A General Framework for Ordering Fuzzy Alternatives with Respect to Fuzzy Orderings

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Abstract

Orderings of fuzzy sets play an important role in various disciplines. In the last 25 years, a lot a different approaches to this problem have been introduced, ranging from rather simple ones to quite exotic ones. The purpose of this paper is to present a new framework for comparing fuzzy sets with respect to fuzzy orderings which already includes several known techniques based on the extension principle. This approach, however, is applicable to fuzzy subsets of any kind of universe for which a fuzzy ordering, no matter whether linear or not, is known.

Keywords: Fuzzy ordering, ordering of fuzzy sets.

1 Introduction

There is no doubt that orderings and rankings are essential in any field related to decision making. Admitting vagueness or impreciseness naturally results in the need for specifying vague preferences in crisp domains, but also in the demand for a framework in which it is even possible to decide between fuzzy alternatives. It is, therefore, not surprising that orderings and rankings of fuzzy sets have become main objects of study in fuzzy decision analysis.

Albeit only scarcely recognized, orderings of fuzzy sets are also important in areas related to fuzzy systems and fuzzy control, where the ordering of a numerical domain is usually taken into account when defining fuzzy sets. There might only be a minority of fuzzy systems or controllers in which expressions like “small”, “medium”, or “large” do not occur. This ordering, however, is also crucial as soon as automatic tuning procedures are concerned, which are supposed to give interpretable, i.e. understandable, results. Similar questions arise in linguistic approximation [6, 15] and rule interpolation [11, 12].

Since the 1970s, a host of different methods for ordering or ranking fuzzy sets has been published (see [5, 16] for detailed reviews). In order to find profound motivations for adding yet another approach, let us sum up common characteristics of these methods:

1. As long as linguistic expressions are represented by fuzzy subsets of numerical domains, there is a certain context-dependent notion of indistinguishability. It could be desirable to take this indistinguishability into account, since not only the ranking of alternatives itself, but also the information, that the difference between two alternatives is, more or less, negligible, could be of interest. All existing methods, however, do not offer the opportunity to integrate indistinguishability which often leads to undesired, counter-intuitive preciseness.

2. All methods are defined for so-called *fuzzy quantities*—fuzzy subsets of the real numbers. Not only from the theoretical, but also from the practical point of view, it could be interesting to consider arbitrary ordered domains, without any restriction concerning analytical properties, cardinality, or linearity of the ordering.

3. The applicability of many ordering methods is restricted to fuzzy quantities having special

properties, such as convexity, normality, or continuity. The motivation for such restrictions is to guarantee some desirable properties, for example, antisymmetry.

The purpose of this paper is to introduce and investigate an ordering method for arbitrary fuzzy subsets of an arbitrary (fuzzy) ordered domain where indistinguishability is taken into account, too. For proof details and theoretical basics, the reader is referred to [4].

Throughout the whole text, the symbol T is supposed to denote a left-continuous t-norm.

2 Preliminaries

Since, according to the above discussions, the ordering method should be able to cope with vagueness and indistinguishability, all studies in this paper will be based on the similarity-based definition of fuzzy orderings (for an extensive study, see [4]). We only recall the very basic definitions.

Definition 1. A binary fuzzy relation E on a domain X is called *fuzzy equivalence relation* with respect to T, for brevity *T-equivalence*, if and only if the following three axioms are fulfilled for all $x, y, z \in X$:

1. Reflexivity: $E(x, x) = 1$
2. Symmetry: $E(x, y) = E(y, x)$
3. T-transitivity: $T(E(x, y), E(y, z)) \leq E(x, z)$

Definition 2. Let $L : X^2 \to [0, 1]$ be a T-transitive fuzzy relation. L is called *fuzzy ordering* with respect to T and a T-equivalence E, for brevity *T-E-ordering*, if and only if it additionally fulfills the following two axioms for all $x, y \in X$:

1. E-reflexivity: $E(x, y) \leq L(x, y)$
2. T-E-antisymmetry:

$$T(L(x, y), L(y, x)) \leq E(x, y)$$

A subclass, which will be of special importance in the following, are so-called direct fuzzifications.

