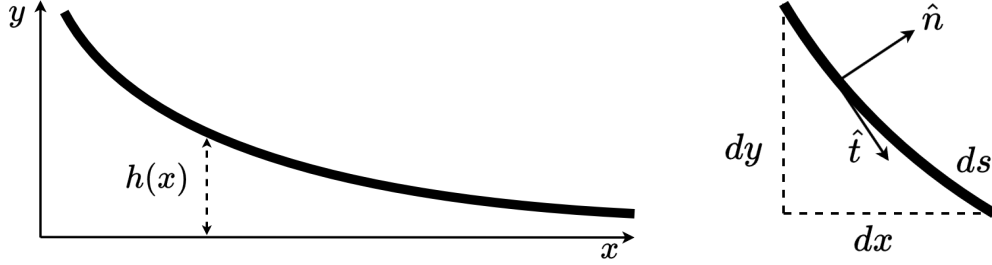


# Steady State Thin Film Model

2D cartesian coordinates  
1D lubrication approximation



## I Curvature Function

Along a curve defined by  $h = f(x)$ , for an infinitesimal arc length,  $ds$ , with unit normal vector  $\hat{n}$ , and unit tangent vector,  $\hat{t}$ , as  $ds \rightarrow 0$ ,

$$ds^2 = dx^2 + dy^2 \quad (1)$$

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \quad (2)$$

let  $\vec{r}(x)$  be a position vector function for the curve  $h = f(x)$ , such that  $\vec{r}(x) = \langle x, f(x) \rangle$  and  $\|\vec{r}(x)\| = \sqrt{x^2 + (f(x))^2}$ , then the tangent vector along the curve is defined as,

$$\vec{t} = \frac{d\vec{r}}{ds} \quad (3)$$

and the unit tangent vector,

$$\hat{t} = \frac{d\vec{r}/ds}{\|d\vec{r}/ds\|} \quad (4)$$

where,  $\frac{d\vec{r}}{ds} = \langle 1, f'(x) \rangle$  and  $\frac{d\vec{r}}{ds} = \sqrt{1 + (f'(x))^2}$ ; expanding,

$$\hat{t} = \left\langle \frac{1}{\sqrt{1 + (f'(x))^2}}, \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} \right\rangle \quad (5)$$

Curvature  $\kappa$  is defined as,

$$\kappa = \frac{1}{R} = \left\| \frac{\hat{t}}{ds} \right\| = \left\| \frac{\hat{t}}{dx} \cdot \frac{dx}{ds} \right\| \quad (6)$$

calculating the derivatives, first,

$$\frac{d\hat{t}}{dx} = < \frac{d}{dx} \left( \frac{1}{\sqrt{1 + (f'(x))^2}} \right), \frac{d}{dx} \left( \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} \right) > \quad (7)$$

$$\frac{d}{dx} \left( \frac{1}{\sqrt{1 + (f'(x))^2}} \right) = -\frac{1}{2} (1 + (f'(x))^2)^{-3/2} \cdot 2f'(x)f''(x) \quad (8)$$

$$= -f'(x) f''(x) (1 + (f'(x))^2)^{-3/2} \quad (9)$$

$$\frac{d}{dx} \left( \frac{f'(x)}{\sqrt{1 + (f'(x))^2}} \right) = \frac{f''(x)}{(1 + (f'(x))^2)^{1/2}} - \frac{(f'(x))^2 f''(x)}{(1 + (f'(x))^2)^{3/2}} \quad (10)$$

therefore,

$$\frac{d\hat{t}}{dx} = < -\frac{f'(x)f''(x)}{(1 + (f'(x))^2)^{3/2}}, \frac{f''(x)}{(1 + (f'(x))^2)^{1/2}} - \frac{(f'(x))^2 f''(x)}{(1 + (f'(x))^2)^{3/2}} > \quad (11)$$

further simplifying,

$$\frac{d\hat{t}}{dx} = \frac{< -f'(x)f''(x), f''(x) >}{(1 + (f'(x))^2)^{3/2}} \quad (12)$$

second, from equation 2,

$$\frac{dx}{ds} = (1 + (f'(x))^2)^{-1/2} \quad (13)$$

from equations 12 and 13,

$$\frac{d\hat{t}}{ds} = \frac{< -f'(x)f''(x), f''(x) >}{(1 + (f'(x))^2)^{3/2}} \cdot (1 + (f'(x))^2)^{-1/2} \quad (14)$$

then curvature can be written as,

$$\kappa = \left\| \frac{d\hat{t}}{ds} \right\| = \frac{\| < -f'(x)f''(x), f''(x) > \|}{(1 + (f'(x))^2)^{3/2}} \cdot (1 + (f'(x))^2)^{-1/2} \quad (15)$$

$$\kappa = \frac{\left[ (-f'(x)f''(x))^2 + (f''(x))^2 \right]^{1/2}}{(1 + (f'(x))^2)^{3/2}} \cdot (1 + (f'(x))^2)^{-1/2} \quad (16)$$

$$\boxed{\kappa = \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}}} \quad (17)$$

