exchange graph)

Thm(FST) $\underline{CEG}(\mathcal{S})$ where the class \mathcal{S} is given by:

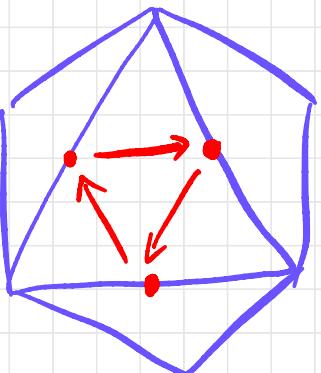
\forall associated QP (Q_T, W_T) of $\forall T$ of \mathcal{S} :

- $(Q_T)_0 = \{\text{open arcs}\}$
- $(Q_T)_1 = \{\text{angles in triangles in } T\}$
- $W_T = \sum \text{3-cycles induced by triangles in } T$

e.g.



$\mathcal{S} = (n+3)\text{-gon}$
 \uparrow
 type A_n



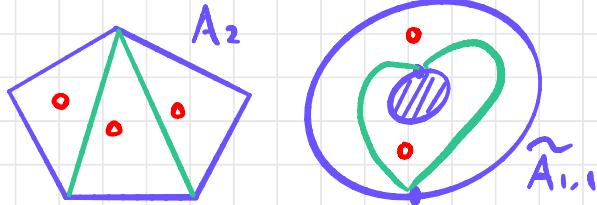
Question: $\pi_1 \text{CEG}(S) = \pi_1 \text{EG}(S) = ?$

§ Decorated Marked Surfaces (\mathbb{Q})

Def. $S_\Delta = S$ with a set Δ of decorations $\subset S^\circ$

with $|\Delta| = \# \text{ triangles in } \mathcal{V} \mathcal{T} \text{ of } S$.

e.g.

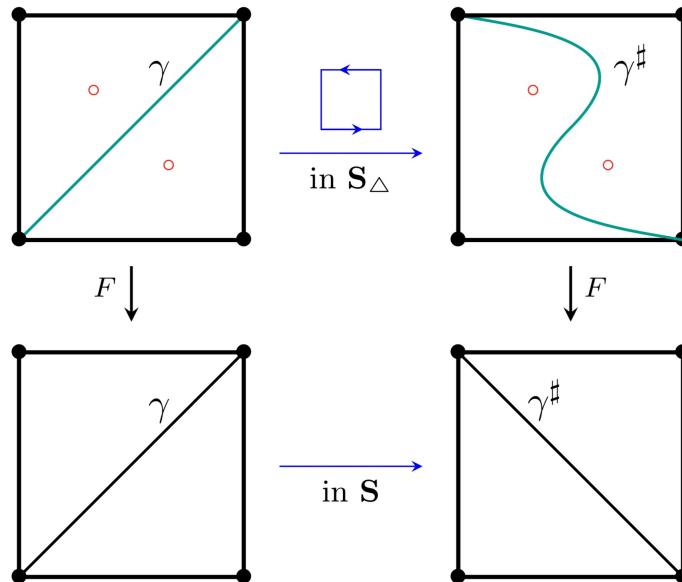


A closed arc η in S_Δ is a curve conn. pts in Δ .

Still have open arcs \vee conn. pts in M .

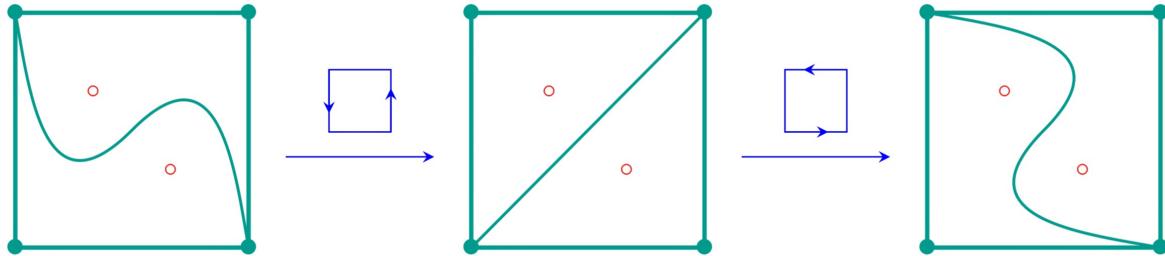
Def. A (decorated) triangulation Π of S_0 is a max.
 collection of open arcs that divides S_0 into once-decorated
 triangles.

