Thapar Institute of Engineering and Technology, Patiala
School of Mathematics

Optimization Methods (UMA-035)
Lab Experiment - 1 (Graphical Method)

Algorithm of Graphical Method to solve LPP

Step 1: Enter the provided data (coefficients of variables in objective function, coefficients of variables in constraints, right hand side elements of the constraints) in array's (namely C, A, B).

Step 2: Select the range of x_1 in which graph to be plotted.

Step 3: Find non-negative value of x_2 from the ith constraint (say x_2^i) in terms of x_1

Step 4: Plot the graph between x_{2i} and x_1

Step 5: Assume an empty array (say, solution) matrix to store the obtained solution

Step 6: Store the i^{th} row of A in an array (say, A1)

Step 7: Store the ith row of B in an array (say, B1)

Step 7: Store the (i+1)th row of A in an array (say, A2)

Step 8: Store the (i+1)th row of B in an array (say, B2)

Step 9: Combine A1 and A2 (say, A3)

Step 10: Combine B1 and B2 (say, B3)

Step 11: Find solution of system of equations A3X=B3

Step 12: Store solution in the considered empty array

Step 13: Assume first column of solution as $x1$

Step 14: Assume second column of solution as $x2$

Step 15: Find such rows of solution which does not satisfy the first constraint

Step 16: Delete such rows from Solution

Step 17: Assume first column of solution as $x1$

Step 18: Assume second column of solution as $x2$

Step 19: Find such rows of solution which does not satisfy the first constraint

Step 20: Repeat for all the constraints

Step 21: Find value of objective function (say, OBJ) corresponding to i^{th} row of solution

Step 22: Find maximum or minimum value of objective function

Step 23: Find that row corresponding to which OBJ is max or min

Step 24: The row represents an optimal solution

Write a MATLAB code to solve the following LPPs by graphical method:

1. Maximize/Minimize $(3x_1 + 2x_2)$ Subject to $2x_1 + 4x_2 <= 8$, $3x_1 + 5x_2 >= 15$,

   $x_1 >= 0$, $x_2 >= 0$.

2. Maximize/Minimize ($3x_1 + 2x_2$) Subject to $2x_1 + 4x_2 >= 8$, $3x_1 + 5x_2 >= 15$,

$x_1 >= 0$, $x_2 >= 0$.

3. Maximize/Minimize ($3x_1 + 2x_2$) Subject to $2x_1 + 4x_2 <= 8$, $3x_1 + 5x_2 <= 15$,

$x_1 >= 0$, $x_2 >= 0$

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Optimization Techniques (UMA-035)
Lab Experiment- 2 (Basic feasible solutions)

Algorithm to find BFS

Consider the LPP:

$$\begin{array}{rc} & \text{Max } z = C^t X \\ \text{subject to} & AX = b, \quad X \geq 0 \end{array}$$

Initially define Input paramenters:

1. Number of constraints as m,
2. Number of unknowns n,
3. Entries bi of the R.H.S. vector b and entries a_{ij} of matrix A,
4. Position of basic variables as p_i where $1 \leq i \leq m$.

Step 1: Construct the basis matrix B from the already defined basic variables p_i's.

Step 2: If $det(B) \neq 0$, then find $X_B = B^{-1}b$, otherwise display (' Not a Basic solution') and STOP.

Step 3: If $X_{B_i} < 0$ for some i then display (' Not a B.F.S') and STOP.
If $X_{B_i} = 0$ for some i then display (' Degenerate B.F.S') and STOP.
If $X_{B_i} > 0$ for all i then display (' Non-degenerate B.F.S') and STOP.

