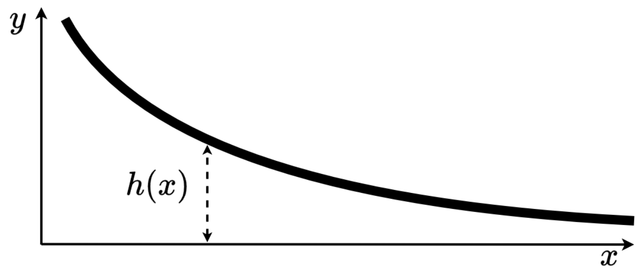
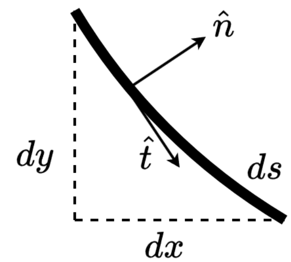
Steady State Thin Film Model

2D cartesian coordinates
1D lubrication approximation





I Curvature Function

Along a curve defined by $h = f(x)$, for an infinitesimal arc length, ds, with unit normal vector $\hat{n}$, and unit tangent vector, $\hat{t}$, as $ds \to 0$,

$$ds^2 = dx^2 + dy^2 \tag{1}$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{2}$$

let $\vec{r}(x)$ be a position vector function for the curve $h = f(x)$, such that $\vec{r}(x) =< x, f(x) >$ and $\|\vec{r}(x)\| = \sqrt{x^2 + (f(x))^2}$, then the tangent vector along the curve is defined as,

$$\vec{t} = \frac{d\vec{r}}{ds} \tag{3}$$

and the unit tangent vector,

$$\hat{t} = \frac{d\vec{r}/ds}{\|d\vec{r}/ds\|} \tag{4}$$

where, $\dfrac{d\vec{r}}{ds} =< 1, f'(x) >$ and $\dfrac{d\vec{r}}{ds} = \sqrt{1 + (f'(x))^2}$; expanding,

$$\hat{t} =< \frac{1}{\sqrt{1 + (f'(x))^2}}, \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} > \tag{5}$$

Curvature κ is defined as,

$$\kappa = \frac{1}{R} = \left\|\frac{\hat{d}t}{ds}\right\| = \left\|\frac{\hat{d}t}{dx} \cdot \frac{dx}{ds}\right\| \tag{6}$$

calculating the derivatives, first,

$$\frac{d\hat{t}}{dx} =< \frac{d}{dx}\left(\frac{1}{\sqrt{1+(f'(x))^2}}\right), \frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+(f'(x))^2}}\right) > \tag{7}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{1+(f'(x))^2}}\right) = -\frac{1}{2}\left(1+(f(x))^2\right)^{3/2} \cdot 2f'(x)f''(x) \tag{8}$$

$$= -f'(x)\ f''(x)\left(1+(f(x))^2\right)^{3/2} \tag{9}$$

$$\frac{d}{dx}\left(\frac{f'(x)}{\sqrt{1+(f'(x))^2}}\right) = \frac{f''(x)}{(1+(f(x))^2)^{1/2}} - \frac{(f'(x))^2\ f''(x)}{(1+(f(x))^2)^{3/2}} \tag{10}$$

therefore,

$$\frac{d\hat{t}}{dx} = < -\frac{f'(x)f''(x)}{(1+(f(x))^2)^{3/2}}, \frac{f''(x)}{(1+(f(x))^2)^{1/2}} - \frac{(f'(x))^2\ f''(x)}{(1+(f(x))^2)^{3/2}} > \tag{11}$$

further simplifying,

$$\frac{d\hat{t}}{dx} = \frac{< -f'(x)f''(x), f''(x) >}{(1+(f(x))^2)^{3/2}} \tag{12}$$

second, from equation 2,

$$\frac{dx}{ds} = \left(1+(f(x))^2\right)^{-1/2} \tag{13}$$

from equations 12 and 13,

$$\frac{d\hat{t}}{ds} = \frac{< -f'(x)f''(x), f''(x) >}{(1+(f(x))^2)^{3/2}} \cdot \left(1+(f(x))^2\right)^{-1/2} \tag{14}$$

then curvature can be written as,

$$\kappa = \left\|\frac{d\hat{t}}{ds}\right\| = \frac{\|< -f'(x)f''(x), f''(x) >\|}{(1+(f(x))^2)^{3/2}} \cdot \left(1+(f(x))^2\right)^{-1/2} \tag{15}$$

