Appendix A: Mathematical double duadratic queue model applied for community mobility analysis

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This section introduces the proposed Double Quadratic Queue (DQQ) model. By adopting the fluid queue derivation process, the model considers the two distinct stages and differentiates itself from a typical single-stage fluid queue model. The fluid queue modeling approach is a method used to analyze and model the behavior of queues in a dynamic setting, taking into account varying arrival and discharge rates. Newell's approach combines Taylor expansion, calculus, and geometric representation to derive an algebraic expression for the queue length at any given time. The DQQ model incorporates both geometric representation and algebraic expression. This fluid queue process helps visualize the relationships between arrival and discharge rates, queue length, and the passage of time. The ultimate goal of the subsequent derivation process is to obtain an algebraic expression for the queue length or cumulative change, which can be employed to study the queue's or system's behavior analytically.

Table 1 summarizes the notations used in the DQQ model.

Table 1 Notations and definitions used in this paper

Notations	**Definitions**
$\pi(t)$	net flow rate at time t
$Q(t)$	ridership rate at time t, $\frac{dQ(t)}{dt} = \pi(t)$
t_0	start time of disruption, $t_0 = 0$
t_1	time instance with maximum negative net flow rate
t_2	time instance with maximum cumulative ridership change
t_3	time instance with maximum positive net flow rate
t_4	end time of recovery process, where reaching a new equilibrium condition
D	disruption duration, $D = t_2 - t_0$
R	recovery duration, R $= t_4 - t_2$
P	entire duration of two normal/equilibriums, $P = t_4 - t_0$
m	Relative disruption duration ratio, $m = \frac{t_2 - t_0}{t_4 - t_0} = \frac{D}{P} = \frac{D}{D+R}$
r	recovery to disruption (ROD) ratio, $r = \frac{R}{D}$, which is the proportion of recovery in relation to the disruption, indicating how well the system recovers from the disruption
Q_{max}	maximum ridership changes at time t_2
Δ_e	permanent loss at end time t_4

ρ	parameter in the single quadratic-form net flow rate function
γ	parameter in the cubic-form net flow rate function
α, β	parameters in the double quadratic-form net flow rate function

To maintain generality, we normalize both time and ridership together such that their cumulative changes from the baseline condition at the beginning of the disruption are set to 0, i.e., $t_0 = 0$. By doing so, the entire duration of the disruption and recovery can be consistently represented as:

$$P = t_4 \tag{1}$$

$$D = t_2 = mP \tag{2}$$

$$r = \frac{R}{D} = \frac{1-m}{m} \tag{3}$$

Typically, parameters t_1, t_2, t_3, t_4, Q_{max}, Δ_e are observable directly from time series datasets. In the DQQ model, we assume the net flow rate, $\pi(t)$, can be represented as double quadratic forms, as shown in **Figure 1**, which has three intersection points, t_0, t_2, and t_4, with the horizontal axis.

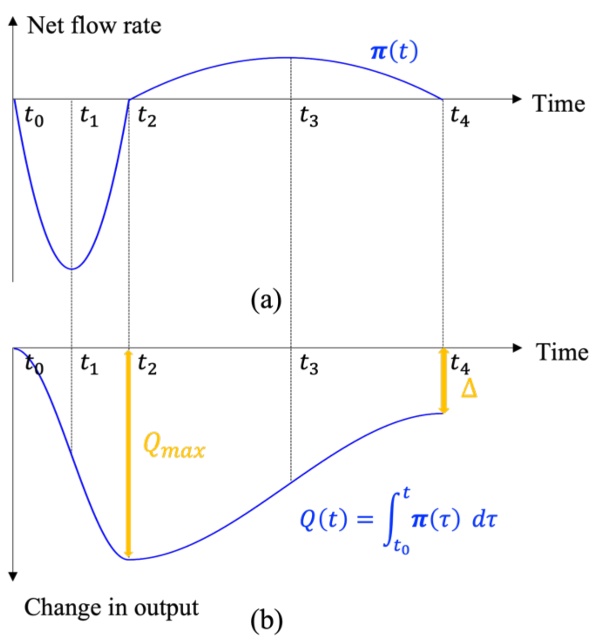

Figure 1 Graphical illustration of DQQ model for a single wave in transit cumulative ridership changes

The first-stage net flow rate, $\pi_D(t)$, could be approximated by Taylor's Theorem with $k = 2$ at $t = t_1$:

$$\pi_D(t) = \pi_D(t_1) + \pi_D{}'(t_1) \cdot (t - t_1) + \frac{\pi_D{}''(t_1)}{2}(t - t_1)^2 \tag{4}$$

Considering the parabola opens downwards, we have:

$$\alpha = \frac{\pi_D{}''(t_1)}{2} \tag{5}$$

At the first stage of DQQ model, $\pi_D(t_0) = \pi_D(t_2) = 0$.

$$\pi_D(t_0) = \pi_D(t_1) - \alpha(t_0 - t_1)^2 \tag{6}$$

$$\pi_D(t_2) = \pi_D(t_1) - \alpha(t_2 - t_1)^2 \tag{7}$$

We have

$$\pi_D(t_1) = \alpha(t_0 - t_1)^2 \tag{8}$$

$$t_0 + t_2 = 2t_1 \tag{9}$$

By substituting **Eq. (8)** and **Eq. (9)** to **Eq. (4)**, the factored form of $\pi_D(t)$ can be represented by:

$$\pi_D(t) = \alpha(t - t_0)(t - t_2) \tag{10}$$

Similarly, considering Taylor's Theorem with $k = 2$ at $t = t_3$, we let $\beta = \frac{\pi_R{}''(t_1)}{2}$, then have $\pi_R(t) = \beta(t - t_2)(t - t_4)$.

