STAT 201A

Homework 3 Markov Process

(1) Simulation of Markov process

(a) Writing the Markov process in matrix representation.

$P \in \mathbb{R}^{3\times 3}$ Solo

$$P \in \mathbb{R}^{3\times 3} = \begin{bmatrix} P_{1,1} & P_{1,2} & P_{1,3} \\ P_{2,1} & P_{2,2} & P_{2,3} \\ P_{3,1} & P_{3,2} & P_{3,3} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

(P_{ij} above the first matrix; P_{ij} above the second matrix)

⇒ The symbol P_{ij} denotes the probability of transitioning from State i to State j.
Since each row of the matrix adds up to 1, then the markov transition probability matrix is obtained.

(b) ~~Code in Jupyter~~ Code in Jupyter.

(2) (a) Convergence of the probability distribution of the markov process.

$$\Pi_\infty^T P = \Pi_\infty^T$$

$$\Rightarrow \begin{bmatrix} \Pi_{\infty,1} & \Pi_{\infty,2} & \Pi_{\infty,3} \end{bmatrix} \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix} = \begin{bmatrix} \Pi_{\infty,1} & \Pi_{\infty,2} & \Pi_{\infty,3} \end{bmatrix}$$

$$0.2\Pi_1 + 0.2\Pi_2 + 0.2\Pi_3 = \Pi_1$$
$$\Rightarrow 0.7\Pi_1 + 0.5\Pi_2 + 0.4\Pi_3 = \Pi_2$$
$$0.1\Pi_1 + 0.3\Pi_2 + 0.4\Pi_3 = \Pi_3$$

$$\Rightarrow -0.8\Pi_1 + 0.2\Pi_2 + 0.2\Pi_2 = 0 \quad ---(1)$$
$$0.7\Pi_1 - 0.5\Pi_2 + 0.4\Pi_3 = 0 \quad ---(2)$$
$$0.1\Pi_1 + 0.3\Pi_2 - 0.6\Pi_3 = 0 \quad ---(3)$$

from eqn (3)

$$\frac{0.6\Pi_3}{0.6} = \frac{0.1\Pi_1}{0.6} + \frac{0.3\Pi_2}{0.6}$$

$$\left(-8.9\Pi_1 + 46\Pi_2 + 5\Pi_1 + 23\Pi_1\right) = 0$$

$$\Pi_3 = \frac{1}{6}\Pi_1 + \frac{1}{2}\Pi_2$$

From eqn (2)

$$\Pi_3 = \frac{0.5}{0.4}\Pi_2 - \frac{0.7\Pi_1}{0.4}$$

$$\frac{1}{6}\Pi_1 + \frac{1}{2}\Pi_2 = \frac{5}{4}\Pi_2 - \frac{7}{4}\Pi_1$$

Ajar.

$$\frac{1}{6}\pi_1 + \frac{7}{4}\pi_1 = \frac{5}{4}\pi_2 - \frac{1}{2}\pi_2$$

$$\frac{23}{12}\pi_1 = \frac{3}{4}\pi_2$$

$$\pi_2 = \left(\frac{23}{12}\pi_1\right) \div \frac{3}{4}$$

~~$\pi_2 = \frac{23}{9}$~~

$$\boxed{\pi_2 = \frac{23}{9}\pi_1}$$

$$\pi_3 = \frac{1}{6}\pi_1 + \frac{1}{2}\pi_2$$

$$\pi_3 = \frac{1}{6}\pi_1 + \frac{1}{2}\left(\frac{23}{9}\pi_1\right)$$

$$\pi_3 = \frac{1}{6}\pi_1 + \frac{23}{18}\pi_1$$

$$\boxed{\pi_3 = \frac{13}{9}\pi_1}$$

$$\pi_P = \left[\pi_1 \quad \frac{23}{9}\pi_1 \quad \frac{26}{18}\pi_1\right]$$

$$\Rightarrow \pi_P = \left[\frac{1}{5} \quad \frac{23}{45} \quad \frac{13}{45}\right]$$

(3) (b) Mean Arrival time

$M_1 = 1 + 0.2M_1 + 0.7M_2 + 0.1M_3$
$M_2 = 1 + 0.2M_1 + 0.5M_2 + 0.3M_3$
$M_3 = 0$

Substituting M_3 in both M_1 and M_2.

$M_1 = 1 + 0.2M_1 + 0.7M_2 + 0.1(0)$
$M_1 = 1 + 0.2M_1 + 0.7M_2$
$M_1 - 0.2M_1 - 0.7M_2 = 1$

$0.8M_1 - 0.7M_2 = 1$ —— (1)

$M_2 = 1 + 0.2M_1 + 0.5M_2 + 0.3(0)$
$M_2 = 1 + 0.2M_1 + 0.5M_2$

$M_2 - 0.5M_2 - 0.2M_1 = 1$

$-0.2M_1 + 0.5M_2 = 1$ —— (2)

Solving eqn (1) and (2) Simultaneously.

$0.8M_1 - 0.7M_2 = 1$
$-0.2M_1 + 0.5M_2 = 1$

$\frac{0.8M_1}{0.8} = \frac{1 + 0.7M_2}{0.8} \Rightarrow M_1 = \frac{1 + 0.7M_2}{0.8}$

$$-0.2\left(\frac{1+0.7u_2}{0.8}\right)+0.5u_2=1$$

$$\frac{-0.2}{0.8}-\frac{0.14u_2}{0.8}+0.5u_2=1$$

$$-\frac{0.14u_2}{0.8}+0.5u_2=1+\frac{0.2}{0.8}$$

$$-0.175u_2+0.5u_2=1.25$$

$$\frac{0.325}{0.325}u_2=\frac{1.25}{0.325}$$

$$\boxed{u_2=3.84615}$$

$$u_1=\frac{1+0.7u_2}{0.8}$$

$$u_1=\frac{1+0.7(3.84615)}{0.8}$$

$$\boxed{u_1=4.61703}$$