$$\kappa = \frac{\left[(-f'(x)f''(x)^2 + (f''(x))^2\right]^{1/2}}{(1+(f(x))^2)^{3/2}} \cdot \left(1+(f(x))^2\right)^{-1/2} \tag{16}$$

$$\boxed{\kappa = \frac{f''(x)}{(1+(f(x))^2)^{3/2}}} \tag{17}$$

The derivative of curvature,

$$\frac{d\kappa}{dx} = \frac{d}{dx}\left(f''(x)\right) \cdot \frac{1}{\left(1+(f'(x))^2\right)^{3/2}} + \frac{d}{dx}\left(\frac{1}{\left(1+(f'(x))^2\right)^{3/2}}\right) \cdot f''(x) \tag{18}$$

$$= \frac{f'''(x)}{\left(1+(f'(x))^2\right)^{3/2}} - \frac{3}{2}\left(1+(f'(x))^2\right)^{-5/2} \cdot 2f'(x)f''(x) \cdot f''(x) \tag{19}$$

$$\boxed{\frac{d\kappa}{dx} = \frac{f'''(x)}{\left(1+(f'(x))^2\right)^{3/2}} - \frac{3f'(x)(f''(x))^2}{\left(1+(f'(x))^2\right)^{5/2}}} \tag{20}$$

II Evolution Equation

Starting with the Young-Laplace equation,

$$p_v - p_l = \sigma\kappa + \Pi \tag{21}$$

$$\frac{dp_v}{dx} - \frac{dp_l}{dx} = \frac{d}{dx}\left(\sigma\kappa\right) + \frac{d\Pi}{dx} \tag{22}$$

assuming vapor pressure to be constant, $dp_v/dx = 0$,

$$\frac{d}{dx}\left(\sigma\kappa\right) = -\left(\frac{dp_l}{dx} + \frac{d\Pi}{dx}\right) \tag{23}$$

$$\left(\frac{d\sigma}{dx}\right)\kappa + \sigma\left(\frac{d\kappa}{dx}\right) = -\left(\frac{dp_l}{dx} + \frac{d\Pi}{dx}\right) \tag{24}$$

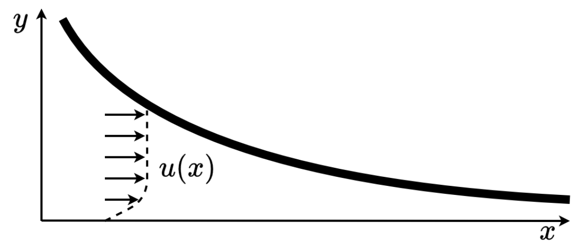
with $\sigma = \sigma_o + \gamma \cdot T_i$ and $\dfrac{d\sigma}{dx} = \gamma \cdot \dfrac{\partial T_i}{\partial x}$, as well as equations 17 and 20,

$$\gamma\frac{\partial T_i}{\partial x}\left(\frac{h''(x)}{\left(1+(h'(x))^2\right)^{3/2}}\right) + \sigma\left(\frac{h'''(x)}{\left(1+(h'(x))^2\right)^{3/2}} - \frac{3h'(x)(h''(x))^2}{\left(1+(h'(x))^2\right)^{5/2}}\right) = -\left(\frac{dp_l}{dx} + \frac{d\Pi}{dx}\right) \tag{25}$$

$$\frac{\gamma}{\sigma}\frac{\partial T_i}{\partial x}h''(x) + h'''(x) - \frac{3h'(x)(h''(x))^2}{1+(h'(x))^2} = -\frac{1}{\sigma}\left(1+(h'(x))^2\right)^{3/2}\left(\frac{dp_l}{dx} + \frac{d\Pi}{dx}\right) \tag{26}$$

$$\boxed{h'''(x) = \frac{3h'(x)(h''(x))^2}{1+(h'(x))^2} - \frac{\gamma}{\sigma}\frac{\partial T_i}{\partial x}h''(x) - \frac{1}{\sigma}\left(1+(h'(x))^2\right)^{3/2}\left(\frac{dp_l}{dx} + \frac{d\Pi}{dx}\right)} \tag{27}$$