Thus, the DQQ model can be represented as **Eq. (4)**. This paper aims to calibrate the unknown second-order gradient parameters, α and β.

$$\pi(t) = \begin{cases} \pi_D(t) = \alpha(t - t_0)(t - t_2), & t_0 < t < t_2 \\ \pi_R(t) = \beta(t - t_2)(t - t_4), & t_2 < t < t_4 \end{cases} \tag{4}$$

As depicted in **Figure 1(a)**, the disruption occurs, and ridership starts to decline at time t_0. At time t_1, the flow loss rate reaches its peak. Consequently, the transit system experiences the greatest passenger loss, Q_{max}, at time t_2, as illustrated in **Figure 2(b)**. Following t_2, transit ridership begins to recover. At t_3, the net flow recovery rate attains its maximum value. Eventually, the transit system reaches a new normal/equilibrium at t_4, where the Δ_e represents the gap between this new equilibrium and the initial equilibrium status. The duration of disruption and recovery processes determines the horizontal temporal interval and the vertical steepness of the adaptivity evolution curve, i.e. the relative location of roots in the DQQ model. Generally, the durations and the permanent loss can be used to assess the system's recovery capability.

As shown in **Figure 1**, the boundary conditions can be represented as Eq. (7)-(14).

$$\frac{d\pi_D(t_1)}{dt} = 0 \tag{5}$$

$$\frac{d\pi_R(t_3)}{dt} = 0 \tag{6}$$

$$Q(t_0) = 0 \tag{7}$$

$$Q(t_2) = Q_{max} \tag{8}$$

$$Q(t_4) = \Delta_e \tag{9}$$

(1) When $t_0 < t < t_2$, the disruption process is represented by $\pi_D(t)$.

The first order derivative of net flow rate in disruption process, $\pi_D(t)$, is obtained:

$$\frac{d\pi_D(t)}{dt} = \alpha(2t - t_2) \tag{10}$$

Substituting Eq. (4) and $\alpha \neq 0$ leads to the following symmetrical relationships between t_0 and t_2:

$$t_2 = 2t_1 \tag{11}$$

Virtual queue length or cumulative flow change at time t, $Q(t)$ can be obtained:

$$Q(t) = \int_{t_0}^{t} \pi_D(t)\, d\tau \tag{12}$$

By substituting $\pi_D(t)$ function in Eq. (4), $Q(t)$ can be represented in terms of t_2 and α:

$$Q(t) = \frac{\alpha}{6} t^2 (2t - 3t_2) \quad (13)$$

As stated above, the maximum queue length is achieved at time t_2:

$$Q(t_2) = \frac{\alpha}{6} [2{t_2}^3 - 3{t_2}^3] = -\frac{\alpha}{6} {t_2}^3 \quad (14)$$

The Eq. (14) dedicates that Q_{max} can be represented by the time instance t_2 and the curvature parameter α. Since Q_{max} and t_2 can be observed from real-life data, the curvature rate α can be calibrated.

(2) When $t_2 < t < t_4$, the recovery process is represented by $\pi_R(t)$.

The first order derivative of $\pi_R(t)$ is derived as:

$$\frac{d\pi_R(t)}{dt} = \beta(2t - t_2 - t_4) \quad (15)$$

By substituting Eq. (6) and $\beta \neq 0$, t_2 and t_4 are symmetric about t_3:

$$t_2 + t_4 = 2t_3 \quad (16)$$

The queue length $Q(t)$ during the recovery process is expressed as:

$$Q(t) = Q(t_2) + \int_{t_2}^{t} \pi_R(t)\, d\tau \quad (17)$$

Substituting Eq. (14) and $\pi_R(t)$ function in Eq. (4) leads to $Q(t)$ expression:

$$Q(t) = -\frac{\alpha}{6} {t_2}^3 - \beta(t - t_2)^2 \left[\frac{t_4 - t_2}{2} - \frac{t - t_2}{3}\right] \quad (18)$$

The final permanent loss is determined at time t_4, i.e. $\Delta_e = Q(t_4)$:

$$Q(t_4) = -\frac{\alpha}{6} {t_2}^3 - \frac{\beta}{6} (t_4 - t_2)^3 \quad (19)$$

Since Δ_e can be observed from real-life data, and the α can be obtained using Eq. (14), the β can be calibrated.

Especially, when the ridership is fully recovered at the end of the adaptative evolution, i.e. $\Delta_e = 0$. Substituting Eq. (19) leads to the internal relationship between the α and β in terms of parameter m or r.

$$\beta = \alpha(\frac{m}{m-1})^3 = -\alpha r^{-3} \tag{20}$$