Definition 3. A T-E-ordering L is called a *direct fuzzification* of a crisp ordering $\preceq$ if and only if it admits the following resolution:

$$L(x, y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x, y) & \text{otherwise} \end{cases}$$

It is worth to mention that there is a one-to-one correspondence between direct fuzzifications of crisp linear orderings and so-called fuzzy weak orderings, i.e. reflexive and T-transitive binary fuzzy relations which fulfill strong completeness [3].

The modifiers 'at least' and 'at most' with respect to a fuzzy ordering L will be essential for all further investigations. They can be defined by means of hulls/closures.

Definition 4. Suppose that X is supplied with some T-E-ordering L. Then, for a fuzzy subset A of X the fuzzy sets 'at least A' and 'at most A' (with respect to L), abbreviated $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$, respectively, can be defined as follows:

$$\mu_{\mathrm{ATL}(A)}(x) = \sup\{T(\mu_A(y), L(y, x)) \mid y \in X\}$$
$$\mu_{\mathrm{ATM}(A)}(x) = \sup\{T(\mu_A(y), L(x, y)) \mid y \in X\}$$

If L coincides with a crisp ordering $\preceq$, we will use the symbols $\mathrm{LTR}(A)$ and $\mathrm{RTL}(A)$ instead (synonymously for left-to-right and right-to-left continuation), where one can easily show the following identities:

$$\mu_{\mathrm{LTR}(A)}(x) = \sup\{\mu_A(y) \mid y \preceq x\}$$
$$\mu_{\mathrm{RTL}(A)}(x) = \sup\{\mu_A(y) \mid x \preceq y\}$$

In words, $\mathrm{LTR}(A)$ is the smallest superset of a fuzzy set A the membership function of which is non-decreasing with respect to $\preceq$. Correspondingly, $\mathrm{RTL}(A)$ is the smallest superset of a fuzzy set A whose membership function is non-increasing. In direct analogy, $\mathrm{ATL}(A)$ can be regarded as the smallest superset of A the membership function of which is non-decreasing with respect to the fuzzy ordering L, analogously for $\mathrm{ATM}(A)$ (for a detailed and mathematically exact argumentation, see [4]).

Theorem 5. *Provided that a T-E-ordering L is a direct fuzzification of a crisp ordering $\preceq$, the following holds for all fuzzy sets A*

$$\mathrm{ATL}(A) = \mathrm{LTR}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{LTR}(A)),$$
$$\mathrm{ATM}(A) = \mathrm{RTL}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{RTL}(A)),$$

where $\mathrm{EXT}(A)$ is the so-called extensional hull of A with respect to E [13] defined as

$$\mu_{\mathrm{EXT}(A)}(x) = \sup\{T(\mu_A(y), E(y, x)) \mid y \in X\}.$$

3 The Basic Idea

In order to have a clear motivation, let us start from a well-known ordering procedure for real intervals (with respect to the usual ordering of real numbers):

$$[a,b] \leq_I [c,d] \iff a \leq c \wedge b \leq d$$

It is easy to check that $\leq_I$ is an ordering. The inequality $a \leq c$ means that there are no elements of the set $[c,d]$ which are below the entire interval $[a,b]$. The inequality $b \leq d$, analogously, means that there are no elements of $[a,b]$ which lie completely above $[c,d]$. This criterion can be generalized to arbitrary crisp subsets of an ordered set $(X, \preceq)$ as follows:

$$\begin{aligned} M \preceq_I N \iff & \big(\forall x \in N\ \exists y \in M:\ y \preceq x\big) \wedge \\ & \big(\forall x \in M\ \exists y \in N:\ x \preceq y\big) \end{aligned}$$

The following lemma provides an equivalent formulation by means of hulls which will be the basis of all our further generalizations.