Write a MATLAB code to compute the basic feasible solutions of an LPP and test your program on the following set of examples:

1. Convert the following linear programming problem in standard form and find all bfs.
$$Max. \;\; z = x_1 + 2x_2, \;\; subject\ to \;\; -x_1 + x_2 \leq 1, \; x_1 + x_2 \leq 2, \; x_1, x_2 \geq 0.$$
2. Check if the the variables (1) (x_1, x_4) (2) (x_3, x_2) can be in the basis for the following problem
$$Max. \;\; z = x_1 + 2x_2 - x_3 + x_4,$$
$$subject\ to \;\; x_1 + x_2 - x_3 + 3x_4 = 15, \; 5x_1 + x_2 + 4x_3 + 15x_4 = 12, \; x_1, x_2, x_3, x_4 \geq 0.$$
3. Solve the following LPP by using finding all its BFS.
$$Max. \;\; z = -x_1 + 2x_2 - x_3, \;\; subject\ to \;\; x_1 \leq 4, \; x_2 \leq 4, \; -x_1 + x_2 \leq 6, \; -1x_1 + 2x_3 \leq 4, \; x_1, x_2, x_3 \geq$$
4. Check if the following LPP has a degenerate BFS, Find all basis corresponding to this solution.
$$Max. \;\; z = x_1 + x_2 + x_3, \;\; subject\ to \;\; x_1 + x_2 \leq 1, \; -x_2 + x_3 \leq 0, \; x_1, x_2, x_3 \geq 0.$$

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Optimization Techniques (UMA-035)
Lab Experiment- 3 (The simplex method)

The Simplex Algorithm (with slack variables only)

Consider an LPP with all constraints of $(\leq)$ type given as

$$\text{Max } z = C^t X$$
$$\text{subject to} \qquad AX \leq b, \quad X \geq 0$$

Convert the above problem to standard form by adding slack variables:

$$\text{Max } z = C^t X$$
$$\text{subject to} \qquad AX + IX_s = b, \quad X \geq 0$$

Where $X_s = (s_1, s_2, \ldots, s_m)$ is a vector of slack variables.

Initially definethe following Input paramenters:

1. Enter the Matrix $A = [A \quad I]$, where I is an identity matrix of order m.
2. Entries bi of the R.H.S. vector b and entries a_{ij} of matrix A
3. Define [m,n]=size (A)
4. Input the variables $s_1, s_2, \ldots s_m$ as basic variables.

Step 1: Construct the basis matrix B from the already defined basic variables.

Step 2: Calculate $Z_j - c_j = C_B^t(B^{-1}A_j) - c_j$ for each $j \in 1, 2 \ldots n$
Find $[val, k] = \min\{(Z_j - c_j), j \in 1, 2 \ldots n\}$
While $val < 0$ calculate $\theta_k = \min\limits_{i \in 1,2 \ldots m} \left\{ \frac{X_{B_i}}{\alpha_i^k} | \ \alpha_i^k > 0 \right\} = \frac{x_{B_r}}{\alpha_r^k}$ and go to next step

Step 3 Update the basis by leaving the variable x_k and entering the variable $x_l = (x_{B_r})$ and go to Step 1.

Note: The variables in the basis are named as $(x_{B_1}, x_{B_2}, \ldots x_{B_m})$ and in the present case we have initial basic variables as slack variables taken in the order $s_1, s_2, \ldots s_m$. So one can understand that initially $x_{B_1} = s_1, x_{B_2} = s_2, \ldots x_{B_m} = s_m$. Similarly when the basis is updated by adding a variable x_k to the basis by replacing a variable $x_l = x_{B_r}$, then the new r^{th} basic variable will be $x_{B_r} = x_l$.

Write a MATLAB code for the simplex method and test your program on the following examples:

1 $Max. \ \ z = x_1 + 2x_2, \ \ subject\ to \ \ -x_1 + x_2 \leq 1, \ x_1 + x_2 \leq 2, \ x_1, x_2 \geq 0.$

2 $Max. \ \ z = 4x_1 + 6x_2 + 3x_3 + x_4,$ (**Ans:** $x_1 = 1/3, x_3 = 4/3 s_2 = 3; z = 16/3$
subject to
$x_1 + 4x_2 + 8x_3 + 6x_4 \leq 11, \ 4x_1 + x_2 + 2x_3 + x_4 \leq 7, \ 2x_1 + 3x_2 + x_3 + 2x_4 \leq 2, \ x_1, x_2, x_3 \geq 0.$