- New flip :

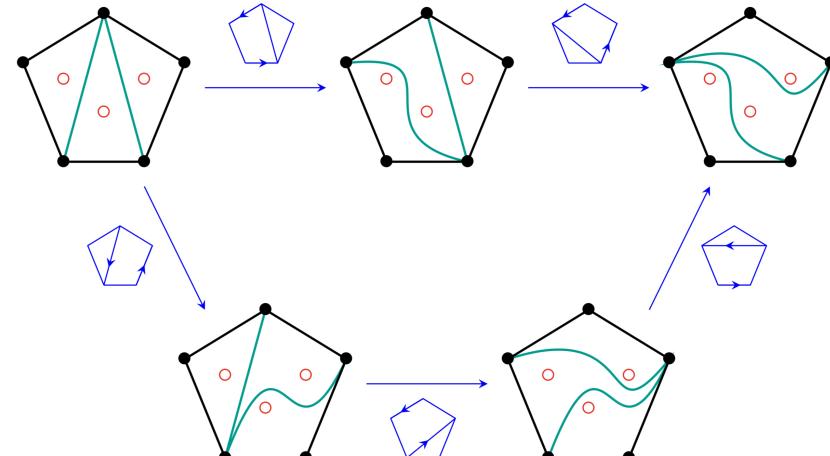


$EG(S_\Delta)$ = exchange graph of tri. of S_Δ

e.g.



Some pieces



braid twist

oriented
← Pentagon relation

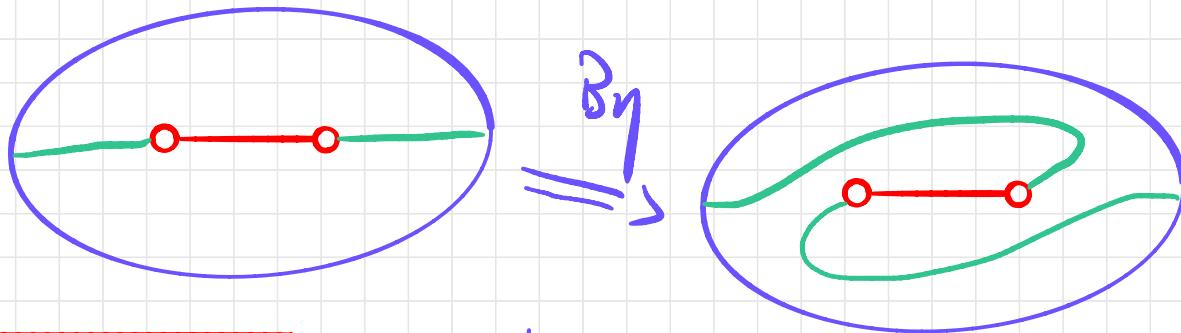
Remark $EG(S_\delta)$ is usually not connected.

$CA(S_\delta) = \{\text{simple closed arcs with different endpoints}\}$

\Downarrow
 $\forall \eta \xrightarrow[\text{group}]{\text{induces}} \text{braid twist}$

$B\eta \in MCG(S_\delta)$

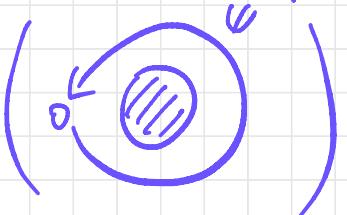
mapping class group of S_δ
fixing Δ & M setwise.



$$BT(S_\delta) = \langle B\eta \mid \eta \in CA(S_\delta) \rangle \subset MCG(S_\delta)$$

Def $SBr(S_\Delta) = \ker(MCQ(S_\Delta) \rightarrow MCG(S_\Delta))$

$$U$$

$$BT(S_\Delta)$$


Thm (King-Q): $EG(S_\Delta)$ consists of

$SBr(S_\Delta)/_{BT(S_\Delta)}$ many connected components. Each of which is isomorphic s.t.

$$EG^\pi(S_\Delta)/_{BT(S_\Delta)} = EG(S)$$

e.g. $BT(S_\Delta) = SBr(S_\Delta) = Br_{n+1}$ for $S = (n+3)$ -gon.

