

# Homework 3

December 3, 2023

## 1. Simulation of Markov Process.

a.

The transition matrix,  $P$ , is given by

$$P = \begin{pmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{pmatrix}$$

b.

```
[1]: import numpy as np
import scipy.linalg
import matplotlib.pyplot as plt

[2]: P = np.array([[0.2, 0.7, 0.1], [0.2, 0.5, 0.3], [0.2, 0.4, 0.4]])

np.random.seed(112523)
states = np.arange(1, len(P) + 1)

def markov_process(P, nrun, x0):
    xt = []
    current_state = x0

    for _ in range(nrun):
        current_state = np.random.choice(
            states,
            p = [P[current_state - 1, next_state - 1] for next_state in states]
        )
        xt.append(current_state)

    return x0, xt

out = markov_process(P, 10, 1)
print("Initial state:", out[0], "\nSimulated chain:", out[1])
```

Initial state: 1  
 Simulated chain: [2, 1, 3, 1, 2, 2, 2, 3, 3, 3]

## 2. Stationary Distribution.

a.

First,

$$P^T = \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.7 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.4 \end{pmatrix}$$

Then

$$\begin{aligned} P^T \cdot \pi &= \pi \\ (P^T - I)\pi &= 0 \\ \left[ \begin{pmatrix} 0.2 & 0.2 & 0.2 \\ 0.7 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} -0.8 & 0.2 & 0.2 \\ 0.7 & -0.5 & 0.4 \\ 0.1 & 0.3 & -0.6 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 & -\frac{9}{13} \\ 0 & 1 & -\frac{23}{13} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

On the other hand, we have,  $\pi_1 + \pi_2 + \pi_3 = 1$ . Hence

$$\pi^T = \left( \frac{1}{5}, \frac{23}{45}, \frac{13}{45} \right)$$

Check:

$$\pi^T \cdot P = \pi^T$$

```
[3]: pi_ = np.array([1/5, 23/45, 13/45])
      np.dot(pi_.T, P)
```

```
[3]: array([0.2, 0.51111111, 0.28888889])
```

Since the stationary distribution is the (appropriately normalized) left eigenvector of  $P$  corresponding to an eigenvalue of 1, we can calculate it in python with the following:

```
[4]: evals, evecs = np.linalg.eig(P.T)
      evec1 = evecs[:, np.isclose(evals, 1)] # np.isclose returns an array
      evec1 = evec1[:,0]
```

```
# normalize the eigenvector to sum to 1
stationary = evec1 / evec1.sum()
stationary = stationary.real
stationary
```

```
[4]: array([0.2          , 0.51111111, 0.28888889])
```

b.

```
[5]: def pi_norm(pi_0, n):
    norm = [np.linalg.norm(pi_0 - stationary)**2]
    P_mul = P.copy()

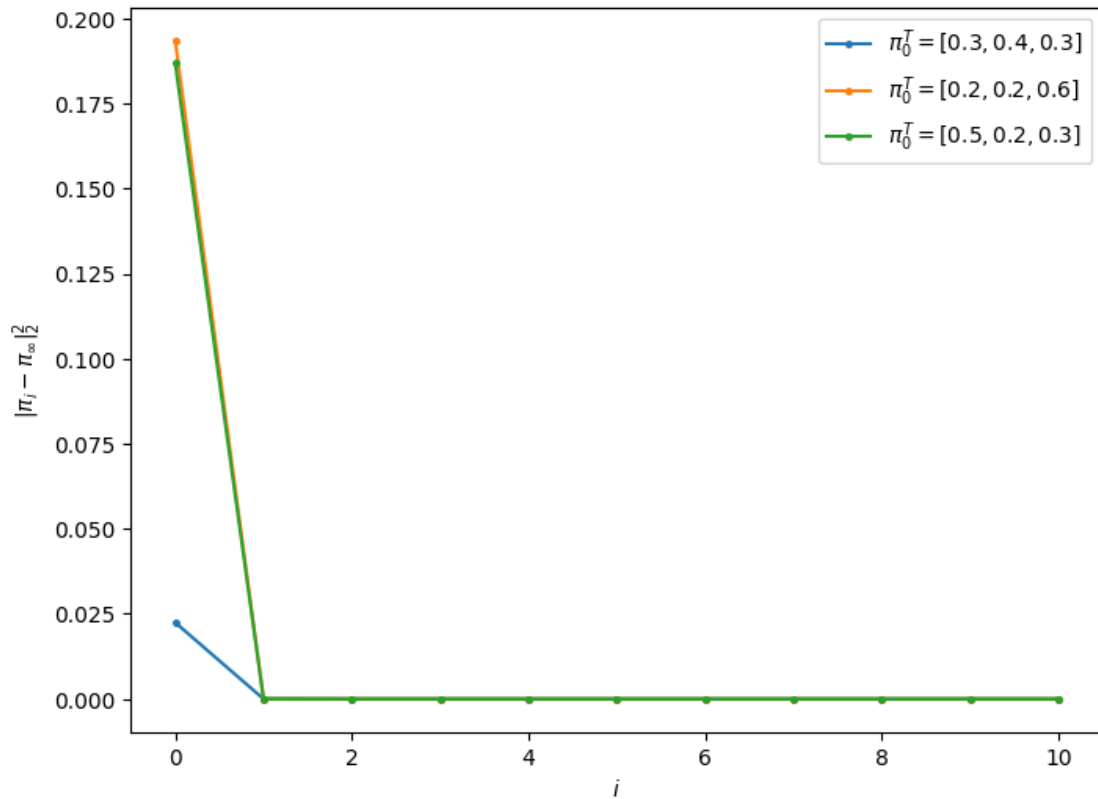
    for i in range(n):
        P_mul = np.matmul(P_mul, P)
        pi_t = np.matmul(pi_0, P_mul)
        norm.append(np.linalg.norm(pi_t - stationary)**2)

    return norm

initial = [np.array([0.3, 0.4, 0.3]),
           np.array([0.2, 0.2, 0.6]),
           np.array([0.5, 0.2, 0.3])]

n = 10
plt.figure(figsize = (8,6))

for pi_0 in initial:
    norm = pi_norm(pi_0, n)
    plt.plot(np.arange(0, n+1), norm, marker = ".", markersize = 5,
             label = r"$\pi_0^T = [{}]"$.format(', '.join(map(str, pi_0))))
plt.xlabel(r"$i$")
plt.ylabel(r"$\|\pi_i - \pi_{\infty}\|^2$")
plt.legend()
plt.show()
```