3 $Min. \ \ z = -3/4x_4 + 20x_5 - 1/2x_6 + 6x_7,$ (**Ans:** $x_1 = 3/4, x_4 = 1, x_6 = 1; z = -5/4$
subject to
$x_1 + 1/4x_4 - 8x_5 - x_6 + 9x_7 = 0, \ x_2 + 1/2x_4 - 12x_5 - 1/6x_6 + 3x_7 = 0, \ x_3 + x_6 = 1,$
$x_i \geq 0, i = 1, 2 \ldots 7$

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Optimization Techniques (UMA-035)
Lab Experiment- 4 (The Big-M method)

The Big M method

Consider an LPP with mixed type of constraints ($\leq, \ \geq, \ =$) and then convert the problem into standard form as given below.

$$\text{(P)} \qquad \text{Max } z = C^t X$$
$$\text{subject to} \qquad AX = b, \quad X \geq 0$$

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Assume that the matrix A does not have an identity submtrix in it. Now, add artificial variables R_i for $i \in \{1, 2, \ldots, \ m\}$ so that the new problem has a identity submatrix in it, and the matrix A is now updated as $[AI]$. Assign a value $M > 10^5 max(c_i)$, for convenience, the value of M can be taken as $M = 10^6$.
Now the new problem is of the following form:

$$(P_s) \qquad \text{Max } z = C^t X + 10^6.e^t R$$
$$\text{subject to} \qquad AX + IR_i = b, \quad X, R \ \geq 0$$

Where $R = (R_1, R_2, \ldots, R_m)$ is a vector of artificial variables and $e = (1, 1, \ldots, 1)^t$ is a vector of one's.

Initially define the following Input paramenters:

1. Enter the Matrix $A = [A \ \ I]$, where I is an identity matrix of order m.
2. Entet the the R.H.S. vector b and the cost matrix $C = [C \ \ 10^6 e]$.
3. Define [m,n]=size (A)
4. Input the variables $R_1, R_2, \ldots R_m$ as basic variables.

Now apply simplex method to solve the problem (P_s). Suppose an optimal basic feasible solution of the problem (P_s) so obtained is X_B.

If ($X_{B_i} = R_j > 0$, for some $i, \ j \in \{1, 2, \ldots, \ m\}$) %i.e. *artificial variable appear in basis*
  display(The problem (P) is infeasible)
Else ($X_{B_i} \neq R_j$, for all $i, \ j \in \{1, 2, \ldots, \ m\}$)
  display (X_B is an optimal basic feasible solution of problem (P))

Note: The variables in the basis are named as $(x_{B_1}, x_{B_2}, \ldots x_{B_m})$ and in the present case we have initial basic variables as artificial variables taken in the order $R_1, R_2, \ldots R_m$. So one can understand that initially $x_{B_1} = R_1, x_{B_2} = R_2, \ldots x_{B_m} = R_m$. The idea behind Big M method is to remove artifical variables from the basis

Similarly when the basis is updated by adding a variable x_k to the basis by replacing a variable $x_l = x_{B_r}$, then the new r^{th} basic variable will be $x_{B_r} = x_l$.

Write a MATLAB code for the simplex method and test your program on the following examples:

1. $Min. \ \ z = 3x_1 + 5x_2, \quad S.T. \ \ x_1 + 3x_2 \geq 3, \ x_1 + x_2 \geq 2, \ x_1, x_2 \geq 0.$
2. $Min. \ \ z = 12x_1 + 10x_2, \quad S.T. \ \ 5x_1 + x_2 \geq 10, \ 6x_1 + 5x_2 \geq 30, \ x_1 + 4x_2 \geq 8, \ x_1, x_2 \geq 0.$
3. $Max. \ \ z = 3x_1 + 2x_2, \quad S.T. \ \ x_1 + x_2 \leq 2, \ x_1 + 3x_2 \leq 3, \ x_1 - x_2 = 1 \ x_1, x_2 \geq 0.$

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Optimization Techniques (UMA-035)
Lab Experiment- 5 (The Two phase method)

The Two phase method

Consider an LPP with mixed type of constraints ($\leq$, $\geq$, $=$) and then convert the problem into standard form as given below.

