1.4 Concentration Bounds Proof (Chebyshev & Chernoff)

$\{X_1, \dots, X_n\}$ iid $N(0,1)$ $\mu = 0, \sigma^2 = 1$

Markov's inequality: $P(X > \varepsilon) \le \frac{E(X)}{\varepsilon}$ for $X > 0$

set $X = (\bar{X}_n - E(\bar{X}_n))^2 = (\bar{X}_n)$ where $E(X) = E((\bar{X}_n - E(\bar{X}_n))^2) = Var(\bar{X}_n) = \frac{1}{n}$

Then $P(\underbrace{(\bar{X}_n - E(\bar{X}_n))^2 > \varepsilon^2}_{=}) \le \frac{E((\bar{X}_n - E(\bar{X}_n))^2)}{\varepsilon^2} = \frac{1}{n\varepsilon^2}$

$P(\overbrace{|\bar{X}_n| > \varepsilon}) \le \frac{1}{n\varepsilon^2}$ Chebyshev inequality for $\bar{X}_n$ where $\{X_1, \dots, X_n\} \sim N(0,1)$

Now we want to show $P(|\bar{X}_n| > \varepsilon) < 2e^{-\frac{\varepsilon^2 n}{2}}$

for $P(X \ge \varepsilon) = P(e^X \ge e^\varepsilon) \le \frac{E(e^X)}{e^\varepsilon}$ By Markov's Ineq.

MGF for $X \sim N(0,1)$ $M_X(t) = E(e^{tx}) = \int e^{tx} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = e^{\frac{t^2}{2}}$

$\bar{X}_n = \frac{1}{n} \sum X_i$ $M_{\sum X_i}(t) = (M_X(t))^n = e^{nt^2/2}$

$M_{\bar{X}}(t) = (M_X(\frac{t}{n}))^n = e^{(\frac{t}{n})^2 \cdot \frac{1}{2} \cdot n} = e^{\frac{t^2}{2n}}$

$P(\bar{X}_n \ge \varepsilon) \le \frac{E(e^{\bar{X}_n})}{e^\varepsilon} \Rightarrow P(|\bar{X}_n| \ge \varepsilon) \le 2 \frac{E(e^{\bar{X}_n})}{e^\varepsilon} = e^{\frac{1}{2n} - \varepsilon}$

for any t $P(e^{tx} \ge e^{t\varepsilon}) \le \frac{E(e^{tx})}{e^{t\varepsilon}} = e^{\frac{t^2 - t\varepsilon 2n}{2n}}$ ← optimize for t

$\frac{d}{dt} \exp\left(\frac{t^2 - t\varepsilon 2n}{2n}\right) = \frac{t - \varepsilon n}{n} \cdot \cancel{e^{\frac{t^2 - t\varepsilon 2n}{2n}}} = 0 \rightarrow t = \varepsilon n$

$e^{\frac{t^2 - t\varepsilon 2n}{2n}}\Big|_{t = \varepsilon n} = e^{-\frac{\varepsilon^2 n}{2}} \Rightarrow P(|X_n| > \varepsilon) \le 2e^{-\frac{\varepsilon^2 n}{2}}$