homework3

December 2, 2023

```
[19]: #1a
```

$$P = \begin{bmatrix} 0.2 & 0.7 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

```
[23]: #1b
import numpy as np

def simulate_markov_chain(transition_matrix, initial_state, num_steps):
    current_state = initial_state
    states_result = [current_state]

    for _ in range(num_steps - 1):
        current_state = np.random.choice(range(1,4), p=transition_matrix[current_state-1])
        states_result.append(current_state)

    return states_result

# define transition matrix
transition_matrix = np.array([[0.2, 0.7, 0.1],
                              [0.2, 0.5, 0.3],
```

```
                                [0.2, 0.4, 0.4]])

# set initial state and the number of steps
initial_state = 1  # initial state is 1
num_steps = 20  # the number of steps from simulation

# simulation
result = simulate_markov_chain(transition_matrix, initial_state, num_steps)

# print the result
print(f"Results of a {num_steps}-step Markov Chain Simulation:{result}")
```

```
Results of a 20-step Markov Chain Simulation:[1, 2, 3, 2, 3, 3, 3, 3, 2, 2, 2,
3, 2, 1, 2, 2, 3, 3, 3, 2]
```

[20]:
```
#2a
```

(a) According to 1.(a), the transpose of transition matrix P is:

$$P^T = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.7 & 0.5 & 0.4 \\ 0.1 & 0.3 & 0.4 \end{bmatrix}$$

then $(P^T - I) = \begin{bmatrix} -0.8 & 0.2 & 0.2 \\ 0.7 & -0.5 & 0.4 \\ 0.1 & 0.3 & -0.6 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & \frac{13}{28} & -\frac{23}{28} \\ 0 & -\frac{13}{4} & \frac{23}{4} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{23}{13} \\ 0 & 1 & -\frac{23}{13} \end{bmatrix}$

$$= \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{23}{13} \\ 0 & 0 & 0 \end{bmatrix}$$

let $\pi_\infty = [\pi_1 \ \pi_2 \ \pi_3]^T$

$(P^T - I)\pi_\infty = 0 \Rightarrow \begin{bmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 1 & -\frac{23}{13} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = 0$

$$\Rightarrow \begin{cases} \pi_1 - \frac{1}{4}\pi_2 - \frac{1}{4}\pi_3 = 0 \\ \pi_2 - \frac{23}{13}\pi_3 = 0 \end{cases}$$

let $\pi_3 = a$, then $\pi_2 = \frac{23}{13}a$, $\pi_1 = \frac{1}{4}a + \frac{1}{4} \times \frac{23}{13}a = \frac{9}{13}a$

thus, $\pi_\infty = \begin{bmatrix} \frac{9}{13}a \\ \frac{23}{13}a \\ a \end{bmatrix}$ where $a \in R$

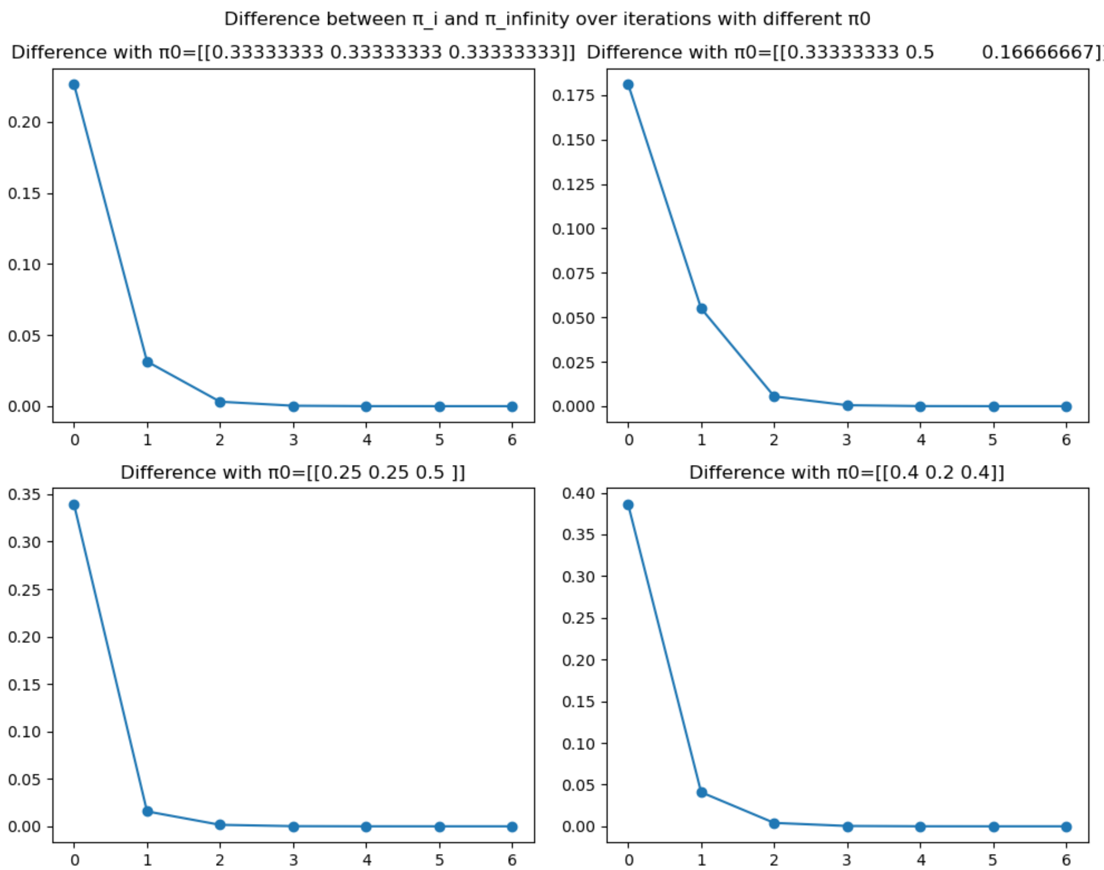
[44]:
```python
#2b
import numpy as np
def compute_pi_i_transpose(transition_matrix, pi_0_transpose, i):
    pi_i_transpose=pi_0_transpose
    for j in range(i):
        pi_i_transpose=pi_i_transpose@transition_matrix
    return pi_i_transpose


#initialize transition matrix and initial probability distribution
transition_matrix = np.array([[0.2, 0.7, 0.1],
                              [0.2, 0.5, 0.3],
```

```
                              [0.2, 0.4, 0.4]])

pi_0=np.array([[1/3],[1/3],[1/3]])
pi_0_transpose=np.transpose(pi_0)

#set steps of markov chain
i=1000

result=compute_pi_i_transpose(transition_matrix, pi_0_transpose, i)
#print the result
print(f"pi_{i}_transpose is: {result}")

#when the first element of pi_100000_transpose is 0.2, "a" in pi_infinity␣
 ↪equals to 13/45
pi_infinity_transpose=np.array([[9/13*(13/45),23/13*(13/45),13/45]])
print("pi_infinity_transpose is:",pi_infinity_transpose)
```

```
pi_10_transpose is: [[0.2        0.51111111 0.28888889]]
pi_infinity_transpose is: [[0.2        0.51111111 0.28888889]]
```