$$\text{(P)} \qquad \text{Max } z = C^t X \qquad \text{subject to} \quad AX = b, \quad X \geq 0$$

Assume that the matrix A does not have an identity submatrix in it. Now, add artificial variables R_i for $i \in \{1, 2, \dots, m\}$ so that the new problem has a identity submatrix in it, and the matrix A is now updated as $[A \ I]$. Now construct the Phase-I problem as

$$(PH_1) \qquad \min z = O^t X + e^t R \qquad \text{subject to} \quad AX + IR_i = b, \quad X, R \geq 0$$

Where $R = (R_1, R_2, \dots, R_m)^t$ is a vector of artificial variables and $e = (1, 1, \dots, 1)_{m \times 1}$ is a vector of one's and $O = (0, 0 \dots, 0)_{n \times 1}$ is a vector of zeros.

Initially define the following Input paramenters:

1. Enter the Matrix $A = [A \ \ I]$, where I is an identity matrix of order m.
2. Entet the the R.H.S. vector b and the cost matrix $C = [O \ \ e]_{(n+m) \times 1}$.
3. Define [m,n]=size (A)
4. Input the variables $R_1, R_2, \dots R_m$ as initial basic variables.

Now apply simplex method to solve the problem (PH_1). Suppose an optimal basic feasible solution of the problem (P_s) so obtained is X_B.

If ($X_{B_i} = R_j > 0$, for some $i, \ j \in \{1, 2, \dots, m\}$) %i.e. artificial variable appear in basis
display(The problem (P) is infeasible)
Else ($X_{B_i} \neq R_j$, for all $i, \ j \in \{1, 2, \dots, m\}$)
display (X_B is an optimal basic feasible solution of problem (PH_1) and goto Phase II)

Phase-II

Treat the optimal basic feasible solution of Phase-I as initial basic feasible solution of Phase-II, while incorporating the folloiwng:

1. Update the cost matrix $C = (c_1, c_2 \dots c_n)^t$ from original variables.
2. Update the matrix Alpha =Alpha(1:n,:) (where Alpha=inv(B)*A)
   (means ignore the columns of artifial variables and update Alpha matrix by considering only first n columns of original variables)
3. Calculate $Z_j - c_j$ and follow the simplex procedure to obtain an optimal basic feasible solution of the problem (P).

Note: The variables in the basis are named as $(x_{B_1}, x_{B_2}, \dots x_{B_m})$ and in the present case we have initial basic variables as artificial variables taken in the order $R_1, R_2, \dots R_m$. So one can understand that initially $x_{B_1} = R_1, x_{B_2} = R_2, \dots x_{B_m} = R_m$. The idea behind Phase-I method is to remove artifical variables from the basis and obtain an initial basic feasible solution of the problem (P). Then Phase-II uses the BFS obtained in Phase-I to get an optimal BFS of problem (P).

Write a MATLAB code for the two phase method and test your program on the following examples:

1. $Min.\ \ z = 3x_1 + 5x_2, \quad S.T.\ \ x_1 + 3x_2 \geq 3,\ x_1 + x_2 \geq 2,\ x_1, x_2 \geq 0.$
2. $Min.\ \ z = 12x_1 + 10x_2, \quad S.T.\ \ 5x_1 + x_2 \geq 10,\ 6x_1 + 5x_2 \geq 30,\ x_1 + 4x_2 \geq 8,\ x_1, x_2 \geq 0.$
3. $Max.\ \ z = 3x_1 + 2x_2, \quad S.T.\ \ x_1 + x_2 \leq 2,\ x_1 + 3x_2 \leq 3,\ x_1 - x_2 = 1\ x_1, x_2 \geq 0.$

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Optimization Techniques (UMA-035)
Lab Experiment- 6 (The dual simplex method)

The dual simplex method

Convert the given linear programming problem in the following form:

$$(P) \qquad \min / \max z = C^t X + O^t s$$
$$\text{subject to} \qquad AX + Is = b, \quad X, s \geq 0$$