III Liquid Pressure Gradient



The Navier-Stokes momentum equation with lubrication approximation,

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x} \tag{28}$$

integrating with respect to y,

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + c_1 \tag{29}$$

applying Marangoni condition at the free surface, i.e., at $y = h$, $\dfrac{\partial u}{\partial y} = -\dfrac{1}{\mu}\dfrac{\partial \sigma}{\partial x}$,

$$\left.\frac{\partial u}{\partial y}\right|_{y=h} = -\frac{1}{\mu}\frac{\partial \sigma}{\partial x} = \frac{1}{\mu}\frac{\partial p}{\partial x} + c_1 \tag{30}$$

$$c_1 = -\frac{1}{\mu}\left(\frac{\partial \sigma}{\partial x} + h\frac{\partial p}{\partial x}\right) \tag{31}$$

integrating equation 29,

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 \cdot y + c_2 \tag{32}$$

applying no-slip condition at the solid wall, i.e., at $y = 0$, $u = 0$,

$$u(0) = \frac{1}{2\mu}\frac{\partial p}{\partial x}\,(0)^2 + c_1(0) + c_2 = 0 \tag{33}$$

$$\therefore c_2 = 0 \tag{34}$$

substituting equations 31 and 34 into equation 32,

$$\boxed{u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 - \frac{1}{\mu}\left(\frac{\partial \sigma}{\partial x} + h\frac{\partial p}{\partial x}\right)y} \tag{35}$$

the liquid mass flow is defined by the integral,

$$\dot{\Gamma} = \int_0^h \rho L \cdot u(y)\, dy \tag{36}$$

where, L is the depth in z,

$$= \rho L \int_0^h \left(\frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 - \frac{1}{\mu} \left(\frac{\partial \sigma}{\partial x} + h \frac{\partial p}{\partial x} \right) y \right)\, dy \tag{37}$$

$$= \rho L \left\{ \left[\frac{1}{2\mu} \frac{\partial p}{\partial x} \cdot \frac{y^3}{3} \right] \Bigg|_0^h \; - \; \left[\frac{1}{\mu} \left(\frac{\partial \sigma}{\partial x} + h \frac{\partial p}{\partial x} \right) \frac{y^2}{2} \right] \Bigg|_0^h \right\} \tag{38}$$

$$= \rho L \left\{ \left[\frac{1}{2\mu} \frac{\partial p}{\partial x} \cdot \frac{h^3}{3} \right] \; - \; \left[\frac{1}{\mu} \left(\frac{\partial \sigma}{\partial x} + h \frac{\partial p}{\partial x} \right) \frac{h^2}{2} \right] \right\} \tag{39}$$

$$\dot{\Gamma} = -\frac{\rho L}{\mu} \left(\frac{1}{3} \frac{\partial p}{\partial x} h^3 + \frac{1}{2} \frac{\partial \sigma}{\partial x} h^2 \right) \tag{40}$$

inverting,

$$\boxed{\frac{\partial p}{\partial x} = -\frac{3}{h^3} \left(\frac{\dot{\Gamma} \mu}{\rho L} + \frac{h^2}{2} \frac{\partial \sigma}{\partial x} \right)} \tag{41}$$

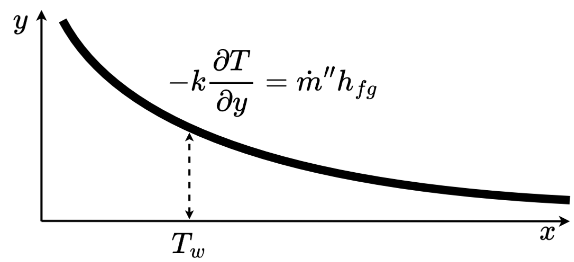
In a control volume, the net evaporation mass is,

$$\dot{m} = \dot{m}'' \times \Delta x \times L \tag{42}$$

which is locally the change in the liquid mass flow,

$$-\Delta \dot{\Gamma} = \dot{m} \tag{43}$$

IV Interface Temperature Distribution



Using steady-state one-dimensional approximation,

$$\frac{\partial^2 T}{\partial y^2} = 0 \qquad \rightarrow \qquad \frac{\partial T}{\partial y} = c_1 \tag{44}$$

at $y = h$, $\dfrac{\partial T}{\partial y} = -\dot{m}'' \cdot \dfrac{h_{\text{fg}}}{k}$

$$\therefore c_1 = -\dot{m}'' \cdot \frac{h_{\text{fg}}}{k} \tag{45}$$

integrating,

$$T(y) = -\dot{m}'' \cdot \frac{h_{\text{fg}}}{k} y + c_2 \tag{46}$$

at $y = 0$, $T = T_w$, $\quad \therefore c_2 = T_w$,

$$T(y) = -\dot{m}'' \cdot \frac{h_{\text{fg}}}{k} y + T_w \tag{47}$$

then surface temperature,

$$\boxed{T_i = T(h) = -\frac{h_{\text{fg}}}{k}\; h(x)\; \dot{m}''(x) + T_w(x)} \tag{48}$$

V Computational Logic