[24]:
```
#compare with pi_infinity
import matplotlib.pyplot as plt

##define Euclidean norm between two vectors
def Euclidean_norm(vector1,vector2):
    difference = vector1 - vector2
    norm_value = np.linalg.norm(difference)
    return norm_value

##define difference between every pi_i and pi_infinity
def difference(transition_matrix, pi_0_transpose, pi_infinity_transpose, i):
    diff=[]
    pi_i_transpose=pi_0_transpose
    diff.append(Euclidean_norm(pi_i_transpose,pi_infinity_transpose))
    for j in range(i):
        pi_i_transpose=pi_i_transpose@transition_matrix
        diff.append(Euclidean_norm(pi_i_transpose,pi_infinity_transpose))
    return diff

##set the value
i=6
transition_matrix = np.array([[0.2, 0.7, 0.1],
                              [0.2, 0.5, 0.3],
                              [0.2, 0.4, 0.4]])
pi_infinity_transpose=np.array([[9/13*(13/45),23/13*(13/45),13/45]])

pi_01=np.array([[1/3],[1/3],[1/3]]) #first initial probability distribution
```

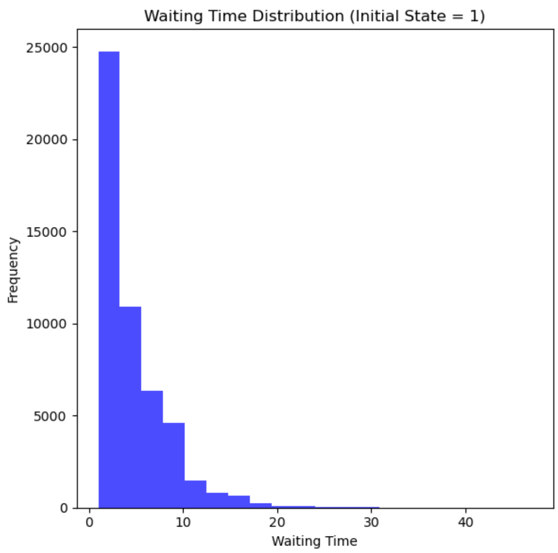
```python
pi_01_transpose=np.transpose(pi_01)

pi_02=np.array([[1/3],[1/2],[1/6]]) #second initial probability distribution
pi_02_transpose=np.transpose(pi_02)

pi_03=np.array([[1/4],[1/4],[1/2]]) #third initial probability distribution
pi_03_transpose=np.transpose(pi_03)

pi_04=np.array([[2/5],[1/5],[2/5]]) #fourth initial probability distribution
pi_04_transpose=np.transpose(pi_04)

##result
diff_values1=difference(transition_matrix, pi_01_transpose, pi_infinity_transpose, i)
diff_values2=difference(transition_matrix, pi_02_transpose, pi_infinity_transpose, i)
diff_values3=difference(transition_matrix, pi_03_transpose, pi_infinity_transpose, i)
diff_values4=difference(transition_matrix, pi_04_transpose, pi_infinity_transpose, i)
##plot the difference
fig, axs = plt.subplots(2, 2, figsize=(10, 8))

axs[0, 0].plot(range(i + 1), diff_values1, marker='o')
axs[0, 0].set_title(f'Difference with  0={pi_01_transpose}')

axs[0, 1].plot(range(i + 1), diff_values2, marker='o')
axs[0, 1].set_title(f'Difference with  0={pi_02_transpose}')

axs[1, 0].plot(range(i + 1), diff_values3, marker='o')
axs[1, 0].set_title(f'Difference with  0={pi_03_transpose}')

axs[1, 1].plot(range(i + 1), diff_values4, marker='o')
axs[1, 1].set_title(f'Difference with  0={pi_04_transpose}')

plt.suptitle('Difference between  _i and  _infinity over iterations with different  0')
plt.tight_layout()
plt.show()
```