Where $s = (s_1, s_2, \ldots, s_m)^t$ is a vector of slack variables and $O = (0, 0 \ldots, 0)_{n\times 1}$ is a vector of zeros. Also Assume that atleast one of the component b_i of the RHS vector $b = (b_1, b_2, \ldots, b_m)$ is negative

Initially define the following Input paramenters:

1. Enter the Matrix $A = [A \quad I]$, where I is an identity matrix of order m.
2. Entet the the R.H.S. vector b and the cost matrix $C = [c \quad O]_{(n+m)\times 1}$.
3. Define [m,n]=size (A)
4. Input the variables $s_1, s_2, \ldots s_m$ as initial basic variables.

Now construct the simplex table using $s_1, s_2, \ldots, s_m$ as initial basic variables. If the simplex table depicts **an optimal but Infeasible solution**, then dual simplex method is applicable. So apply the following procedure.

1. Select the leaving variable as $X_{B_r} = \min\limits_i \{X_{B_i} \mid X_{B_i} < 0\}$
2. Select the entering variable x_k using the formula $\dfrac{z_k - c_k}{y_{rk}} = \min_j \left\{ \dfrac{|z_j - c_j|}{|y_{rj}|} \quad : \; y_{rj} < 0 \right\}$
3. Now update the basis as by removing r^{th} basic variable with k^{th} nonbasic variable. Again construct the simplex table and repeat the above procedure until an optimal basic feasible soluiton is not obtained.

Write a MATLAB code for the dual simplex method and test your program on the following examples:

1. $Min. \;\; z = 3x_1 + 5x_2, \;\; S.T. \;\; x_1 + 3x_2 \geq 3, \; x_1 + x_2 \geq 2, \; x_1, x_2 \geq 0.$
2. $Min. \;\; z = 12x_1 + 10x_2, \;\; S.T. \;\; 5x_1 + x_2 \geq 10, \; 6x_1 + 5x_2 \geq 30, \; x_1 + 4x_2 \geq 8, \; x_1, x_2 \geq 0.$
3. $min. \;\; z = 3x_1 + 2x_2, \;\; S.T. \;\; x_1 + x_2 \leq 1, \; x_1 + 2x_2 \geq 3, \; x_1, x_2 \geq 0.$

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Optimization Techniques (UMA-035)
Lab Experiment- 7 (Least cost method)

Least cost method of Transportation problem

Consider a cost matrix representation of a Transportation problem:

				$u_i \downarrow$
x_{11} c_{11}	x_{12} c_{12}	$\cdots$	x_{1m} c_{1n}	a_1
x_{21} c_{21}	x_{22} c_{22}	$\cdots$	x_{2m} c_{2n}	a_2
$\vdots$				$\vdots$
x_{m1} c_{m1}	x_{m2} c_{m2}	$\cdots$	x_{mn} c_{mn}	a_m
b_1	b_2	$\cdots$	b_n	

Here a_i is the availability of the product at source S_i and b_j is the requirement of the same at destination D_j, c_{ij} represents the cost of trasporting a unit product from source S_i to destination D_j. The variable x_{ij} is the quantity to be transported from the source i to destination j.

Initially define Input paramenters:

1. Enter the number of sources as m, and destinations as n.
2. Enter the cost coefficients c_{ij}, the availabilty at i^{th} source as a_i and demand at j^{th} destination as b_j for each $i = 1, 2 \ldots m$, $j = 1, 2, \ldots n$.

Intially take $k = 1$

Step 1: Define $c_{pq} = \min(c_{ij})$, and assign $x_{pq} = \min(a_p, b_q)$, go to Step 2.

Step 2: If $\min(a_p, b_q) = a_p$, then update $b_q = b_q - a_p$, $a_p = a_p - x_{pq}$ else $\min(a_i, b_j) = b_q$, then update $a_p = a_p - b_q$, $b_q = b_q - x_{pq}$.

Step 3 Assigne $c_{pq} = 10^5$ (a very large no.) , Set$k = k + 1$, if $k = m + n - 1$ go to Step 4 else go to Step 1.