Thm (king-Q) upgrade to groupoid version: $\mathcal{EG}(S)$

$\mathcal{EG}^{\pi}(S_{\Delta})$ is the uni.cover of $\mathcal{EG}(S)$
with covering group $BT(S_{\Delta})$

($\exists \textcircled{1}, \textcircled{2}, \textcircled{3}$ relations)

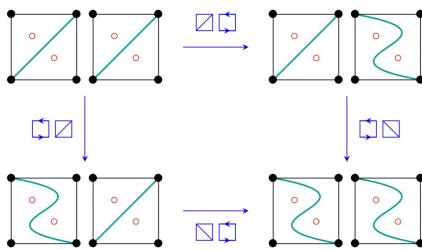
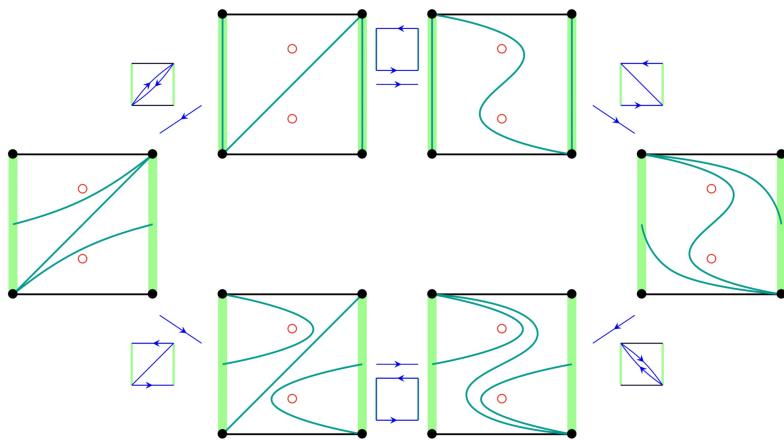


FIGURE 3.9. The square relation for $\mathcal{EG}(S_{\Delta})$

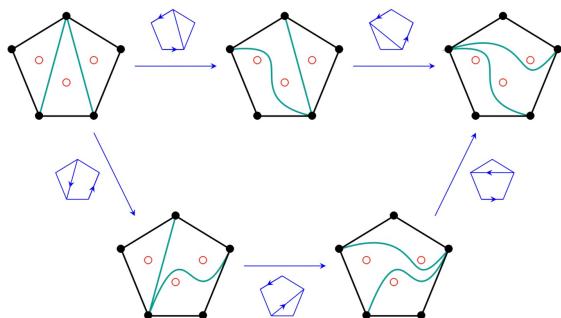
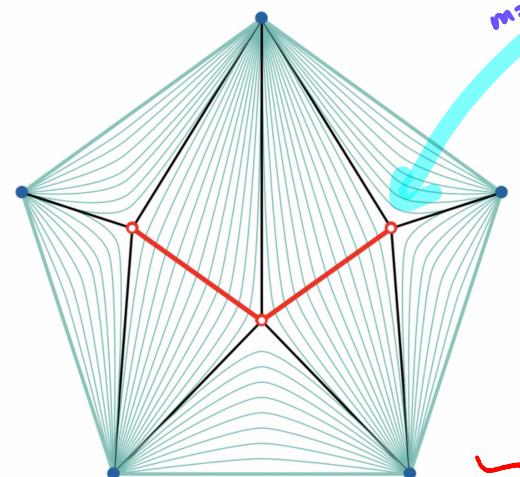


FIGURE 3.10. The pentagon relation for $\mathcal{EG}(S_{\Delta})$

§ Framed Quadratic Differentials

Recall quad. diff $\phi(z) = g(z) dz^{\otimes 2}$

& local foliations:



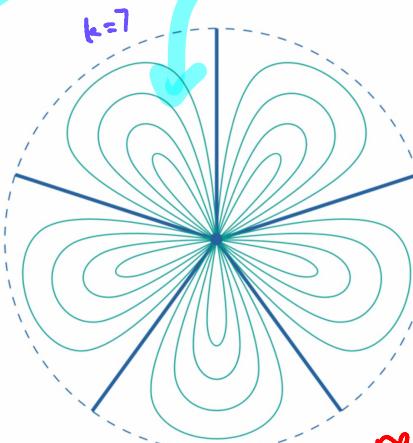
$$\mathbb{CP}^1(1,1,1,-7)$$

\mathbb{CP}^1

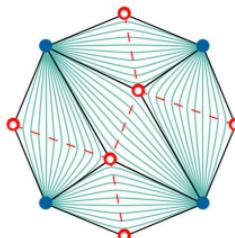
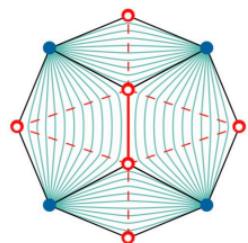
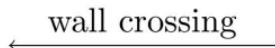
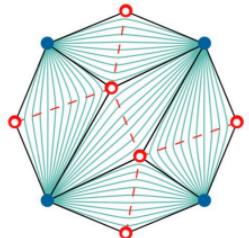
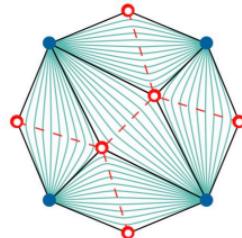
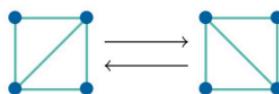
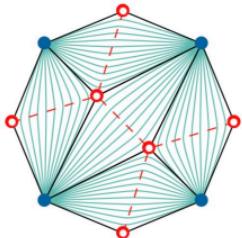
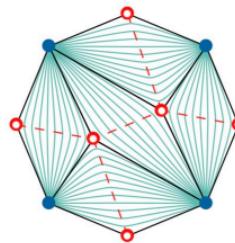
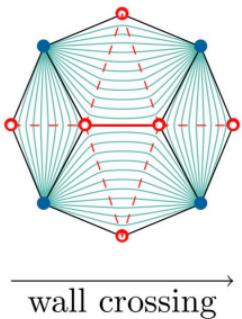
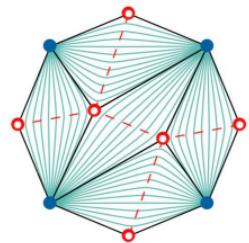
$\hookrightarrow \exists$ zero $g(z) = z^m$
pole $g(z) = z^{-k}$

$$m > 0$$

$$k > 0$$



real blow-up $p \rightsquigarrow \partial p$



$$\text{Quad}(S) = \{(S, \phi, \xi) : \xi : S^\phi \rightarrow S\}$$

- ϕ : a GMN diff on S
for a Riemann surface S

GMN diff: a mero quad. diff s.t.