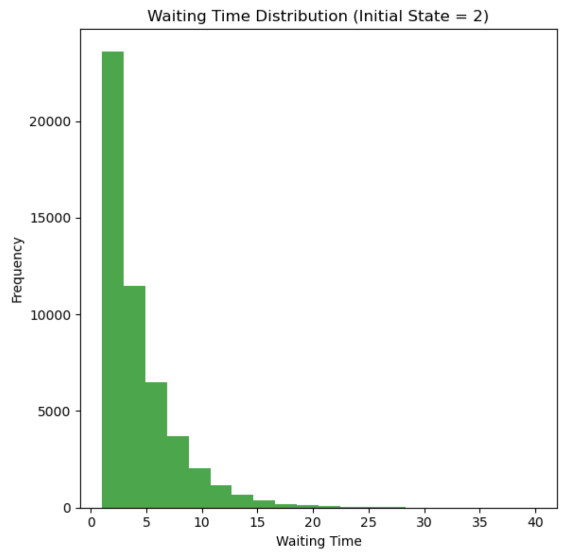


[10]:
```python
#3a
import time
import numpy as np

def wait_time(transition_matrix, initial_state):
    waiting_time = []
    for i in range(50000):
        current_state = initial_state
        states_result = [current_state]
        while current_state !=3:
            current_state = np.random.choice(range(1,4),
 ↪p=transition_matrix[(current_state-1),:])
            states_result.append(current_state)

        waiting_time.append(len(states_result)-1)
    return waiting_time

#set transition_matrix
transition_matrix = np.array([[0.2, 0.7, 0.1],
```

```
                              [0.2, 0.5, 0.3],
                              [0.2, 0.4, 0.4]])

#when initial state is 1
initial_state1 =1
waiting_time1=wait_time(transition_matrix, initial_state1)

#when initial state is 2
initial_state2 =2
waiting_time2=wait_time(transition_matrix, initial_state2)
```

[11]:
```
#plot the result
import matplotlib.pyplot as plt

plt.figure(figsize=(12, 6))

# subplot of initial state is 1
plt.subplot(1, 2, 1)
plt.hist(waiting_time1, bins=20, color='blue', alpha=0.7)
plt.title('Waiting Time Distribution (Initial State = 1)')
plt.xlabel('Waiting Time')
plt.ylabel('Frequency')

# subplot of initial state is 2
plt.subplot(1, 2, 2)
plt.hist(waiting_time2, bins=20, color='green', alpha=0.7)
plt.title('Waiting Time Distribution (Initial State = 2)')
plt.xlabel('Waiting Time')
plt.ylabel('Frequency')

plt.tight_layout()
plt.show()
```





[9]:
```python
#mean of arrival time
##initial state is 1
mean_arrival_time1=np.mean(waiting_time1)
print("Mean of the arrival time with X0=1 is:",mean_arrival_time1)
##initial state is 2
mean_arrival_time2=np.mean(waiting_time2)
print("Mean of the arrival time with X0=2 is:",mean_arrival_time2)
```

```
Mean of the arrival time with X0=1 is: 4.59907
Mean of the arrival time with X0=2 is: 3.84398
```

[ ]:
```python
#3(b)
```

Based on transition matrix P:

$$\mu_1 = E[T_1] = 1 + p_{11}\mu_1 + p_{12}\mu_2 + p_{13}\mu_3 = 1 + 0.2\mu_1 + 0.7\mu_2 + 0.1\mu_3$$

$$\mu_2 = E[T_2] = 1 + p_{21}\mu_1 + p_{22}\mu_2 + p_{23}\mu_3 = 1 + 0.2\mu_1 + 0.5\mu_2 + 0.3\mu_3$$

$$\mu_3 = E[T_3] = E[0] = 0$$

thus, solve the following linear system of equations:

$$\begin{cases} 0.8\mu_1 - 0.7\mu_2 = 1 \\ -0.2\mu_1 + 0.5\mu_2 = 1 \end{cases} \Rightarrow \begin{cases} \mu_1 = \frac{60}{13} \approx 4.6154 \\ \mu_2 = \frac{50}{13} \approx 3.8462 \end{cases}$$