Q1

$\max_{W2\in[0,W_1]} u(W_1 - W_2)$

Condition: $W_2 = 0$

Q2

The condition that characterizes the optimal choice in period 2:

$\max_{W_3\in[0,W2]} u(W_2 - W_3)$

The condition that characterizes the optimal choice in period 1:

$\max_{W_2\in[0,W1]} [u(W_1 - W_2) + \max_{W_3\in[0,W2]} \beta u(W_2 - W_3)]$

Q3

Period 1: $u'(W_1 - W_2) = \beta u'(W_2 - W_3)$

Period 2: $u'(W_2 - W_3) = \beta u'(W_3 - W_4)$

Period 3: $W_4 = 0,\ c_3 = W_3 - W_4 = W_3$

Since $\beta = 0.9$, solved the equation:

$W_1 = 1, W_2 = 0.630996, W_3 = 0.29889, W_4 = 0$

$c_1 = 0.369004, c_2 = 0.332103, c_3 = 0.29889$

(Calculation process and plots are in jupyter notebook.)

Q4

The condition that characterize the optimal choice in period T-1:

$-u'\big(W_{T-1} - \psi_{T-1}(W_{T-1})\big) + \beta u'\big(\psi_{T-1}(W_{T-1})\big) = 0$

$V_{T-1}(W_{T-1}) = u\big(W_{T-1} - \psi_{T-1}(W_{T-1})\big) + \beta u(\psi_{T-1}(W_{T-1}))$

Q5

Since $W_{T+1} = \psi_T(\bar{W}) = 0$, if we suppose $\psi_T(\bar{W}) = \psi_{T-1}(\bar{W})$, we can get $\psi_{T-1}(\bar{W}) = W_T = 0$

Also, since we know $u'(W_{T-1} - W_T) = \beta u'(W_T)$, and $(1+\beta)W_T = \beta W_{T-1}$, We can make the assumption if $W_T = 0$, W_{T-1}=0. Then we will get all equal to zero from W_{T-2} to W_1, which is contracted from the conditions. Thus, $\psi_T(\bar{W}) \neq \psi_T(\bar{W})$**.**

Sinc $V_T(\bar{W}) = u(\bar{W}) = \ln(\bar{W})\, , if\ we\ suppose\ V_T(\bar{W}) = V_{T-1}(\bar{W}), we\ can\ get\ V_{T-1}(\bar{W}) = \ln(\bar{W} - W_T) + \beta \ln(W_T) = \ln(\bar{W})$

Also, since we know $u'(W_{T-1} - W_T) = \beta u'(W_T)$, we can get $\left(1 - \frac{W_T}{\bar{W}}\right)(W_T)^\beta = \left(1 - \frac{\beta}{\beta+1}\right)(W_T)^\beta = 1$ from solving the two equations. As we get $(W_T)^\beta = \beta + 1$ and $\beta > 0$, $(W_T)^\beta > 1$. This is contradicted from the conditions. Thus, $V_T(\bar{W}) \neq V_{T-1}(\bar{W})$.

Q6

$V_{T-2}(W_{T-2}) = \max_{W_{T-1}} \ln\,(W_{T-2} - W_{T-1} + \beta \ln\left(\frac{W_{T-1}}{1+\beta}\right) + \beta^2 \ln\,(\frac{\beta W_{T-1}}{1+\beta})$

Solve the equation for V_{T-2} is :

$$V_{T-2}(W_{T-2}) = \ln\left(\frac{W_{T-2}}{1+\beta+\beta^2}\right) + \beta ln \ln\left(\frac{\beta W_{T-2}}{1+\beta+\beta^2}\right) + \beta^2 \ln\left(\frac{\beta^2 W_{T-2}}{1+\beta+\beta^2}\right)$$

Q7

Analytical solution for $\psi_{T-s}(W_{T-s})$: $\psi_{T-s}(W_{T-s}) = \frac{\sum_{i=1}^{s}\beta^i}{1+\sum_{i=1}^{s}\beta^i} W_{T-s}$

Analytical solution for $V_{T-s}(W_{T-s})$: $V_{T-s}(W_{T-s}) = \sum_{i=0}^{s}\beta^i \ln\left(\frac{\beta^i W_{T-s}}{1+\sum_{i=1}^{s}\beta^i}\right)$

Since the horizon becomes infinite, the equation would be like:

$$\lim_{s\to\infty}\psi_{T-s}(W_{T-s}) = \beta(W_{T-s}) = \psi(W_{T-s})$$

$$\lim_{s\to\infty}V_{T-s}(W_{T-s}) = \left(\frac{1}{1-\beta}\right)\ln\big((1-\beta)W_{T-s}\big) + \frac{\beta}{(1-\beta)^2}\ln(\beta) = V(W_{T-s})$$

Q8

$$V(W) \equiv \max_{W'\in[0,W]} u(W-W') + \beta V(W')$$

Q9-Q22 Please see the jupyter notebook