{ zeros with order 1
poles with order ≥ 3

S^ϕ : real blow-up
of S w.r.t. ϕ

$$(S, \phi, \xi) \sim (S', \phi', \xi')$$

if $\exists f : S \rightarrow S'$,

$$\phi' = f^*(\phi)$$

$$\& \xi' \circ f \circ \xi^{-1} \in \text{Diff}(S)$$

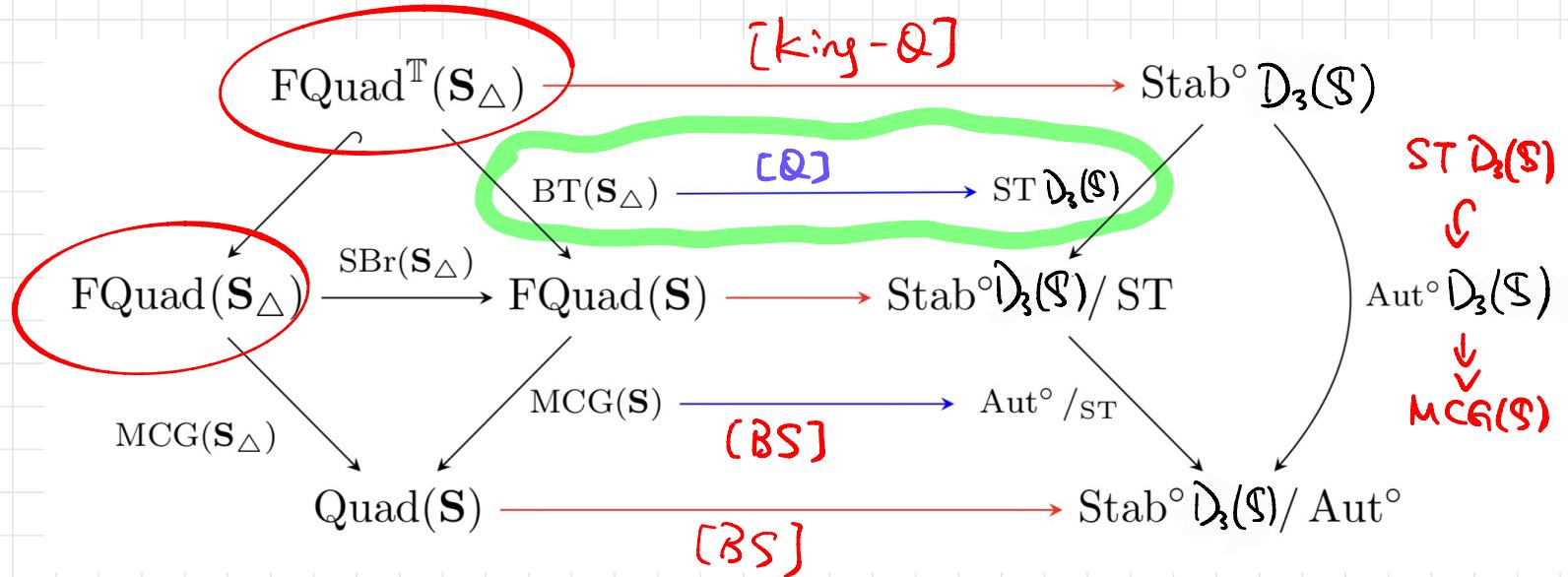
$= \text{MCG}(S)$

$$F\text{Quad}(S) = \dots \dots$$

$$\text{Diff}_0(S)$$

$$\Rightarrow \text{Quad}(S) = F\text{Quad}(S) \cancel{\text{Diff}_0(S)}$$

Decorated / Framed version :



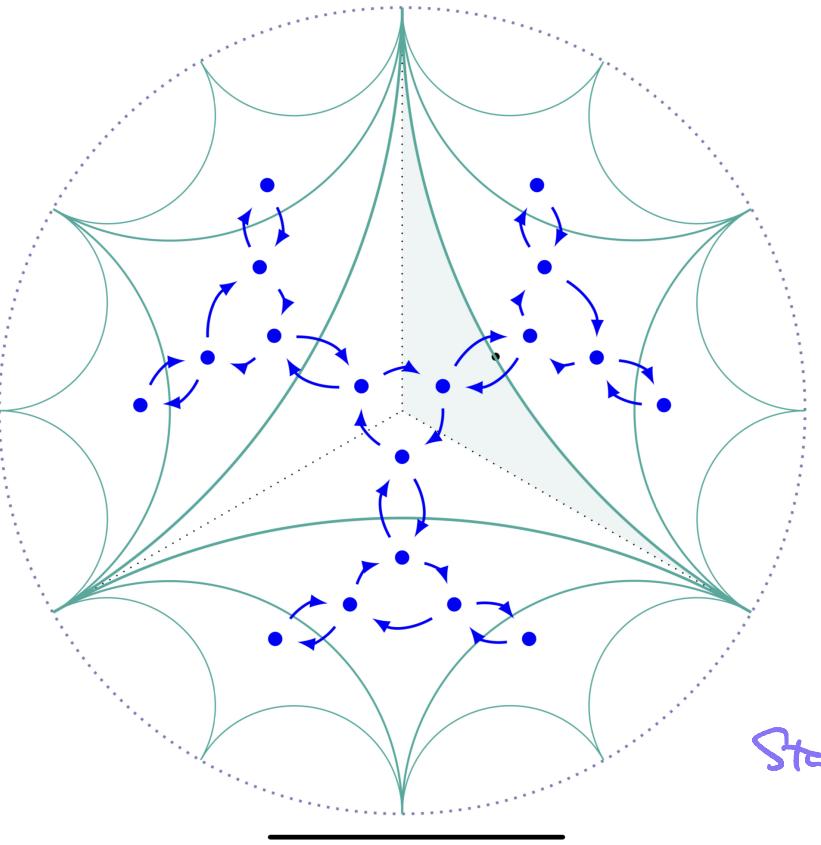
FACT: $\mathcal{CEG}(S) \& \mathcal{EG}^T(S_\delta)$

 are skeletons of
 $\mathcal{Quad}(S)$ & $F\mathcal{Quad}^T(S_\delta)$

Thm ($k, \mathbb{R}, \mathbb{Q}$) $\text{Stab}^\circ D_S(S) \cong F\mathcal{Quad}^T(S_\delta)$
is simply connected.

Ref. 1805.00030, 1407.0806, 1407.5986, 1111.1010

cf. 1712.09585



$$\text{Stab}^{\circ} D_3(A_2) \underset{\mathbb{C}}{\sim} \overline{\mathbb{CP}(1,1,1,-7)} \underset{\mathbb{C}}{\sim}$$

Thank
you