The derivative of curvature,

$$\frac{d\kappa}{dx} = \frac{d}{dx} (f''(x)) \cdot \frac{1}{(1 + (f'(x))^2)^{3/2}} + \frac{d}{dx} \left( \frac{1}{(1 + (f'(x))^2)^{3/2}} \right) \cdot f''(x) \quad (18)$$

$$= \frac{f'''(x)}{(1 + (f'(x))^2)^{3/2}} - \frac{3}{2} (1 + (f'(x))^2)^{-5/2} \cdot 2f'(x)f''(x) \cdot f''(x) \quad (19)$$

$$\boxed{\frac{d\kappa}{dx} = \frac{f'''(x)}{(1 + (f'(x))^2)^{3/2}} - \frac{3f'(x)(f''(x))^2}{(1 + (f'(x))^2)^{5/2}}} \quad (20)$$

## II Evolution Equation

Starting with the Young-Laplace equation,

$$p_v - p_l = \sigma\kappa + \Pi \quad (21)$$

$$\frac{dp_v}{dx} - \frac{dp_l}{dx} = \frac{d}{dx} (\sigma\kappa) + \frac{d\Pi}{dx} \quad (22)$$

assuming vapor pressure to be constant,  $dp_v/dx = 0$ ,

$$\frac{d}{dx} (\sigma\kappa) = - \left( \frac{dp_l}{dx} + \frac{d\Pi}{dx} \right) \quad (23)$$

$$\left( \frac{d\sigma}{dx} \right) \kappa + \sigma \left( \frac{d\kappa}{dx} \right) = - \left( \frac{dp_l}{dx} + \frac{d\Pi}{dx} \right) \quad (24)$$

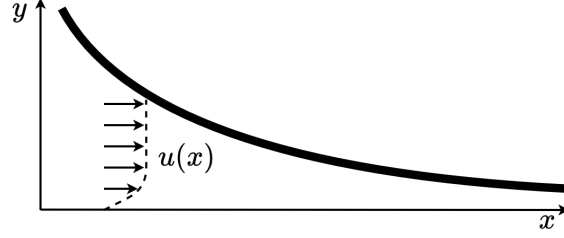
with  $\sigma = \sigma_o + \gamma \cdot T_i$  and  $\frac{d\sigma}{dx} = \gamma \cdot \frac{\partial T_i}{\partial x}$ , as well as equations 17 and 20,

$$\gamma \frac{\partial T_i}{\partial x} \left( \frac{h''(x)}{(1 + (h'(x))^2)^{3/2}} \right) + \sigma \left( \frac{h'''(x)}{(1 + (h'(x))^2)^{3/2}} - \frac{2h'(x)(h''(x))^2}{(1 + (h'(x))^2)^{5/2}} \right) = - \left( \frac{dp_l}{dx} + \frac{d\Pi}{dx} \right) \quad (25)$$

$$\frac{\gamma}{\sigma} \frac{\partial T_i}{\partial x} h''(x) + h'''(x) - \frac{3h'(x)(h''(x))^2}{1 + (h'(x))^2} = - \frac{1}{\sigma} (1 + (h'(x))^2)^{3/2} \left( \frac{dp_l}{dx} + \frac{d\Pi}{dx} \right) \quad (26)$$

$$\boxed{h'''(x) = \frac{3h'(x)(h''(x))^2}{1 + (h'(x))^2} - \frac{\gamma}{\sigma} \frac{\partial T_i}{\partial x} h''(x) - \frac{1}{\sigma} (1 + (h'(x))^2)^{3/2} \left( \frac{dp_l}{dx} + \frac{d\Pi}{dx} \right)} \quad (27)$$

### III Liquid Pressure Gradient



The Navier-Stokes momentum equation with lubrication approximation,

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x} \quad (28)$$

integrating with respect to y,

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + c_1 \quad (29)$$

applying Marangoni condition at the free surface, i.e., at  $y = h$ ,  $\frac{\partial u}{\partial y} = -\frac{1}{\mu} \frac{\partial \sigma}{\partial x}$ ,

$$\left. \frac{\partial u}{\partial y} \right|_{y=h} = -\frac{1}{\mu} \frac{\partial \sigma}{\partial x} = \frac{1}{\mu} \frac{\partial p}{\partial x} + c_1 \quad (30)$$