Step 4: Stop and note the BFS and calculate the objective function value $z = \sum_{i,j} c_{ij} x_{ij}$

Write a MATLAB code to compute the basic feasible solutions of a Transportation problem using Northwest corner rule and test your program on the following set of examples:

1. Consider the cost matrix of the following transportation roblem

	D_1	D_2	D_3	D_4	a_i
S_1	2	10	4	5	12
$S_2$1	6	12	8	11	25
S_3	3	9	5	7	20
b_j	25	10	15	5	

2. Consider the cost matrix of the following transportation roblem

	D_1	D_2	D_3	D_4	D_5	a_i
S_1	3	11	4	14	15	15
S_2	6	16	18	2	28	25
S_3	10	13	15	19	17	10
S_4	7	12	5	8	9	15
b_j	20	10	15	15	5	

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Optimization Methods (UMA-035)
Lab Experiment - 8 (Multi-objective LPP)

Algorithm of Weighted Sum Method to multi-objective LPP

Step 1: Transform the multi-objective linear programming problem (P1) into its equivalent single objective linear programming problem (P2).

(P1)

Maximize/Minimize (C_iX), i=1,2,…,m

Subject to

AX<= or =or >= b

X>=0.

(P1)

Maximize/Minimize (C_1X+ C_2X+…+ C_mX)/m

Subject to

AX<= or =or >= b

X>=0.

Step 2: Find an optimal solution of the problem i.e., an efficient solution of the problem (P1) by an appropriate method (Simplex method or Big-M method or Two-Phase method).

Write a MATLAB code to solve the following multi-objective LPPs by weighted sum method:

1. Maximize ($3x_1 + 2x_2+4x_3$)
   Maximize ($x_1 + 5x_2+3x_3$)
   Subject to
   $2x_1 +4x_2 +x_3 <= 8$,
   $3x_1 + 5x_2 +4x_3 >= 15$,
   $x_1>=0$, $x_2>= 0$, $x_3>= 0$

2. Maximize ($x_1 + 4x_2+x_3$)
   Maximize ($2x_1 + 7x_2+5x_3$)
   Subject to
   $x_1+x_2 +x_3 <= 8$,
   $x_1+ 5x_2 +4x_3 >= 15$,
   $x_1>=0$, $x_2>= 0$, $x_3>= 0$

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Optimization Methods (UMA-035)
Lab Experiment - 9 (Fibonacci Search Technique)

Algorithm of Fibonacci Search Technique

Fibonacci numbers

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
1	1	2	3	5	8	13	21	34

$F_0 = 1, F_1 = 1, F_n = F_{n-1} + F_{n-2}, n > 1.$

Step 1: Using the relation, Measure of effectiveness= $\frac{Interval\ \ of\ uncertainty}{L_0}$, find the value of Measure of effectiveness.

Step 2: Using the relation $\frac{1}{F_n} \leq$ Obtained value of measure of effectiveness, find the smallest natural number n.

Step 3: Store the given interval$[a, b]$

Step 3: Find $L_0 = b - a$

Step 4:for $i = n$, find

$x_1 = a + \frac{F_{i-2}}{F_i} L_0,$ and $x_2 = a + \frac{F_{i-1}}{F_i} L_0,$

Step 5: If $f(x_1) > f(x_2)$ and the problem is of minimum. Then, repeat Step 3 with $n = n - 1 a = x_1$and $b = b$.

If $f(x_1) < f(x_2)$ and the problem is of minimum. Then, repeat Step 3 with $n = n - 1 a = a$and $b = x_2$.

If $f(x_1) > f(x_2)$ and the problem is of maximum. Then, repeat Step 3 with $n = n - 1 a = a$and $b = x_2$.

If $f(x_1) < f(x_2)$ and the problem is of maximum. Then, Then, repeat Step 3 with $n = n - 1 a = x_1$and $b = b$.

Step 6: Repeat Step 3 to upto $i = 2$.

Write a MATLAB code to solve the following problem

Example:

Minimize the function $x(x-2)$, $0 \leq x \leq 1.5$ within the interval of uncertainty $0.25L_0$.