Lemma 6. *Let a domain X be equipped with an ordering $\preceq$. With the above notations, the following holds for all crisp subsets $M, N \subseteq X$:*

$$\begin{aligned} \mathrm{LTR}(M) \supseteq \mathrm{LTR}(N) &\Leftrightarrow \forall x \in N\ \exists y \in M:\ y \preceq x \\ \mathrm{RTL}(M) \subseteq \mathrm{RTL}(N) &\Leftrightarrow \forall x \in M\ \exists y \in N:\ x \preceq y \end{aligned}$$

As an immediate consequence, we obtain that the assertion $M \preceq_I N$ is always equivalent to

$$\mathrm{LTR}(M) \supseteq \mathrm{LTR}(N) \wedge \mathrm{RTL}(M) \subseteq \mathrm{RTL}(N). \quad (1)$$

Since the operators LTR and RTL are not restricted to crisp sets, we can write down an extension of $\preceq_I$ to fuzzy subsets immediately:

$$\begin{aligned} A \preceq_I B \iff & \big(\mathrm{LTR}(A) \supseteq \mathrm{LTR}(B) \wedge \\ & \mathrm{RTL}(A) \subseteq \mathrm{RTL}(B)\big) \end{aligned}$$

This means that we are able to order fuzzy sets with respect to a crisp ordering $\preceq$. The generalization to an arbitrary fuzzy ordering is now straightforward.

Definition 7. Let L be a fuzzy ordering on X. Then the relation $\preceq$ on $\mathcal{F}(X)$ is defined in the following way:

$$\begin{aligned} A \preceq_L B \iff & \big(\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \wedge \\ & \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)\big) \end{aligned}$$

4 Analysis

In order to study the properties of relations of type $\preceq_L$, let us assume that L is an arbitrary but fixed fuzzy ordering with respect to some t-norm T and a fuzzy equivalence relation E.

First of all, it is trivial to see that the relation $\preceq_L$ is reflexive and transitive, since it is nothing else than the intersection of two inclusion relations, which are, of course, reflexive and transitive. The next result characterizes antisymmetry, or better, non-antisymmetry in a unique way.

Theorem 8. *The following holds for all fuzzy subsets $A, B \in \mathcal{F}(X)$, where $\mathrm{ECX}(A)$ stands for the minimum intersection of $\mathrm{ATL}(A)$ and $\mathrm{ATM}(A)$:*

$$A \preceq_L B \wedge A \succeq_L B \Leftrightarrow \mathrm{ECX}(A) = \mathrm{ECX}(B)$$

Theorem 8, however, only provides a valuable insight if we know more about the inside of the operator ECX. If we go back to crisp orderings for a moment, we see that the above property corresponds to the equality of the convex hulls. In contrast to vector space-based definitions of convexity [14, 17], we will use a general ordering-based definition of convexity. It is left as an exercise to the reader to check that, for the real number and their natural ordering, the two definitions are equivalent.

Definition 9. Assume we are given a crisp ordering $\preceq$. Then a fuzzy set A is called *convex* (with respect to $\preceq$) if and only if the following property holds for all $x, y, z \in X$:

$$x \preceq y \preceq z \Rightarrow \mu_B(y) \geq \min\big(\mu_B(x), \mu_B(z)\big)$$

The smallest convex superset—the *convex hull*—of A is denoted with $\mathrm{CVX}(A)$.

Lemma 10. *For any crisp ordering $\preceq$ and any fuzzy set A, $\mathrm{CVX}(A)$ and $\mathrm{ECX}(A)$ are equal.*

Hence, we obtain that, for some crisp ordering $\preceq$, the relation $\preceq_I$ is unable to distinguish between two fuzzy sets A and B if and only their convex hulls coincide.

In the case that we are dealing with an ordering relation L which is not crisp, this argumentation becomes more subtle. It is intuitively clear that $\mathrm{ECX}(A)$ is always something like a generalized convex hull, but this should be made more precise. If L is a direct fuzzification, a clear answer can be given.

Theorem 11. *Let L be a T-E-ordering which is a direct fuzzification of some crisp ordering $\preceq$. Then the following equalities hold for all $A \in \mathcal{F}(X)$:*

$$\mathrm{ECX}(A) = \mathrm{CVX}(\mathrm{EXT}(A)) = \mathrm{EXT}(\mathrm{CVX}(A))$$

In words, for direct fuzzifications, the operator ECX can be interpreted as a kind of 'extensional convex hull'.