### 3. Absorbing State.

a.

```
[6]: np.random.seed(112523)

def markov_process2(P, x0):
    current_state = x0
    arrival_time = 0

    while current_state != 3:
        current_state = np.random.choice(
            states, p = [P[current_state - 1, next_state - 1] for next_state in
↪states]
        )
        arrival_time += 1
    return arrival_time
```

```
[7]: nsim = 1000
x0_1 = [markov_process2(P, x0 = 1) for _ in range(nsim)]
x0_2 = [markov_process2(P, x0 = 2) for _ in range(nsim)]
```

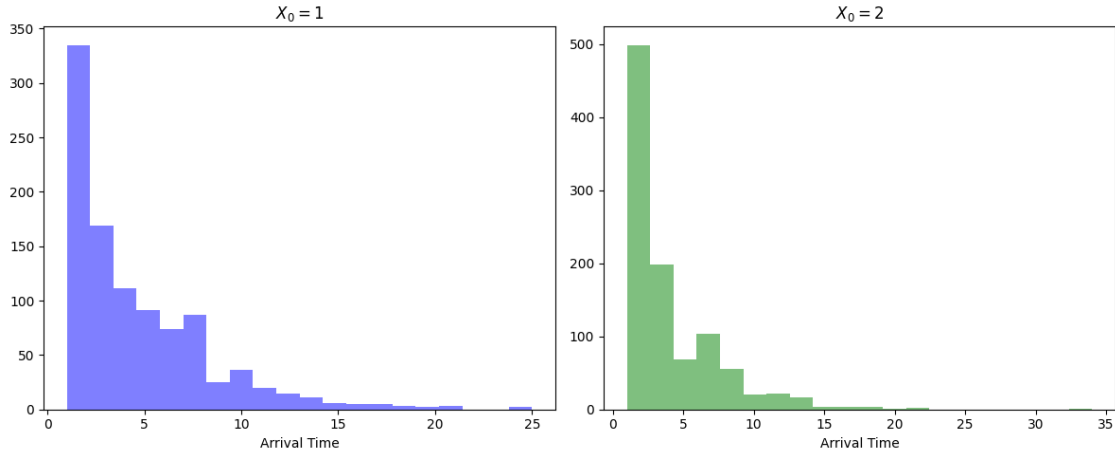
```

plt.figure(figsize = (12,5))
plt.subplot(1,2,1)
plt.hist(x0_1, bins = 20, color = "blue", alpha = 0.5)
plt.title(r"$X_0 = 1$")
plt.xlabel("Arrival Time")

plt.subplot(1,2,2)
plt.hist(x0_2, bins=20, color = "green", alpha = 0.5)
plt.title(r"$X_0 = 2$")
plt.xlabel("Arrival Time")

plt.tight_layout()
plt.show()

```



**b.**

Given

$$\mu_i = 1 + \sum_{j=1}^3 p_{ij} \mu_j$$

with  $\mu_i = \mathbb{E}[T_i]$ .

For an absorbing state 3 ( $\mu_3 = 0$ ), then we have

$$\begin{aligned} \mu_1 &= 1 + 0.2\mu_1 + 0.7\mu_2 + 0.1\mu_3 \\ \mu_2 &= 1 + 0.2\mu_1 + 0.5\mu_2 + 0.3\mu_3 \\ \mu_3 &= 0 \end{aligned}$$

This implies

$$\begin{aligned}\mu_1 &= \frac{60}{13} \\ \mu_2 &= \frac{50}{13} \\ \mu_3 &= 0\end{aligned}$$

```
[8]: A = np.array([[0.8, -0.7, -0.1], [-0.2, 0.5, -0.3], [0, 0, 1]])
      b = np.array([1, 1, 0])

      mu = np.linalg.solve(A, b)
      mu
```

```
[8]: array([4.61538462, 3.84615385, 0.        ])
```

```
[9]: [np.mean(x0_1), np.mean(x0_2)]
```

```
[9]: [4.648, 3.811]
```

The results we found in part (a) is close to the theoretical mean arrival time to the absorbing state.