$$c_1 = -\frac{1}{\mu} \left( \frac{\partial \sigma}{\partial x} + h \frac{\partial p}{\partial x} \right) \quad (31)$$

integrating equation 29,

$$u(y) = \frac{1}{\mu} \frac{\partial p}{\partial x} y^2 + c_1 \cdot y + c_2 \quad (32)$$

applying no-slip condition at the solid wall, i.e., at  $y = 0$ ,  $u = 0$ ,

$$u(0) = \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot 0 + c_1(0) + c_2 \quad (33)$$

$$\therefore c_2 = 0 \quad (34)$$

substituting equations 31 and 34 into equation 32,

$$\boxed{u(y) = \frac{1}{\mu} \frac{\partial p}{\partial x} y^2 - \frac{1}{\mu} \left( \frac{\sigma}{x} + h \frac{\partial p}{\partial x} \right) y} \quad (35)$$

the liquid mass flow is defined by the integral,

$$\dot{\Gamma} = \int_0^h \rho L \cdot u(y) dy \quad (36)$$

where,  $L$  is the depth in  $z$ ,

$$= \rho L \int_0^h \left( \frac{1}{\mu} \frac{\partial p}{\partial x} y^2 - \frac{1}{\mu} \left( \frac{\sigma}{x} + h \frac{\partial p}{\partial x} \right) y \right) dy \quad (37)$$

$$= \rho L \left\{ \left[ \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{y^3}{3} \right] \Big|_0^h - \left[ \frac{1}{\mu} \left( \frac{\partial \sigma}{\partial x} + h \frac{\partial p}{\partial x} \right) \frac{y^2}{2} \right] \Big|_0^h \right\} \quad (38)$$

$$= \rho L \left\{ \left[ \frac{1}{\mu} \frac{\partial p}{\partial x} \cdot \frac{h^3}{3} \right] - \left[ \frac{1}{\mu} \left( \frac{\partial \sigma}{\partial x} + h \frac{\partial p}{\partial x} \right) \frac{h^2}{2} \right] \right\} \quad (39)$$

$$\boxed{\dot{\Gamma} = -\frac{\rho L}{\mu} \left( \frac{1}{6} \frac{\partial p}{\partial x} h^3 + \frac{1}{2} \frac{\partial \sigma}{\partial x} h^2 \right)} \quad (40)$$

For an evaporation mass flux,  $\dot{m}''$ , the evaporation rate,  $\dot{m}$ ,

$$\frac{\partial \dot{m}}{\partial x} = \int_0^L \dot{m}'' dz = L \cdot \dot{m}'' \quad (41)$$

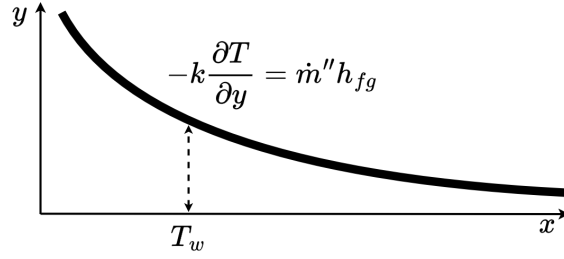
then mass conservation dictates,

$$\frac{\partial \dot{\Gamma}}{\partial x} + \frac{\partial \dot{m}}{\partial x} = 0 \quad (42)$$

$$\dot{m} = -\frac{\rho}{\mu} \left( \frac{1}{6} \frac{\partial p}{\partial x} h^3 + \frac{1}{2} \frac{\partial \sigma}{\partial x} h^2 \right) \quad (43)$$

$$\boxed{\frac{\partial p}{\partial x} = 6 \frac{\dot{m}''}{h^3} \frac{\mu}{\rho} - 3h \frac{\partial \sigma}{\partial x}} \quad (44)$$

## IV Interface Temperature Distribution



Using steady-state one-dimensional approximation,

$$\frac{\partial^2 T}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial T}{\partial y} = c_1 \quad (45)$$

$$\text{at } y = h, \quad \frac{\partial T}{\partial y} = -\dot{m}'' \cdot \frac{h_{fg}}{k}$$

$$\therefore c_1 = -\dot{m}'' \cdot \frac{h_{fg}}{k} \quad (46)$$

integrating,

$$T(y) = -\dot{m}'' \cdot \frac{h_{fg}}{k} y + c_2 \quad (47)$$

$$\text{at } y = 0, T = T_w, \quad \therefore c_2 = T_w,$$

$$T(y) = -\dot{m}'' \cdot \frac{h_{fg}}{k} y + T_w \quad (48)$$

then surface temperature,

$$\boxed{T_i = T(h) = -\frac{h_{fg}}{k} h(x) \dot{m}''(x) + T_w(x)} \quad (49)$$

## V Computational Logic