In any case, we see that neither $\preceq_I$ nor $\preceq_L$ is guaranteed to be antisymmetric. However, we have found equivalence relations uniquely describing non-antisymmetry. Of course, we can obtain orderings by factorization with respect to the symmetric kernels of $\preceq_I$ and $\preceq_L$, respectively. From the above consideration, we know that the symmetric kernels can be represented as

$$\begin{aligned} A \cong_I B &\iff \mathrm{CVX}(A) = \mathrm{CVX}(B) \\ A \cong_L B &\iff \mathrm{ECX}(A) = \mathrm{ECX}(B) \end{aligned}$$

and we obtain the following result.

Theorem 12. *The relation $\preceq_I$ is an ordering on $\mathcal{F}(X)_{/\cong_I}$ which is isomorphic to the set of convex fuzzy subsets:*

$$\mathcal{F}_I(X) = \{A \in \mathcal{F}(X) \mid A = \mathrm{CVX}(A)\}$$

Analogously, the relation $\preceq_L$ is an ordering on $\mathcal{F}(X)_{/\cong_L}$ which is isomorphic to the set of extensional convex fuzzy subsets:

$$\mathcal{F}_L(X) = \{A \in \mathcal{F}(X) \mid A = \mathrm{ECX}(A)\}$$

The above results have a different quality if we compare them with the existing approaches which restricted to some special classes of fuzzy subsets from the beginning (e.g. [10, 12]) just to preserve properties, such as antisymmetry. The new method is not restricted to (extensional) convex fuzzy sets. It can distinguish between any two arbitrary fuzzy subsets as long as their (extensional) convex hulls do not coincide. Since non-antisymmetry is characterized by an equivalence relation, however, it is possible to define orderings of the equivalence classes in order to obtain a broader class of fuzzy subsets for which antisymmetry is satisfied.

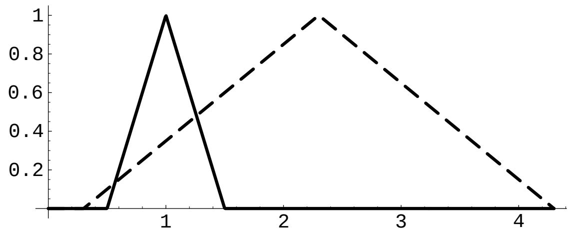

Figure 1: Two convex fuzzy quantities which are incomparable.

5 Fuzzification

If we consider the two convex fuzzy quantities in Figure 1, it is easy to see that, if we construct $\preceq_I$ by means of the natural ordering of real numbers, these two triangular fuzzy quantities are incomparable. Even if a fuzzy ordering L is taken, which directly fuzzifies $\leq$, the situation cannot be better. The question is whether it is natural at all to compare vague phenomena crisply or if, as the example in Figure 1 suggests, this directly leads to artificial preciseness.

In this section, we want to overcome this problem by allowing intermediate degrees to which a fuzzy set is smaller or equal than another. For this purpose, let us reconsider the definition of $A \preceq_L B$:

$$\mathrm{ATL}(A) \supseteq \mathrm{ATL}(B) \wedge \mathrm{ATM}(A) \subseteq \mathrm{ATM}(B)$$

If we want to make this crisp expression fuzzy, we have to specify (1) a fuzzy concept of subsethood and (2) a conjunction operation.

Definition 13. For a left-continuous t-norm T, the *residual implication* (residuum) is defined as

$$\vec{T}(x, y) = \sup\{z \in [0,1] \mid T(x, z) \leq y\},$$

while the corresponding *biimplication* (equivalence) is defined as

$$\overset{\leftrightarrow}{T}(x, y) = \min\left(\vec{T}(x, y), \vec{T}(y, x)\right).$$

In the framework of many-valued predicate logics based on residuated lattices [7, 9], it is natural to define the degree of inclusion of a fuzzy set A in another fuzzy set B as [1, 2, 8]

$$\mathrm{INCL}_T(A, B) = \inf_{x \in X} \vec{T}(\mu_A(x), \mu_B(x)),$$

where the infimum is, more or less, a generalization of a universal quantifier.

Theorem 14. *The relation* INCL_T *is a fuzzy ordering on* $\mathcal{F}(X)$ *with respect to* T *and the fuzzy equivalence relation*

$$\mathrm{SIM}_T(A,B) = \inf_{x\in X} \overset{\leftrightarrow}{T}(\mu_A(x), \mu_B(x)).$$

If we replace the Boolean conjunction $\wedge$ by the minimum t-norm $T_{\mathbf{M}}(x,y) = \min(x,y)$ and the usual inclusion $\subseteq$ by INCL_T, we obtain the following generalization of $\preceq_L$:

$$\begin{aligned}\mathcal{L}_L(A,B) = \min\big(&\mathrm{INCL}_T(\mathrm{ATL}(B), \mathrm{ATL}(A)),\\ &\mathrm{INCL}_T(\mathrm{ATM}(A), \mathrm{ATM}(B))\big)\end{aligned}$$

which is, in fact, a fuzzy ordering of fuzzy sets—perfectly fitting to the results of Theorems 8 and 12:

Theorem 15. *The fuzzy relation* $\mathcal{L}_L$ *is fuzzy ordering on* $\mathcal{F}(X)$ *with respect to* T *and the fuzzy equivalence relation*

$$\mathcal{E}_L(A,B) = \mathrm{SIM}_T\big(\mathrm{ECX}(A), \mathrm{ECX}(B)\big).$$

So far, it remains an open question in which way the crisp ordering $\lesssim_L$ and the fuzzy ordering $\mathcal{L}_{\tilde{T},L}$ are related to each other. The next result gives an exhaustive answer:

Theorem 16. *The following characterization of the kernel of* $\mathcal{L}_L$ *holds:*

$$\forall A, B \in \mathcal{F}(X): \; \mathcal{L}_L(A,B) = 1 \Longleftrightarrow A \preceq_L B$$

The relationship between $\mathcal{E}_L$ *and* $\cong_L$ *is given analogously:*

$$\forall A, B \in \mathcal{F}(X): \; \mathcal{E}_L(A,B) = 1 \Longleftrightarrow A \cong_L B$$

In particular, this entails that $\preceq_L$ is a subrelation of $\mathcal{L}_L$ which implies that the comparability of two fuzzy sets with respect to $\mathcal{L}_L$ cannot be worse than comparability with respect to $\preceq_L$. The following example will show that the problem of artificial strictness when comparing fuzzy sets with $\preceq_L$ is perfectly solved if they are compared with the relation $\mathcal{L}_L$.

Example 17. Let us reconsider the example shown in Figure 1, where we denote the left fuzzy quantity with A and the right one with B, i.e.

$$\begin{aligned}\mu_A(x) &= \max\big(0, 1 - 2\cdot|x-1|\big),\\ \mu_B(x) &= \max\big(0, 1 - \tfrac{1}{2}\cdot|x-2.3|\big).\end{aligned}$$

For $L = \chi_{\leq}$ and $T = T_{\mathbf{L}} = \max(x+y-1, 0)$, we obtain

$$\begin{aligned}\mathcal{L}_L(A,B) &= 0.9\\ \mathcal{L}_L(B,A) &= 0\\ \mathcal{E}_L(A,B) &= 0\end{aligned}$$

which seems to be quite a reasonable result.

6 Concluding Remarks

In this paper, a general method for ordering fuzzy sets with respect to fuzzy orderings was introduced. We have seen that the restriction to certain subclasses of fuzzy sets is not necessary in this approach.

Since it is often not desirable or natural to compare fuzzy sets crisply, a straightforward fuzzification of the ordering approach has been carried out—leading to fuzzy orderings of fuzzy sets.

The reader should be aware that, in any case, different heights of two fuzzy sets immediately imply incomparability with respect to $\preceq_L$. If the fuzzy variant $\mathcal{L}_L$ is taken, this problem is slightly solved, but still in a way which is far from being satisfactory. For a detailed discussion and a possible solution, the reader is referred to [4].

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